Synthese (2013) 190:891–908
DOI 10.1007/s11229-012-0191-y

Substantive assumptions in interaction: a logical perspective

Olivier Roy · Eric Pacuit

Received: 19 July 2012 / Accepted: 19 July 2012 / Published online: 5 October 2012

Abstract In this paper we study substantive assumptions in social interaction. By substantive assumptions we mean contingent assumptions about what the players know and believe about each other's choices and information. We first explain why substantive assumptions are fundamental for the analysis of games and, more generally, social interaction. Then we show that they can be compared formally, and that there exist contexts where no substantive assumptions are being made. Finally we show that the questions raised in this paper are related to a number of issues concerning "large" structures in epistemic game theory.

Keywords Epistemic game theory · Epistemic logic · Harsanyi type spaces · Comparing information · Substantive assumptions · Structural assumptions

1 Introduction

R. Aumann (1987) famously wrote that common knowledge of the partition struture is:

> *not* an assumption, but a 'theorem', a tautology; [...] implicit in the model itself

What are such "theorems", implicit in the model itself? Why are they important? Can one distinguish them from *substantive* assumptions, for instance common knowledge of rationality? Are there models where no substantive assumptions are being made? If

O. Roy (✉)
Munich Center for Mathematical Philosophy, Ludwig-Maximilians-Universität München,
Munich, Germany
e-mail: Olivier.Roy@lmu.de

E. Pacuit
TiLPS, Tilburg University, Tilburg, The Netherlands
e-mail: E.J.Pacuit@uvt.nl

yes, how do such models relate to "large" structures, extensively studied in epistemic game theory?

This paper answers these questions. In Sect. 2 we explain the importance of precisely distinguishing substantive from structural assumptions. In Sect. 3 we use syntactic methods to compare substantive assumptions made in different structures, and we use this comparison, in turn, to show that there are indeed models where no substantive assumptions are being made. Section 5 situates this result within the extensive literature on "large" structures.

We focus on qualitative, also called logical models of information in games and social interaction (Fagin et al. 1995; Aumann 1999; van Ditmarsch et al. 2007). These are models of *all out*, as opposed to *graded* attitudes c.f. (Huber and Schmidt-Petri 2009). They have proved both historically and conceptually important in the analysis of games. Furthermore, they naturally lend themselves to a syntactic analysis, such as the one we provide in Sect. 3.[1]

The main point of this paper is conceptual, rather than a formal one. We wish to clarify some issues often implicitly assumed in epistemic logic and epistemic game theory, and to connect them with current research on larges structures, e.g. (Friedenberg and Meier 2010; Friedenberg 2010). Section 4 makes a modest formal contribution by studying a general ordering of "informativeness" of structures, and using this ordering to show the existence of structures where no substantive assumptions are being made. The existence of such structures is well-known. See e.g. (Heifetz 1999; Meier 2008). But it is not always explicitly acknowledged, and both existence and non-existence results are around (Heifetz and Samet 1998a). Our formal result, and the discussion in Sect. 5, aim at clarifying this issue. Furthermore, the method used to prove the result is new, and is, we think, conceptually illuminating. All in all our contribution can be seen as in connecting the dots, in order to clarify some notions that are crucial for our understanding of social interaction.

2 Motivations and basic definitions

One of the fundamental insights behind the *epistemic* approach to games (Aumann 1999; Brandenburger 2007) is that strategic interaction takes place in specific *contexts*, c.f. (Aumann and Dreze 2008). A context is a description of the players' information about each other, including information about the other's information - beliefs about others' beliefs, knowledge about the knowledge of others, and so on.

Informally[2], a substantive assumption is an assumption made in a specific (class of) contexts of interaction, but one that could be relaxed. In this paper we are interested in *epistemic* substantive assumptions, bearing on what agents *know* or *believe* in a certain context of strategic interaction. Common knowledge of rationality and common priors

[1] Models of graded beliefs, such as Harsanyi type spaces, have also be the subject of syntactic analysis. There are a number of interesting issues here about the appropriate choice of language and axiomatic system (see Fagin and Halpern 1994; Heifetz and Mongin 2001; Zhou 2009; Goldblatt 2008 for different approaches). Porting the present analysis to models of graded attitudes is an important task, but one that we leave for future work.

[2] We give a formal definition below.

are paradigmatic examples of such substantive assumptions. They are often used in epistemic modeling, but one can easily construct models of games where either of these condition fails to obtain.

Substantive assumptions play a key role in the epistemic foundations of game theory.[3] Epistemic characterization results can, arguably, be seen as drawing the behavioral or the normative consequences of making certain substantive assumptions. We have already mentioned common knowledge of rationality, which implies playing rationalizable strategies (Bernheim 1984; Pearce 1984). The more recent characterization of self-admissible strategies in terms of common assumption of admissibility (Brandenburger et al. 2008) provides another clear example.

Not all assumptions are substantive, though. As pointed out in the quote above, some assumptions[4] are "implicit" or built into the (epistemic) model one is working with. In partitional, a.k.a. S5 models of knowledge,[5] for instance, positive and negative introspection[6] are assumptions that cannot be relaxed—without leaving that class of models, of course. But some assumptions are even more tenacious: so-called "logical omniscience" or what philosophers call "extensionality" seem to be deeply entrenched in the kind of models that are common in epistemic game theory and epistemic logic.[7]

It is thus conceptually important to identify formally, and clearly tell apart substantive from structural assumptions. They lie at the very foundation of the epistemic perspective on games and social interaction. This is what we do in the coming sections.

2.1 Finitary epistemic languages and Kripke structures

Our approach is primarily syntactic, and so we start by introducing (finitary) languages, and then move to so-called Kripke structures. Syntactic approaches have a long history in logic (Hintikka 1962) and game theory (Aumann 1999). Such languages give a more coarse-grained view of knowledge and beliefs than probabilistic representations, of course. They are nevertheless sufficient to analyze certain solution concepts epistemically, for instance iterated elimination of strictly dominated strategies and Nash equilibrium (van Benthem et al. 2011). Furthermore, syntax gives us greater generality—formal languages like the one we present below can be interpreted on a large variety of models—and it proves convenient to compare different informational states, something that we will become crucial in the next sections.

Let N be a finite set of agents and PROP a countable set of propositions.

---

3 See for instance the discussion in Samuelson (2004) and the references in Moscati (2009).

4 Of course the quote opens by saying that these are "*not* assumptions". We rather view them as assumptions of a different kind, hence our terminology.

5 See again: Fagin et al. (1995), Aumann (1999), and van Ditmarsch et al. (2007).

6 Positive introspection means that if an agent knows a certain fact ϕ, then she knows that she knows ϕ. Negative introspection means that if an agent doesn't know that ϕ, then she knows that she doesn't know that ϕ.

7 Work on awareness (Fagin and Halpern 1987; Modica and Rustichini 1994; Halpern 2001; Heifetz et al. 2006), however, have made major steps in lifting these assumptions.

Definition 1 (Finitary Epistemic Language) A finitary epistemic language $\mathcal{L}_{EL}$ is recursively defined as follows:

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box_i \phi \mid \Box^*_G \phi$$

where i ranges over N, p over PROP, and $\emptyset \neq G \subseteq N$.

This is the standard multi-agent epistemic logic with a "common knowledge" modality (Fagin et al. 1995; van Ditmarsch et al. 2007). The formula $\Box_i \phi$ can be read as "agent i knows that ϕ" or as "agent i believes that ϕ", depending on the properties of this operator.[8] The formula $\Box^*_G \phi$ should be read as "it is common knowledge/belief among group G that ϕ". Both $\Box_i$ and $\Box^*_G$ have their duals, $\Diamond_i$ and $\Diamond^*_G$, defined respectively as $\neg\Box_i\neg$ and $\neg\Box^*_G\neg$. One could also work with languages containing both (common) knowledge and beliefs operators, as well as with more expressive languages such as Propositional Dynamic Logic (Harel et al. 2000).

This language is finitary in the sense that it allows only finite conjunctions ($\wedge$) and disjunctions, as well as finite, but unbounded stacking of epistemic operators. The $\Box^*_G \phi$ modality has an infinitary character, it being equivalent to the infinite conjunction of $E^{n+1}\phi =_{df} \bigwedge_i (\Box_i E^n \phi)$ for all $n < \omega$, but it can be finitely axiomatized—c.f. the references above. Some, but not all, of the observations below carry over to infinitary versions of this language, for instance those studied in Segerberg (1994) and Heifetz (1999). We leave this generalization for future work.

Epistemic languages can be interpreted in a wide variety of structures, from "Kripke" or relational structures (Blackburn et al. 2001) to partition structures (Heifetz and Samet 1998a) to topological spaces and "neighborhood models" (Blackburn et al. 2006, chap. 1). We use here Kripke structures as an illustrative example.

Let, again, PROP be a countable set of atomic propositions.

Definition 2 (Kripke structure) A *Kripke structure* $\mathcal{M}$ is a tuple $\langle W, N, \mathcal{R}, V \rangle$ where W is a nonempty set of *states*, N is a finite set of *agents*, $\mathcal{R}$ is a collection of *binary relations* on W and $V : W \rightarrow 2^{\text{PROP}}$ is a *valuation* function from W to subsets of PROP. Given a relation $R \in \mathcal{R}$ and state $w \in W$, we write $R[w]$ for $\{w' : wRw'\}$. A *pointed Kripke structure* is a pair $(\mathcal{M}, w)$.

It is usually assumed that $\mathcal{R}$ contains at least one relation R_i for each agent $i \in N$. We write R^+_G for the transitive closure of the union of the relations R_i for $i \in G$. This relation is used to interpret the common belief modality. For common knowledge one uses the reflexive-transitive closure R^*_G. The epistemic language is then interpreted in a Kripke structure as follows, with $\Box^*_G$ read as "common knowledge":

[8] Knowledge is usually assumed to satisfy the K axiom $(\Box_i(\phi \rightarrow \psi) \rightarrow (\Box_i\phi \rightarrow \Box_i\psi))$ and the "necessitation rule": from ϕ a theorem, infer $\Box_i\phi$—although this need not be the case, depending on the underlying class of structures one is working with—as well as the S5 axioms: (T) $\Box_i\phi \rightarrow \phi$; (4) $\Box_i\phi \rightarrow \Box_i\Box_i\phi$ and (5) $\neg\Box_i\phi \rightarrow \Box_i\neg\Box_i\phi$. For beliefs one usually drops (T), beliefs can mistaken, after all, and replace it with (D), $\Box_i\phi \rightarrow \neg\Box\neg_i\phi$, ensuring consistent attitudes. Unless stated otherwise, in what follows we use $\rightarrow$ for the material implication.

Definition 3 Interpretation of $\mathcal{L}_{EL}$ in Kripke structures.

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & p \in V(w) \\
\mathcal{M}, w \Vdash \neg\phi & \text{iff} & \mathcal{M}, w \nVdash \phi \\
\mathcal{M}, w \Vdash \phi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \phi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Box_i \phi & \text{iff} & \forall v (\text{if } w R_i v \text{ then } \mathcal{M}, v \Vdash \phi) \\
\mathcal{M}, w \Vdash \Box^*_G \phi & \text{iff} & \forall v (\text{if } w R^*_G v \text{ then } \mathcal{M}, v \Vdash \phi)
\end{array}$$

Subsets of W are usually called *events*, and an event E is *definable* in a model $\mathcal{M}$ in a given language whenever there is a formula ϕ of that language such that $E = \{w : \mathcal{M}, w \Vdash \phi\}$. It is well-known that, in general, not all events are definable by formulas of finitary epistemic languages, but that there are elegant model-theoretic characterizations of classes of definable Kripke structures.[9]

Kripke structures can be seen as models for games, i.e. contexts as understood in epistemic game theory, in the obvious way. The propositional variables then range over strategy profiles of the underlying game, making the valuation V the usual "strategy function" in interactive epistemology (Aumann 1999; Board 2002; Stalnaker 1999). One can also equip oneself with a range of primitive propositions describing the agents' preferences over profiles in the game[10], and whether a given strategy choice is "rational" at a given state.[11]

3 Substantive and structural assumptions

The first step toward two of our contributions—the proof that structures minimizing substantive assumptions exist and the relation of this proof with known results on large structures—is to pinpoint substantive assumptions formally, and distinguishes them from structural assumptions. Syntactically, there is an obvious way to do this. In completely axiomatizable classes of structures, structural assumptions are, just like Aumann points out in the quote opening this paper, theorems, i.e. provable formals in the logical system. Structural assumptions then become consistent formulas characterizing sub-classes of the given classes of models. To make this precise we briefly review the standard notions of axiom systems, logical consequence, consistency, and completeness with respect to given classes of (Kripke) structures.[12]

An *axiom system* Λ in $\mathcal{L}_{EL}$ is a set of designated formulas, called *axioms*, together with a set of *inference rules*. For instance, the formulas and the rule mentioned in Footnote 8, together with all propositional tautologies and the rule *modus ponens*, form the axiom system known as S5. A *derivation* in an axiom system is a finite sequence

[9] For example, a class of elementary pointed Kripke structures is definable by a (set of) modal formulas iff the class is closed under bisimulations and ultraproducts and its complement is closed under ultrapowers (Blackburn et al. 2001, Theorem 2.75, p. 107). Bisimulations are defined later in the paper. See also (Blackburn et al. 2001) for the relevant definitions.

[10] This can also be done in a "modal" way. See van Benthem et al. (2009).

[11] See e.g. de Bruin (2010).

[12] The treatment in what follows is entirely standard, but for reasons of space is bound to be elliptic. Details can be found in any textbook on modal logic, for instance (Blackburn et al. 2001; van Benthem 2010).

of formulas that are either axioms or obtainable from some formulas earlier in the sequence by one of the inference rules. When a formula ϕ can be derived from a set of formulas Σ in a given axiom system Λ, we write $\Sigma \vdash_\Lambda \phi$. Call a non-empty set T of formula in $\mathcal{L}_{EL}$ a *theory*. A theory is *consistent* given an axiom system Λ, or Λ-consistent, for short, if it is not the case that $T \vdash_\Lambda \bot$.[13] T is *maximally consistent* if it is consistent and there is no other theory $T' \supsetneq T$ that is also consistent.

There is of course a close correspondence between axiom systems and sets of formulas that are "valid" in all Kripke frames satisfying certain properties.[14] S5, for instance, is *sound* and *complete* with respect to the class of Kripke frames where the relations R_i for each agent i are equivalence relations: all axioms are valid and the inference rules preserve validity (soundness) and all valid formulas in that class of structures are provable in S5 (completeness).[15] From now on we will assume that we are working with classes of Kripke frames that are completely axiomatizable, but will come back to this point shortly.

Definition 4 Let Λ be a given axiom system in $\mathcal{L}_{EL}$, T be a maximally Λ-consistent theory and ϕ a formula in T. Then ϕ is a *substantive assumption* if there is another maximally Λ-consistent theory T' that does not contain ϕ. An *epistemic* substantive assumption is a substantive assumption of the form $\Box_i\phi$ for some agent i, or $\Box^*_G\phi$ for some $G \subseteq I$. If $\phi \in T$ is not a substantive assumption we call it a *structural assumption*.

In short, structural assumptions are theorems of the logic Λ, and substantive assumptions are consistent, non-provable formulas. Maximally consistent theories are used here only to make clearer the connection with the ordering of theories that we define in the next section.

This definition is completely standard, but its explicit statement is nevertheless important, for a number of reasons. First, this definition crystallizes an idea that is often implicitly used when substantive and structural assumptions are being discussed (e.g. in Aumann 1987). Second, it shows that these basic notions from logic draw the line between substantive and structural assumptions precisely where it should be: Not only are structural assumptions logical truths, so to speak, valid in all structures of a given class, and but their negations are "logical impossibilities", counting neither as structural nor as substantive. This speaks for our syntactic approach, since such logical impossibilities are difficult to pinpoint at the level of structures, without moving to very large ones.[16] Finally, the considerations in the next section show that, despite its simplicity, equating structural assumptions with provable formulas allows to connect the comparison of substantive assumptions with ordering of "informativeness"

[13] We write $\bot$ for contradiction, for instance $p \wedge \neg p$.

[14] Call a frame a Kripke structure with the valuation V omitted. A formula is valid on a frame if it is true in all states of all structures based on that frame. By "valid in all Kripke structures…" we mean valid on a given class of frames satisfying certain properties.

[15] See again Blackburn et al. (2001) for more on this.

[16] In the words of Aumann (2010) : "In a semantic model it's difficult to express what it means for something to be logically impossible. In a semantic model, something that is logically impossible is represented by an empty set; but if an event is represented by an empty set, that doesn't mean it's logically impossible. You have to have a *universal* semantic model to say that, and these models are large and clumsy."

of structures, and ultimately to show the existence of models where no substantive assumptions are being made.

Another basic, but conceptually important observation directly following from this definition is that what counts as a substantive or a structural assumption is relative to the language one is working with. Which assumptions are substantive or structural is a *language-dependent* question. At a trivial level, what counts as substantive assumptions depends on the choice of logic or class of models one is working with.[17] But the connections goes deeper than that, since properties of structures that are not definable in the formal language at hand are "off the radar", so to speak, neither counting as structural nor as substantive assumptions. This is a conceptually fundamental fact for epistemic modeling of social interaction, to which we return in the Conclusion.

4 Making no substantive assumptions

It is important to know whether there exist structures where no epistemic substantive assumptions are being made. Substantive assumptions about the information the interacting agents can have drastic consequences on what they will/should rationally do. If no such structures exist, then one is never sure whether conclusions drawn from epistemic modeling rest on implicit, perhaps unwarranted substantive assumptions. If, on the other hand, such structures (of a given class) do exist, then they can be seen as weakest possible informational contexts, i.e. contexts where agents have as little information as possible about each other. Such contexts can be used as benchmark cases to "test" behavioral conclusions - or simply to pinpoint precisely which structural assumptions are being made in that class.

Some care needs to be taken here, though, as "making no epistemic substantive assumptions" can have a local and a global reading. Logically, a state in a specific structure makes no substantive assumptions whenever the agents consider possible every consistent sets of literals and, more importantly, every consistent hierarchies of knowledge and beliefs of the other players. All the agents "know" or "believe" in such states are structural assumptions. Globally, a model makes no substantive assumptions whenever it only validates provable formulas. Of course, local minimization at all states of substantive assumptions implies global minimization, but not the other way around.

From a logician's perspective, it is clear that there exists structures that make no substantive assumptions, at least globally. The *canonical model*[18] for a given logical system is a clear example. The argument for this is simply that this model is

[17] In S5, both positive and negative introspection are structural assumptions according to this definition. But if one moves to K, the logic sound and complete with respect to the class of *all* Kripke structures, then these are, of course, substantive assumptions. On the other hand, so-called logical omniscience, embodied by the K axiom and the Necessitation rule, are structural assumptions in a fundamental sense: to drop them one has to move to a different semantics for informational attitudes, for instance neighborhood semantics.

[18] For the precise definition of the canonical model construction, also known as Henkin model in model theory, see the references in Footnote 12.

constructed from *all* maximally consistent theories of a given logical system.[19] When this language has the expressive resources to describe higher-order knowledge and beliefs, like the one above, this means that this structure represents all consistent hierarchies of knowledge and beliefs[20], and so that this structure validates only structural assumptions, i.e. that the only formulas valid in the canonical models are provable ones.[21]

In this section we prove a stronger result, namely that there are structures that make no substantive assumptions, *locally*. The proof of this is in itself illuminating, as it depends on, and make formal, the tight connection between comparisons of substantive assumptions and comparisons of "informativeness of structures", a notion that is non-trivial to describe formally, and that conceptually interesting in its own right. We present it in the next subsection, and then move to the main existence result.

4.1 Comparing substantive assumptions

Given their fundamental role for the epistemic analysis of games, it is not only important to pinpoint substantive and, *a fortiori*, epistemic substantive assumption, but also to be able to compare structures with respect to the assumptions they are making. Intuitively, one makes more substantive assumptions in a structure where rationality is common knowledge than in structures where rationality is not common knowledge, *ceteris paribus*. In the latter structure, the agents, again intuitively, *know or believe less*. There is a tight connection between, on the one hand, the idea that less substantive assumptions are being made in one structure than in another and, on the other hand, the notion of "informativeness" of structures.

Defining an ordering of structures according to their informativeness is not a trivial task, though. At the level of structures, extending the state space is the usual technique to find models where agents know less, but this will not work in full generality. Many extensions of a given space do not translate in changes in informational attitudes.[22] The correct notion here requires a quantification over all possible extensions of a given model, a notion which is more naturally captured by our syntactic definition of substantive assumptions.

The task of comparing informativeness is not completely trivial at the syntactic level either. The naive procedure would consist in checking whether all the formulas of the form $\Box_i \phi$ contained in one maximally Λ-consistent theory are also present in some

[19] Some care is needed here for logical systems that are not *compact*, such as epistemic logic with a common knowledge operator. See (Blackburn et al. 2001, Chap. 4) for a discussion. These technical issues are not crucial for the general point we are making here.

[20] This is also the key idea behind the classic construction of the so-called *canonical type space* of Mertens and Zamir (1985).

[21] The canonical model is also "assumption complete" (Zvesper and Pacuit 2010). We come back to this notion in Sect. 5.

[22] Put in logical terms, epistemic languages like the one defined above are "invariant" under a many different model transformations (Blackburn et al. 2001). This is also true for more expressive languages, for example languages with infinitary conjunctions or probability operators. The same point can also be made model-theoretically, c.f. the notion of knowledge morphism in Heifetz and Samet (1998a).

other one. This would boil down to check whether everything known in one theory is known in the other. This doesn't work in the general case. If agents are negatively introspective, as in S5 structures for instance, ignorance of a given fact induces knowledge of that very ignorance. One obviously wants to discard this kind of self-knowledge in comparing substantive assumptions. They would make most pairs of different maximally Λ-consistent theories incomparable according to the naive ordering. On the other hand, comparing only first-order knowledge or beliefs, i.e. attitudes bearing to non-epistemic facts in a given context, will not do either, as assumptions about the information of *others* is of crucial importance—think again of common knowledge of rationality here.[23]

To circumvent these difficulties we use a notion of comparison of informativeness that has been put forward by Parikh (1991) in order to analyze non-monotonic phenomena tied to epistemic reasoning. Let $\mathcal{L}_{EL}$ be a finitary epistemic language and Λ a logical system in it. We write $sub(\phi)$ for the set of sub-formulas of ϕ, not necessarily proper ones. In this definition it is convenient to assume that all $\Box\phi$ are re-written as $\neg\Diamond\neg\phi$. This can be done without loss of generality.

Definition 5 Given T_1, T_2 two maximally Λ-consistent theories, we say that the agents in T_1 *know at least as much as* in T_2, written, $T_1 \geq T_2$, iff for all formula $\phi \in (T_1 \cup T_2) \setminus (T_1 \cap T_2)$[24], up to provable equivalence in Λ, there is a ψ in $sub(\phi)$ such that $\psi = \Diamond_i \chi$ for some i, and $\psi \in T_2 \setminus T_1$. If $T_1 \neq T_2$ and $T_1 \geq T_2$, then we say that the agents know strictly more in T_1 than in T_2, written $T_1 > T_2$.

This ordering discards the self-knowledge about one's own ignorance mentioned above. Suppose that the only difference between T_1 and T_2 is that i knows that ϕ in the first but not in the second. Formally, this means that $\neg\Box_i\phi$ is in T_2, and so is $\Diamond_i\neg\phi$, by maximal Λ-consistency of T_2.[25] Neither of these formulas is in T_1. But if i is negatively introspective then $\Box_i\Diamond_i\neg\phi$ is also in T_2 but not in T_1 and *vice versa* if he is positively introspective: $\Box_i\neg\Diamond_i\neg\phi$ is in T_1 but not in T_2. In this case one can still say that agents know more in T_1 than in T_2 because, even though i knows about his knowledge and ignorance, respectively, since there is a sub-formula of $\Box_i\Diamond_i\neg\phi$, namely $\Diamond_i\neg\phi$ itself, that is in T_2 but not in T_1.

This ordering is consistent with the naive one, mentioned above, for agents that are not introspective. Take two maximally consistent theories T_1 and T_2 that agree on literals. If all formulas $\Box_i\phi$ in T_2 are also in T_1, something that cannot happen in the introspective case, an easy argument shows that it must be the case that $T_1 \geq T_2$.

Given that there is a natural correspondence between maximally Λ-consistent theories and states in structures for which Λ is sound and complete, this ordering also compares epistemic substantive assumptions at the level of structures. Suppose that

[23] It should be observed that in non-introspective contexts, cases where agents do have information about their own information are of obvious importance, and will indeed count as substantive assumptions according to our definition.

[24] Given two sets X, Y, we write $X \setminus Y$ for the set-theoretic difference between X and Y.

[25] We assume here that Λ contains at least axiom D (c.f. Footnote 8). This can be done without loss of generality. If Λ doesn't contain D, then all theories where one agent has inconsistent beliefs become $>$-minimal. All remarks below would then bear on the $>$-minimal theories with *consistent* beliefs.

$\mathcal{M}_1, w$ and $\mathcal{M}_2, v$ are two pointed Kripke structures in a class $\mathcal{K}$ for which a given Λ is sound and complete. Let $T_1 = \{\phi : \mathcal{M}_1, w \Vdash \phi\}$ and $T_2 = \{\phi : \mathcal{M}_2, v \Vdash \phi\}$. Then if $T_1 > T_2$ we know that some epistemic substantive assumptions are relaxed by moving from $\mathcal{M}_1, w$ to $\mathcal{M}_2, v$. The relation $>$ thus provides a means of comparing structures and, as we show presently, it can be used to find models where *no* substantive assumptions are being made.

4.2 Minimizing substantive assumptions

We now show the existence of structures where no substantive assumptions are being made, locally. We do this in three steps. We start by showing that the relation $\geq$, defined in the previous section, is a partial order. Building on this we show that any descending $\geq$-chain starting with a given theory T has a minimal element T^*, and that for any theory T' that agrees with T on literals, the agents know at least as much in T' than as in T^*. The construction of this minimal element is in itself interesting, as it make use of some flexibility that usually goes unnoticed in the standard technique to build maximally consistent sets of formulas. By minimality of T^*, and up to logical equivalence, this last step will be enough to show that the class of pointed structures that satisfy all and only the formulas in T^* is the one where no epistemic, substantive assumptions are being made.[26]

Recall that if $T_1 \geq T_2$ then these two theories agree on literals. Formally, let $md(\phi)$ be the modal depth of ϕ, and $|T_i| = \{\phi \in T_i : md(\phi) = 0\}$. If $T_1 \geq T_2$ then $|T_1| = |T_2|$. This simple observation grounds the fact that $\geq$ is indeed a partial order on theories that share the same literals.

Proposition 1 *Let T_1 be a MCS and $\mathcal{T}_1 = \{T_i : T_i$ is a MCS and $|T_i| = |T_1|\}$. Then $\geq$ is a partial order on $\mathcal{T}_1$.*

Proof Reflexivity is obvious. For anti-symmetry, suppose $T_k \neq T_l$ and $T_k \leq T_l$. We show that $T_l \not\leq T_k$. We know by assumption that $((T_k \cup T_l) \setminus (T_k \cap T_l)) \neq \emptyset$. Take a $\phi = \Diamond_i \psi \in T_k \setminus T_l$ such that all proper sub-formulas of ϕ are in $T_k \cap T_l$. We know that such a formula exists because $T_k \leq T_l$. By our choice of ϕ we know that it has no sub-formula $\Diamond_j \chi'$ in T_l but not in T_k, which means that $T_l \not\leq T_k$.

For transitivity, suppose $T_k \geq T_l \geq T_m$. Because $\geq$ is anti-symmetric we can assume that all these three theories are pairwise different. Take $\phi \in ((T_k \cup T_l) \setminus (T_k \cap T_l))$. We can assume WLOG that ϕ is of the form $\Diamond_i \psi$. We show by induction on the modal depth of ϕ it has a sub-formula $\chi = \Diamond_j \chi'$ such that $\chi \in T_m$ but not in T_k.

- Basic case: $(md(\phi) = 1)$ If $\phi \in T_m \setminus T_k$ we are done. But the other case, $\phi \in T_k \setminus T_m$, is impossible. Suppose indeed that $\phi \in T_k \setminus T_m$. This means that $\Box_i \neg\psi \in T_m$. Now, either $\phi = \Diamond_i \psi$ or $\neg\phi = \Box_i \neg\psi$ is in T_l. In the first case, because $T_l \geq T_m$, there must be a sub-formula χ of ψ of the form $\Diamond_j \chi'$. But this can't be because $md(\phi) = 1$ and thus $md(\psi) = 0$. The reasoning in the other case, viz. when $\Box_i \neg\psi \in T_l$, is the same but uses $T_k \geq T_l$.

[26] Here, again, discarding the agents' information about their own information.

– Inductive step. Our inductive hypothesis is that for all formula $\phi \in ((T_k \cup T_m) \setminus (T_k \cap T_m))$, if $md(\phi) \leq n$ then it has a sub-formula $\chi = \Diamond_j \chi'$ such that $\chi \in T_m$ but not in T_k. Take $\phi \in ((T_k \cup T_m) \setminus (T_k \cap T_m))$ with $md(\phi) = n+1$. Observe again that if $\phi \in T_m \setminus T_k$ we are done. Assume then, otherwise, that $\Diamond_i \psi = \phi \in T_k \setminus T_m$. If there is a sub-formula of ψ in either T_k or T_m but not both, we are done by the inductive hypothesis, because $md(\psi) = n$. Suppose then that this is not the case, i.e. $sub(\psi) \subseteq T_k \cap T_m$. Suppose now that $\phi \in T_l$. Because $T_l \geq T_m$ and $\neg\phi \in T_m$, ϕ must have a proper sub-formula $\Diamond_k \chi' \in T_m$ but not in T_l, i.e. $\Box_k \neg \chi' \in T_l$. We know by assumption that $\Diamond_k \chi'$ must also be in T_k. But then, because $T_k \geq T_l$, $\Diamond_k \chi'$ must itself have a sub-formula $\Diamond_l \chi'' \in T_l$ such that $\neg \Diamond_l \chi'' \in T_k$, and also in T_m by assumption, and so on…At some point this back-and-forth between T_k and T_m is bound to stop, because there are only finitely many sub-formulas of ψ, which will contradict either $T_k \geq T_l$ or $T_l \geq T_m$. The argument for $\neg\phi \in T_l$ follows the same line. $\square$

With this in hand, a slight variation of the construction of maximally consistent sets in Lindenbaum's Lemma gives us the theory we are looking for.

Proposition 2 *Let, as before, T_1 be a maximally consistent theory and $\mathcal{T}_1$ be the set of theories that agree with T_1 on literals. $\mathcal{T}_1$ has a $\geq$-minimal element.*

Proof The idea of the proof is to construct T^* step-wise, per modal depth. This is done by considering successive fragments of $\mathcal{L}_{EL}$, adding formulas of each of these fragments, and checking consistency at each step. Within each fragment, an additional sub-division is required, to make sure that the $\Diamond$ formulas are added first.

Let $\mathcal{L}_0$ be the propositional fragment of $\mathcal{L}_{EL}$, that is, is the smallest set of formulas ϕ such that: (with $p \in \mathcal{L}_{EL}$)

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi$$

Then for all $n < \omega$ define $\mathcal{L}^{\Diamond}_{n+1}$ as the smallest set of formulas ϕ such that:

$$\phi := \Diamond_i \psi$$

with $i \in I$ and $\psi \in \mathcal{L}_n$. $\mathcal{L}^{\Box}_{n+1}$ is defined analogously, i.e. as the smallest set of formulas ϕ such that:

$$\phi := \neg\phi \mid \phi \wedge \phi \mid \Box_i \psi$$

with $i \in I$ and $\psi \in \mathcal{L}_n$.

We are now ready to construct inductively T^*:

– $T^*_0 = |T_1|$
– T^*_{n+1} is defined step-wise:
  1. Take a well-ordering $S = \langle \phi_1, \ldots \rangle$ of $\mathcal{L}^{\Diamond}_{n+1}$. Let $T^0_{n+1} = T^*_n$. For all $\phi_k \in S$ with $k \geq 1$, define:
     – $T^k_{n+1} = T^{k-1}_{n+1} \cup \{\phi_k\}$ if $T^{k-1}_{n+1} \nvdash \neg\phi_k$.

– $T^k_{n+1} := T^{k-1}_{n+1}$ otherwise.
Set $T^{\Diamond}_{n+1} = \bigcup_{k<\omega} T^k_{n+1}$

Claim $T^{\Diamond}_{n+1}$ is consistent and in $\mathcal{T}_1$.

Proof of Claim Membership in $\mathcal{T}_1$ is obvious. Suppose $T^{\Diamond}_{n+1}$ is not consistent. Then there is a finite set Ψ and a formula ϕ_i, both in $T^{\Diamond}_{n+1}$, such that $\vdash \bigwedge \Psi \to \neg\phi_i$. We can assume WLOG that for all $\phi_j \in \Psi$, $j < i$. But then by construction $\phi_i \notin T^i_{n+1}$. ◀

2. Take a well-ordering $S' = \langle \phi_1, \ldots \rangle$ of $\mathcal{L}^{\Box}_{n+1}$. Let $T^0_{n+1} = T^{\Diamond}_{n+1}$. For all $\phi_k \in S$ with $k \geq 1$, define :
 – $T^k_{n+1} = T^{k-1}_{n+1} \cup \{\phi_k\}$ if $T^{k-1}_{n+1} \nvdash \neg\phi_k$.
 – $T^k_{n+1} := T^{k-1}_{n+1}$ otherwise.
 Set $T_{n+1} = \bigcup_{k<\omega} T^k_{n+1}$. T_{n+1} is consistent and in $\mathcal{T}_1$, by a similar argument as before.

– Set $T^* = \bigcup_{n<\omega} T_{n+1}$

Claim T^* is a MCS in $\mathcal{T}_1$.

Proof of Claim Maximality and membership in $\mathcal{T}_1$ are again obvious. If T^* is not consistent, then either there is a finite set Ψ and a formula χ, such that $\vdash \bigwedge \Psi \to \neg\chi$, or inconsistency arises from the breakdown of compactness for the reflexive-transitive closure modality $\Box^*_G$. The first case can't be, since it would contradict the consistency of one of the T^*_n. The second case can be taken care of axiomatically, by introducing an infinitary inference rule for the $\Box^*_G$ modality.[27] ◀

Claim $T_j \geq T^*$, for all $T_j \in \mathcal{T}_1$.

Proof of Claim If $T_j = T^*$ then we're done. Suppose $T_j \neq T^*$ and take ϕ in either T_j or T^* but not both. We can assume WLOG that ϕ is of the form $\Diamond_i\psi$ or $\Box_i\psi$. We show by induction on the modal depth of ϕ that it has a sub-formula ψ of the form $\Diamond_i\chi$ such that $\psi \in T^*$ but not in T_j.

– Basic case ($md(\phi) = 1$). Then ϕ itself must be either $\Diamond_i\chi$ or $\Box_i\chi$ with $md(\chi) = 0$. But by construction[28] it must be that $\Diamond_i\chi$ is in T^*, and so $\Box_i\neg\psi \in T_j$, as required.
– Inductive step. Our inductive hypothesis is that for all ϕ in either T_j or T^* but not both, if ϕ is of modal depth n then we can find the required sub-formula. Take ϕ of modal depth $n + 1$ in either T_j or T^* but not both. Now observe that ϕ is of the form $\Diamond_i\psi$ or $\Box_i\psi$ with $md(\psi) = n$, which means that we're done by the inductive hypothesis.

◀

[27] See the references in Footnote 19.

[28] Assuming that the underlying logic is at least KD, so $\Box_i\phi \to \Diamond_i\phi$ is an axiom scheme.

This result makes formal the claim that, for any pointed Kripke structure, one can maximally relax its substantive assumptions, locally[29], *up to provable equivalence* in the given finitary epistemic languages that we work with. This not only strengthens the common place observation that there are structure where no substantive assumptions are being made at the global level, but it also sheds light on a number of known results regarding the non-existence of so-called "universal knowledge structures." We explain why in the next section, and situate the present analysis with respect to the literature on such "large" structures.

5 Connection with known results on the (non-)existence of large structures

The reader acquainted with the extensive literature on what may be called *large* (or *rich*) type structures[30]— so-called universal or complete structures—can legitimately wonder about the relationship between these structures and those that minimize substantive assumptions as defined in this paper. In this section, we briefly explain this relationship.

5.1 Universal knowledge structures

Intuitively, a *universal Kripke structure* is a Kripke structure that "contains" a copy of every other knowledge structure. This can be made precise as follows: Define maps between Kripke structures that preserve the what each player knows and/or believes at each state. Such maps are called *knowledge morphisms* in the game theory literature (where attention is restricted to classes of Kripke structures where the relations on equivalence relations) and *bounded morphisms*, or *p-morphisms*, in the modal logic literature (see Blackburn et al. 2001, Definition 2.10, p. 59).

Definition 6 A *bounded morphism*, from $\mathcal{M} = \langle W, \mathcal{R}, V\rangle$ to $\mathcal{M}' = \langle W', \mathcal{R}', V'\rangle$ is a (total) function $f : W \to W'$ such that:

1. For all $p \in \text{PROP}$, $p \in V(w)$ iff $p \in V'(f(w))$;
2. For all players i and for all $w, v \in W$, $wR_i v$ implies $f(w)R'_i f(v)$
3. For or all players i and for all $w \in W$ and $v' \in W'$, if $f(w)R'_i v'$ then there is a $v \in W$ such that $wR_i v$ and $f(v) = v'$.

So, bounded morphisms are mappings that preserve not only the basic facts (condition 1), but also each agents' information (conditions 2 & 3). More precisely, a well-known

[29] There is a technical subtlety that should be mentioned here. The minimal element built in the proof of Observation 2 is a state in the *canonical model*. We assume that we work with classes of models that are completely axiomatizable, but it might be that the canonical model is *not* a model of that class, for instance when some of the axioms are not canonical. In this case, however, there are known techniques to *transform* the canonical model in a truth-preserving way such that, after the transformation, the resulting model *is* a member of the required class. We refer to Blackburn et al. (2001, Chap.4) for more on this.

[30] We do not attempt a complete overview of this interesting literature here. See Brandenburger and Keisler (2006, Section 11), Siniscalchi (2008, Section 3) and Pintér (2005) for discussion and pointers to the relevant results.

observation is that if there is a bounded morphism f from $\mathcal{M}$ to $\mathcal{M}'$, then for all states w in $\mathcal{M}$, and all formulas $\phi \in \mathcal{L}_{EL}$, $\mathcal{M}, w \models \phi$ iff $\mathcal{M}', f(w) \models \phi$.

Definition 7 A Kripke structure $\mathcal{M}_U$ is *universal* iff for any knowledge structure $\mathcal{M}$, there is a bounded morphism from $\mathcal{M}$ into $\mathcal{M}_U$.

In the language of category theory, such structures are also called *weakly terminal* objects.

It is not hard to see that a universal Kripke structure (if it exists) must minimize substantive assumptions. If a model contains states that satisfy all and only the formulas in a $>$-minimal element, then no epistemic substantive assumptions will be valid in that model. But since, by definition, that model will have a bounded morphic image in the universal structure, the latter will not validate any epistemic substantive assumptions either. Universal Kripke structures will, in fact, validate only *structural* assumptions.

In general, however, such universal structures do not exist. Heifetz and Samet (1998a) show that a universal *knowledge* structure (i.e., a Kripke structure where the relations are assumed to be equivalence relations) does not exist. This results was used by Meier (2005) to show that there is no universal structure with respect to any class of Kripke structures that contains the class of knowledge structures. That is, Meier showed that relaxing various *structural assumptions*, such as truth and/or the introspective principles, does not suffice to prove that a universal structure exists.[31]

Of course, it does not necessarily follow from this negative result that there are no structures where only structural assumptions are valid. The *canonical model*[32] for a given logical system is a clear example, and the argument for this is simply that this model is constructed from *all* maximally consistent theories of a given logical system[33]. Universality is thus related to the minimization of substantive assumptions, but the two notions are different.

Our results in the previous section show precisely in what sense the *canonical Kripke structure*, familiar in the modal logic literature, minimizes substantive assumptions. This construction of the canonical model depends on the underlying language $\mathcal{L}$ and logical system Λ. The key aspect of this construction is that for any Λ-consistent set Γ of formulas from $\mathcal{L}$, there is a state in the canonical model that satisfies all formulas in Γ.

For all-out attitudes specified in finitary languages, like the one studied in the present paper, the canonical model minimizes substantive assumptions, but is not universal. But, as already observed by Heifetz (1999)[34], for instance, if one redefines universality in terms of truth preservation in a given language, then universal knowledge structures do exist. Of course the question then becomes one of motivating the choice

[31] See Pintér (2010) for a related non-existence result (for "topological" type spaces). These results are best viewed in the context of the general theory of *final coalgebras* (see Venema (2006, Sect. 10) for a general discussion and Goldblatt (2006) for the relevant results).

[32] For the precise definition of the canonical model construction, also known as Henkin model in model theory, see the references in Footnote 12.

[33] The remarks in Footnote 19 apply here as well.

[34] Cf. also the extensive discussion in Fagin et al. (1999).

of a specific *language* to describe the agents' attitudes. This is a difficult question, to which we come back briefly in the Conclusion, but for now it is sufficient to point out that it is not unlike the one of motivating certain topological assumptions on type structures.

Indeed, the situation is much better behaved in the probabilistic setting. For example, the central result of Brandenburger and Dekel (1993) shows that (under the assumption that the space of uncertainty is a Polish space) the canonical type space is, in fact, a *universal* type structure.[35]

5.2 Complete structures

A structure is said to be *assumption-complete* if, for each subset X in a given set of subsets of that structure and each agent i, there is a state where i "assumes" X. "Assumes" is taken here to mean strongest belief. In a Kripke structure, for instance, a set X is assumed by i at a state w if $R_i[w] = X$.

A simple counting argument shows that there cannot exist a complete structure where the set of conjectures is *all* subsets of the set of states (types) (Brandenburger 2003). A deeper result is the *impossibility theorem* from Brandenburger and Keisler (2006, Theorem 5.4) showing that a complete structure does not exist even if the set of events is restricted to first-order definable sets[36]. Some positive results are in sight as well: Mariotti et al. (2005) constructs a complete structure where the set of conjectures are compact subsets of some well-behaved topological space.

The relation between assumption-complete structures and those that minimize substantive assumptions can be encapsulated as a quantifier switch. Let B_{-i} be a consistent set of formulas of the form $B_j\phi$ for $i \neq j$, and $\mathcal{B}_{-i}$ be the set of all such B_{-i}. Local minimization of epistemic substantive assumptions is an $\exists\forall$ statement: Local minimization means that there is a state w where the agent assumes "all" $B_{-i} \in \mathcal{B}_{-i}$ or, to be more precise: there is a w such that for all such B_{-i} there is a state in $R_i[w]$ that satisfies all formulas in B_{-i}. Existence of assumption complete structure, on the other hand, is a $\forall\exists$ statement: If an assumption-complete structure exists for the language at hand, then for all B_{-i} there is a state where agent i assumes $\{w : w \Vdash B_j\phi$ for all $B_j\phi \in B_{-i}\}$. Assumption-complete structures, when they exist, are thus structures that contain states where epistemic substantive assumptions are minimized, but the other way around is not true.

6 Conclusion

In this paper we studied substantive assumptions in games or situations of social interaction. We have explained why they are important, shown how to identify and compare them, formally, and shown that there exist contexts where no substantive assumptions

[35] This result has been generalized in numerous ways. We do not discuss these generalizations here. See Heifetz and Samet (1998b), Meier (2008), Meier (2006), and Pintér (2005) for results and discussion of the relevant literature.

[36] See Abramsky and Zvesper (2012) for an extensive analysis and generalization of this result.

are being made. Towards the end of the paper we briefly explained the relation between such structures and a number of other "large" structures studied in the literature.

Our approach was primarily syntactic, and this was of great importance in determining what count as assumptions at all, substantive or structural, in a given class of structures. Properties of structures that are not definable in the language at hand are simply off the radar for our notion of structural and/or substantive assumptions. This raise a broader conceptual question: which granularity of epistemic analysis is needed or desirable? Or, the other way around, why would one choose to ignore details in favor of more coarse-grained languages? Issues of computational complexity speak in favor of the second approach, while notions of behavioral equivalence seem to point towards the first. We do not take a stance on this question here, but rather leave it as an open question, phrased in the words of the one of the founding fathers of analytic philosophy: *Of what one cannot speak, must one pass over in silence?*

Acknowledgments We are grateful to Ram Ramanujam, Rohit Parikh, Dietmar Berwanger and the audience of the Synthese Conference 2010 held at Columbia University for helpful questions and comments. Olivier Roy gratefully acknowledge the support of the Alexander von Humboldt Foundation. Eric Pacuit gratefully acknowledges support from NWO Vidi grant 276-20-010.

References

Abramsky, S., & Zvesper, J. A. (2012). From Lawvere to Brandenburger-Keisler: Interactive forms of diagonalization and self-reference. *Colagebraic Methods in Computer Science, LNCS, 7399,* 1–19.

Aumann, R. J. (1987). Correlated equilibrium as an expression of bayesian rationality. *Econometrica, 55*, 1–18.

Aumann, R. J. (1999). Interactive epistemology I: Knowledge. *International Journal of Game Theory, 28*, 263–300.

Aumann, R. (2010). Interview on epistemic logic. In Hendricks, V., & Roy, O. (Eds.), *Epistemic logic, 5 questions*. New York: Automatic Press.

Aumann, R., & Dreze, J. (2008). Rational expectations in games. *American Economic Review, 98*, 72–86.

Bernheim, D. (1984). Rationalizable strategic behavior. *Econometrica, 52*, 1007–1028.

Blackburn, P., de Rijke, M., & Venema, Y. (2001). *Modal logic*. Cambridge: Cambridge University Press.

Blackburn, P., van Benthem, J., & Wolter, F. (Eds.). (2006). *Handbook of modal logic*. Amsterdam: Elsevier.

Board, O. (2002). Knowledge, beliefs and game-theoretic solution concepts. *Oxford Review of Economic Policy, 18*, 433–445.

Brandenburger, A. (2003). On the existence of a 'complete' possibility structure. In Basili, M., Dimitri, N., & Gilboa, I. (Eds.), *Cognitive processes and economic behavior* (pp. 30–34). London: Routledge.

Brandenburger, A. (2007). The power of paradox: Some recent developments in interactive epistemology. *International Journal of Game Theory, 35*, 465–492.

Brandenburger, A., & Dekel, E. (1993). Hierarchies of beliefs and common knowledge. *Journal of Economic Theory, 59*, 189–198

Brandenburger, A., Friedenberg, A., & Keisler, H. J. (2008). Admissibility in games. *Econometrica, 76*, 307–352.

Brandenburger, A., & Keisler, H. J. (2006). An impossibility theorem on beliefs in games. *Studia Logica, 84*(2), 211–240.

de Bruin, B. (2010). *Explaining games: The epistemic programme in game theory. Synthese Library* (Vol. 346). New York: Springer.

Fagin, R., & Halpern, J. Y. (1987). Belief, awareness, and limited reasoning. *Artificial Intelligence, 34*(1), 39–76.

Fagin, R., & Halpern, J. (1994). Reasoning about knowledge and probability. *Journal of the ACM, 41*, 340–367.
Fagin, R., Halpern, J., Geanakoplos, J., & Vardi, M. (1999). The hierarchical approach to modeling knowledge and common knowledge. *International Journal of Game Theory, 28*(3), 331–365.
Fagin, R., Halpern, J. Y., Moses, Y., & Vardi, M. (1995). *Reasoning about knowledge*. London: MIT Press.
Friedenberg, A. (2010). When do type structures contain all hierarchies of beliefs? *Games and Economic Behavior, 68*(1), 108–129.
Friedenberg, A., & Meier, M. (2010). On the relationship between hierarchy and type morphisms. *Economic Theory*, 1–23. doi:10.1007/s00199-010-0517-2
Goldblatt, R. (2006). Final coalgebras and the hennessy-milner property. *Annals of Pure and Applied Logic, 183*, 77–93.
Goldblatt, R. (2008). Deductive systems for coalgebras over measurable spaces. *Journal of Logic and Computation Page*. doi:10.1093/logcom/exn092
Halpern, J. Y. (2001). Alternative semantics for unawareness. *Games and Economic Behavior, 37*(2), 321–339.
Harel, D., Kozen, D., & Tiuryn, J. (2000). *Dynamic logic*. London: The MIT Press.
Heifetz, A. (1999). How canonical is the canonical model? A comment on Aumann's interactive epistemology. *International Journal of Game Theory, 28*(3), 435–442.
Heifetz, A., Meier, M., & Schipper, B. C. (2006). Interactive unawareness. *Journal of Economic Theory, 130*(1), 78–94.
Heifetz, A., & Mongin, P. (2001). Probability logic for type spaces. *Games and Economic Behavior, 35*, 31–53.
Heifetz, A., & Samet, D. (1998a). Knowledge spaces with arbitrarily high rank. *Games and Economic Behavior, 22*(2), 260–273.
Heifetz, A., & Samet, D. (1998b). Topology-free typology of beliefs. *Journal of Economic Theory, 82*, 324–341.
Hintikka, J. (1962). *Knowledge and belief: An introduction to the logic of two notions*. Ithaca, NY: ornell University Press.
Huber, F., & Schmidt-Petri, C. (2009). *Degrees of belief. Synthese Library* (Vol. 342). New York: Springer.
Mariotti, T., Meier, M., & Piccione, M. (2005). Hierarchies of beliefs for compact possibility models. *Journal of Mathematical Economics, 41*, 303–324.
Meier, M. (2005). On the nonexistence of universal information structures. *Journal of Economic Theory, 122*(1), 132–139.
Meier, M. (2006). Finitely additive beliefs and universal type spaces. *Annals of Probability, 34*(1), 386–422.
Meier, M. (2008). Universal knowledge-belief structures. *Game and Economic Behavior, 62*, 53–66.
Mertens, J. F., & Zamir, S. (1985). Formulation of Bayesian analysis for games with incomplete information. *International Journal of Game Theory, 14*(1), 1–29.
Modica, S., & Rustichini, A. (1994). Awareness and partitional information structures. *Theory and Decision, 37*(1), 107–124.
Moscati, I. (2009). *Interactive and common knowledge in the state-space model. Cesmep working papers*. University of Turin. http://econpapers.repec.org/RePEc:uto:cesmep:200903.
Parikh, R. (1991). Monotonic and non-monotonic logics of knowledge. *Fundamenta Informaticae, XV*, 255–274.
Pearce, D. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica, 52*, 1029–1050.
Pintér, M. (2005). Type space on a purely measurable parameter space. *Economic Theory, 26*(1), 129–139.
Pintér, M. (2010). The non-existence of a universal topological type space. *Journal of Mathematical Economics, 46*(2), 223–229.
Samuelson, L. (2004). Modeling knowledge in economic analysis. *Journal of Economic Literature, 42*(2), 367–403.
Segerberg, K. (1994). A model existence theorem in infinitary propositional modal logic. *Journal of Philosophical Logic, 23*(4), 337–367.
Siniscalchi, M. (2008). Epistemic game theory: Beliefs and types. In Durlauf, S., & Blume, L. (Eds.), *The new palgrave dictionary of economics*. Basingstoke: Palgrave Macmillan.

Stalnaker, R. (1999). Extensive and strategic forms: Games and models for games. *Research in Economics, 53*, 293–319.
van Benthem, J. (2010). *Modal logic for open minds*. Stanford: CSLI Publications.
van Benthem, J., Girard, P., & Roy, O. (2009). Everything else being equal: A modal logic for ceteris paribus preferences. *Journal of Philosophical Logic, 38*(1), 83–125.
van Benthem, J., Pacuit, E., & Roy, O. (2011). Toward a theory of play: A logical perspective on games and interaction. *Games 2*(1), 52–86. doi:10.3390/g2010052.
van Ditmarsch, H., van de Hoek, W., & Kooi, B. (2007). *Dynamic epistemic logic. Synthese Library* (Vol. 337). New York: Springer.
Venema, Y. (2006). Algebra and coalgebra. In Blackburn, P., van Benthem, J., & Wolter, F. (Eds.), *Handbook of modal logic* (pp. 331–426). Amsterdam: Elsevier
Zhou, C. (2009). A complete deductive system for probability logic. *Journal of Logic and Computation, 19*(6), 1427–1454.
Zvesper, J., & Pacuit, E. (2010). A note on assumption-completeness in modal logic. *Logic and the Foundations of Game and Decision Theory–LOFT, 8*, 190–206.