Aggregating Judgements: Logical and Probabilistic Approaches

Lecture 3

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August 8, 2018

Plan

- ✓ **Monday** Representing judgements; Introduction to judgement aggregation; Aggregation paradoxes I
- ✓ **Tuesday** Aggregation paradoxes II, Axiomatic characterizations of aggregation methods I
- **Wednesday** Axiomatic characterizations of aggregation methods II, Distance-based characterizations
- **Thursday** Opinion pooling; Merging of probabilistic opinions (Blackwell-Dubins Theorem); Aumann's agreeing to disagree theorem and related results
- **Friday** Belief polarization; Diversity trumps ability theorem (The Hong-Page Theorem)

So far...

- Aggregating judgements: single event, multiple issues, logically connected issues, probabilistic opinions, imprecise probabilities, causal models, ...

- May's Theorem: axiomatic characterization of majority rule

- Condorcet Jury Theorem: epistemic analysis of majority rule

- Aggregation paradoxes: multiple election paradox, doctrinal paradox, discursive dilemma, the problem with conjunction, the corroboration paradox

Judgement Aggregation

U. Endriss. *Judgment Aggregation*. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, Handbook of Computational Social Choice, Cambridge University Press, 2016.

C. List. *The theory of judgment aggregation: An introductory review*. Synthese 187(1): 179-207, 2012.

D. Grossi and G. Pigozzi. *Judgement Aggregation: A Primer*. Morgan & Claypool Publishers, 2014.

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Agenda: $A = \{p \mid p \in I\} \cup \{\neg p \mid p \in I\}$

Judgement set for i: $J_i \subseteq A$ that is consistent and complete:

- Consistency: Standard notion of consistency for propositional logic.
- Completeness: For all $\varphi \in I$, $\varphi \in J_i$ or $\neg\varphi \in J_i$.

Notation:

- $\mathcal{J} = \{J \mid J \subseteq A$ is consistent and complete $\}$.
- If $J_i \subseteq \mathcal{L}$, we write $J_i(p) = 1$ when $p \in J_i$ and $J_i(p) = 0$ when $p \notin J_i$.
- If $\mathbf{J} = (J_1, \dots, J_n)$, then let $\mathbf{J}_p = \{i \mid p \in J_i\}$

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- If $\mathbf{J} = (J_1, \ldots, J_n)$, then let $\mathbf{J}_p = \{i \mid p \in J_i\}$

Aggregation function: $F : \mathcal{J}^n \to \wp(A)$

Properties

Universal Domain: The domain of F is the set of all possible profiles of consistent and complete judgement sets.

Collective Rationality: F generates consistent and complete collective judgment sets.

Anonymity: For all profiles $(J_1, \ldots, J_n)$, $F(J_1, \ldots, J_n) = F(J_{\pi(1)}, \ldots, J_{\pi(n)})$ where π is a permutation of the voters.

Unanimity: For all profiles $(J_1, \ldots, J_n)$ if $p \in J_i$ for each i then $p \in F(J_1, \ldots, J_n)$

Responsiveness Conditions

Systematicity: For any $p, q \in A$ and all $\mathbf{J} = (J_1, \ldots, J_n)$ and $\mathbf{J}^* = (J_1^*, \ldots, J_n^*)$ in the domain of F,

if [for all $i \in N$, $p \in J_i$ iff $q \in J_i^*$]

then [$p \in F(\mathbf{J})$ iff $q \in F(\mathbf{J}^*)$].

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Independence: For any $p \in A$ and all $\mathbf{J} = (J_1, \ldots, J_n)$ and $\mathbf{J}^* = (J_1^*, \ldots, J_n^*)$ in the domain of F,

if [for all $i \in N$, $p \in J_i$ iff $p \in J_i^*$]
then [$p \in F(\mathbf{J})$ iff $p \in F(\mathbf{J}^*)$].

Responsiveness Conditions

Monotonicity: For any $p \in X$ and all $(J_1, \ldots J_i, \ldots, J_n)$ and $(J_1, \ldots, J_i^*, \ldots, J_n)$ in the domain of F,

if $[p \notin J_i,\ p \in J_i^*$ and $p \in F(J_1, \ldots, J_i, \ldots J_n)]$
then $[p \in F(J_1, \ldots, J_i^*, \ldots J_n)]$.

Responsiveness Conditions

Non-dictatorship: There exists no $i \in N$ such that, for any profile $(J_1, \ldots, J_n)$, $F(J_1, \ldots, J_n) = J_i$

Agenda Richness

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Definition A set $Y \subseteq \mathcal{L}$ is **minimally inconsistent** if it is inconsistent and every proper subset $X \subsetneq Y$ is consistent.

Agenda Richness

Definition An agenda X is **minimally connected** if

1. (non-simple) it has a minimal inconsistent subset $Y \subseteq X$ with $|Y| \geq 3$
2. (*even-number-negatable*) it has a minimal inconsistent subset $Y \subseteq X$ such that

$$Y - Z \cup \{\neg z \mid z \in Z\} \text{ is consistent}$$

   for some subset $Z \subseteq Y$ of even size.

Impossibility Theorems

Theorem (Dietrich and List, 2007) If (and only if) an agenda is non-simple and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, systematicity and unanimity is a dictatorship (or inverse dictatorship).

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Theorem (Nehring and Puppe, 2002) If (and only if) an agenda is non-simple, every aggregation rule satisfying universal domain, collective rationality, systematicity unanimity, and monotonicity is a dictatorship.

Characterization Result

$p \in X$ conditionally entails $q \in X$, written $p \vdash^* q$ provided there is a subset $Y \subseteq X$ consistent with each of p and $\neg q$ such that $\{p\} \cup Y \vdash q$.

Totally Blocked: X is totally blocked if for any $p, q \in X$ there exists $p_1, \ldots, p_k \in X$ such that

$$p = p_1 \vdash^* p_2 \vdash^* \cdots \vdash^* p_k = q$$

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Theorem (Nehring and Puppe, 2002, 2010) If (and only if) an agenda is totally blocked, every aggregation rule satisfying universal domain, collective rationality, independence unanimity, and monotonicity is a dictatorship.

Proof Sketch, I

$C \subseteq N$ is **winning for** p if for all profiles $\mathbf{A} = (A_1, \ldots, A_n)$, if $p \in A_i$ for all $i \in C$ and $p \notin A_j$ for all $j \notin C$, then $p \in F(\mathbf{A})$

$\mathcal{C}_p = \{C \mid C \text{ is winning for } p\}$

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3. (The agenda has a minimal consistent set with at least three elements.) If $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2 \in \mathcal{C}$.

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5. For all $C \subseteq N$, either $C \in \mathcal{C}$ or $\overline{C} \in \mathcal{C}$.

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4. $N \in \mathcal{C}$.

5. For all $C \subseteq N$, either $C \in \mathcal{C}$ or $\overline{C} \in \mathcal{C}$.

6. There is an $i \in N$ such that $\{i\} \in \mathcal{C}$.

An employee-owned bakery must decide whether to buy a pizza oven (P) or a fridge to freeze their outstanding Tiramisu (F). The pizza oven and the fridge cannot be in the same room. So they also need to decide whether to rent an extra room in the back (R). They all agree that they will rent the room if they decide to buy both the pizza oven and the fridge: $((P \wedge F) \rightarrow R)$, but they are contemplating renting the room regardless of the outcome of the vote on the appliances.

F. Cariani. *Judgement Aggregation*. Philosophy Compass, 6, 1, pgs. 22 - 32.

P, F are reasons for R

$\neg P, \neg F$ are not reasons for $\neg R$

$\neg R, P$ are reasons for $\neg F$

A. Rubinstein and P. Fishburn. *Algebraic Aggregation Theory*. Journal of Economic Theory, 38, pp. 63 - 77, 1986.

$$\langle J_1, J_2, \ldots, J_n \rangle \mapsto J$$

$$\begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{pmatrix} \mapsto J$$

$$\begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{pmatrix} \mapsto (y_1, y_2, \ldots, y_m)$$

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \mapsto (y_1, y_2, \ldots, y_m)$$

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Each x_{ij} is an element of a **field** B.

For $i = 1, \ldots, n$, $x_i = (x_{i1}, \ldots, x_{im})$ is an element of a **vector space** $X \subseteq B^m$ over B.

Aggregator: $f : X^n \to X$

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C1: $(x_{1j}, \ldots, x_{nj}) = (x'_{1j}, \ldots, x'_{nj})$ implies $f_j(x_1, \ldots, x_n) = f_j(x'_1, \ldots, x'_n)$

$$\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2j} & \cdots & x_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix} \mapsto (f_1(x_1, x_2, \ldots, x_n), \ldots, f_j(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$$

$$\begin{pmatrix} x'_{11} & \cdots & x'_{1j} & \cdots & x'_{1m} \\ x'_{21} & \cdots & x'_{2j} & \cdots & x'_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x'_{n1} & \cdots & x'_{nj} & \cdots & x'_{nm} \end{pmatrix} \mapsto (f_1(x'_1, x'_2, \ldots, x'_n), \ldots, f_j(x'_1, x'_2, \ldots, x'_n), \ldots, f_m(x'_1, x'_2, \ldots, x'_n))$$

C2: $(x_{1j}, \ldots, x_{nj}) = (b, \ldots, b)$ implies $f_j(x_1, \ldots, x_n) = b$

$$\begin{pmatrix} x_{11} & \cdots & b & \cdots & x_{1m} \\ x_{21} & \cdots & b & \cdots & x_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & b & \cdots & x_{nm} \end{pmatrix} \mapsto (f_1(x_1, x_2, \ldots, x_n), \ldots, b, \ldots, f_m(x_1, x_2, \ldots, x_n))$$

$$F_C = \{f \in F \mid f \text{ satisfies } C1 \text{ and } C2\}$$

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$$F_A = \{f \in F \mid f \text{ satisfies } C1 \text{ and } f_j(y+z) = f_j(y) + f_j(z) \text{ for all } j \leq m \text{ and column vectors for which } y, z, y+z \in X_j^n \}$$

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$$F_S = \{f \in F \mid f \text{ there exists } \lambda_1, \ldots, \lambda_n \in B \text{ such that } \sum \lambda_i = 1 \text{ and, for all } (x_1, \ldots, x_n) \in X^n, f(x_1, \ldots, x_n) = \sum \lambda_i x_i\}$$

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$$F_P = \{f \in F \mid f \text{ there is an } i = 1, \ldots, n \text{ such that for all } (x_1, \ldots, x_n) \in X^n \; f(x_1, \ldots, x_n) = x_i\}$$

Theorem 1. Suppose $m \geq 3$ and $X = \{(x^1, \ldots, x^m) \in B^m \mid \sum_j b_j x^j = b\}$ with $b_j \neq 0$ for all $j \leq m$. Then, $F_C \subseteq F_A$.

Theorem 1. Suppose $m \geq 3$ and $X = \{(x^1, \ldots, x^m) \in B^m \mid \sum_j b_j x^j = b\}$ with $b_j \neq 0$ for all $j \leq m$. Then, $F_C \subseteq F_A$.

Corollary 1. Suppose $m \geq 3$,
$X = \{(x^1, \ldots, x^m) \in \mathbb{R}^m \mid \sum_j b_j x^j = b$ and $x^j \geq 0$ for all $j\}$ with b and all b_j positive.
Then $F_C \subseteq F_A$.

Corollary 2. Give the hypothesis of Theorem 1, let $f \in F_C$. Then $f \in F_S$ if B is a finite field, or if $B = \mathbb{R}$ and every f_j is continuous or monotone.

Theorem 1. Suppose $m \geq 3$ and $X = \{(x^1, \ldots, x^m) \in B^m \mid \sum_j b_j x^j = b\}$ with $b_j \neq 0$ for all $j \leq m$. Then, $F_C \subseteq F_A$.

y, z and $y + z$ are n-element vectors in B^n

$\mathbf{b}$ is the n-element vector with all components equal to b.

We must show $f(y + z) = f(y) + f(z)$.

We show that $f_1(y + z) = f_1(y) + f_1(z)$ (similar proof for other components)

$M_1 = (y, \mathbf{0}, t)$
$M_2 = \left(\mathbf{0}, \frac{b_1}{b_2}y, t\right)$
with $t = (\mathbf{b} - b_1 y)/b_3$

- For each $j = 1, \ldots, n$, $b_1 y_j + b_2 0 + b_3 t_j = b$
- For each $j = 1, \ldots, n$, $b_1 0 + b_2 \frac{b_1}{b_2} y_j + b_3 t_j = b$
- $b_1 f_1(y) + b_2 0 + b_3 f(t) = b$
- $b_1 0 + b_2 f_2 \left(\frac{b_1}{b_2} y\right) + b_3 f(t) = b$

So, $b_1 f_1(y) = b_2 f_2 \left(\frac{b_1}{b_2} y\right)$

$M_1 = (y + z, \mathbf{0}, w)$
$M_2 = \left(z, \frac{b_1}{b_2} y, w\right)$
with $w = (\mathbf{b} - b_1(y + z))/b_3$

- For each $j = 1, \ldots, n,\ \ b_1(y + z)_j + b_2 0 + b_3 w_j = b$
- For each $j = 1, \ldots, n,\ \ b_1 z_j + b_2 \frac{b_1}{b_2} y_j + b_3 w_j = b$
- $b_1 f_1(y + z) + b_2 0 + b_3 f(w) = b$
- $b_1 f_1(z) + b_2 f_2\left(\frac{b_1}{b_2} y\right) + b_3 f(w) = b$

So, $b_1 f_1(y + z) = b_1 f_1(z) + b_2 f_2\left(\frac{b_1}{b_2} y\right)$

- $b_1 f_1(y) = b_2 f_2\left(\frac{b_1}{b_2} y\right)$
- $b_1 f_1(y + z) = b_1 f_1(z) + b_2 f_2\left(\frac{b_1}{b_2} y\right)$
- So, $b_1 f_1(y + z) = b_1 f_1(z) + b_1 f_1(y)$; hence, $f_1(y + z) = f_1(y) + f_1(z)$

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The aggregation for event j depends only on the experts' probabilities for event j

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The aggregation for event j depends only on the experts' probabilities for event j

If the aggregator satisfies $C1$ and $C2$, then Corollary 1 with $X = \{(p^1, \ldots, p^m) \mid p^j \geq 0 \text{ and } \sum p^j = 1\}$ implies that the aggregator is additive.

If it is also continuous, then Corollary 2 implies that the aggregator is a **weighted average** of the experts' probability vectors.

Aggregating Probabilities

C. Genest and J. V. Zidek. *Combining probability distributions: A critique and an annotated bibliography*. Statistical Science,1(1), pp. 114 - 135, 1986.

F. Dietrich and C. List. *Probabilistic opinion pooling*. in Oxford Handbook of Probability and Philosophy, 2016.

Probability

W is a set of states (or outcomes)

$\mathcal{E}$ is an algebra of events, or propositions: $\mathcal{E} \subseteq \wp(W)$ that is closed under (countable) union and complement. (For present purposes, let $\mathcal{E} = \wp(W)$.)

A **probability measure** is a function $P : \mathcal{E} \to [0, 1]$ such that

- $P(W) = 1$
- Finite Additivity: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ ($E_1 \cap E_2 = \emptyset$)
- Countable Additivity: $P(\bigcup_i E_i) = \sum_i P(E_i)$ ($\{E_i\}$ are pairwise disjoint)

Probability

Let $(W, \mathcal{E})$ be an algebra of events

Let $\mathcal{P}$ be the set of probability functions on $(W, \mathcal{E})$

Probabilistic aggregation function: $F : \mathcal{P}^n \to \mathcal{P}$

Aggregation Functions

Linear pooling: for all $A \in \mathcal{E}$, $f(\mathbf{P})(A) = w_1 P_1(A) + \cdots w_n P_n(A)$, with $\sum_i w_i = 1$

Aggregation Functions

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Geometric pooling: for all $w \in W$, $f(\mathbf{P})(w) = c \cdot [P_1(w)]^{w_1} \cdots [P_n(w)]^{w_n}$ with $\sum_i w_i = 1$ and $c = \frac{1}{\sum_{w' \in W} [P_i(w')]^{w_1} \cdots [P_i(w')]^{w_n}}$

Aggregation Functions

Linear pooling: for all $A \in \mathcal{E}$, $f(\mathbf{P})(A) = w_1 P_1(A) + \cdots w_n P_n(A)$, with $\sum_i w_i = 1$

Geometric pooling: for all $w \in W$, $f(\mathbf{P})(w) = c \cdot [P_1(w)]^{w_1} \cdots [P_n(w)]^{w_n}$ with $\sum_i w_i = 1$ and $c = \frac{1}{\sum_{w' \in W} [P_i(w')]^{w_1} \cdots [P_i(w')]^{w_n}}$

Multiplicative pooling: for all $w \in W$, $f(\mathbf{P})(w) = c \cdot [P_1(w)] \cdots [P_n(w)]$ with $c = \frac{1}{\sum_{w' \in W} [P_i(w')] \cdots [P_i(w')]}$

Note that multiplicative pooling = geometric pooling with weights all equal to 1.

Example, I

$W = \{w_1, w_2\}$

$\mathbf{P} = (P_1, P_2, P_3)$ with $P_1(w_1) = 0.9$, $P_2(w_1) = 0.1$, $P_3(w_1) = 0.6$

Example, I

$W = \{w_1, w_2\}$

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$f_{lin}(\mathbf{P})(w_1) = \frac{1}{3} * 0.9 + \frac{1}{3} * 0.1 + \frac{1}{3} * 0.6 = 0.5333$

Example, I

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$f_{geom}(\mathbf{P})(w_1) = \frac{\sqrt{0.9*0.1*0.6}}{\sqrt{0.9*0.1*0.6} + \sqrt{0.1*0.9*0.4}} = 0.5337$

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$W = \{w_1, w_2\}$

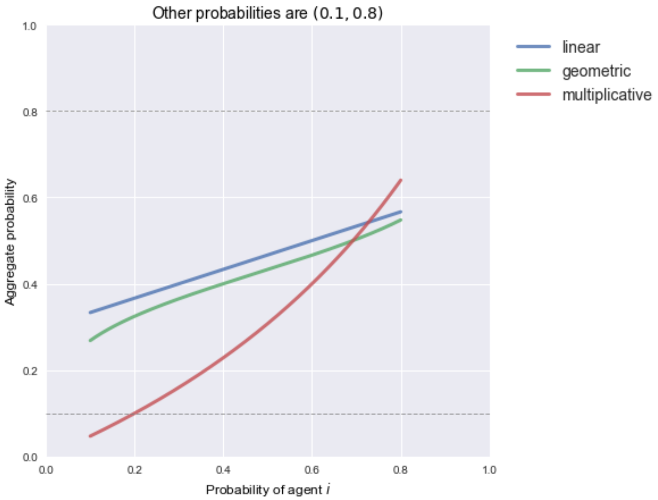
$\mathbf{P} = (P_1, P_2, P_3)$ with $P_1(w_1) = 0.9$, $P_2(w_1) = 0.1$, $P_3(w_1) = 0.6$

$f_{lin}(\mathbf{P})(w_1) = \frac{1}{3} * 0.9 + \frac{1}{3} * 0.1 + \frac{1}{3} * 0.6 = 0.5333$

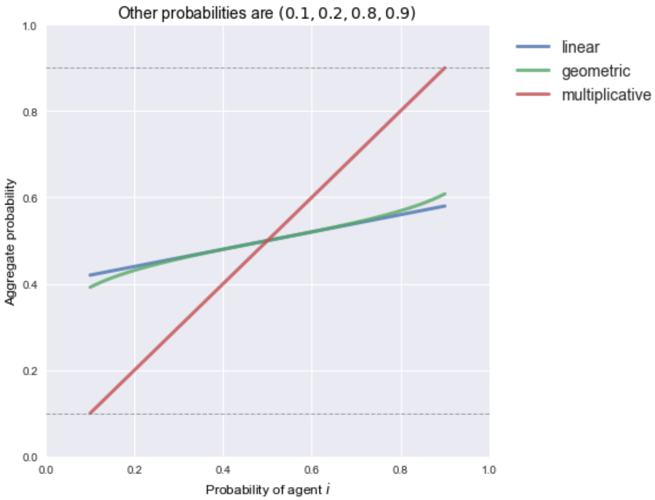
$f_{geom}(\mathbf{P})(w_1) = \frac{\sqrt{0.9*0.1*0.6}}{\sqrt{0.9*0.1*0.6} + \sqrt{0.1*0.9*0.4}} = 0.5337$

$f_{mult}(\mathbf{P})(w_1) = \frac{0.9*0.1*0.6}{0.9*0.1*0.6+0.1*0.9*0.4} = 0.6$

Example, II

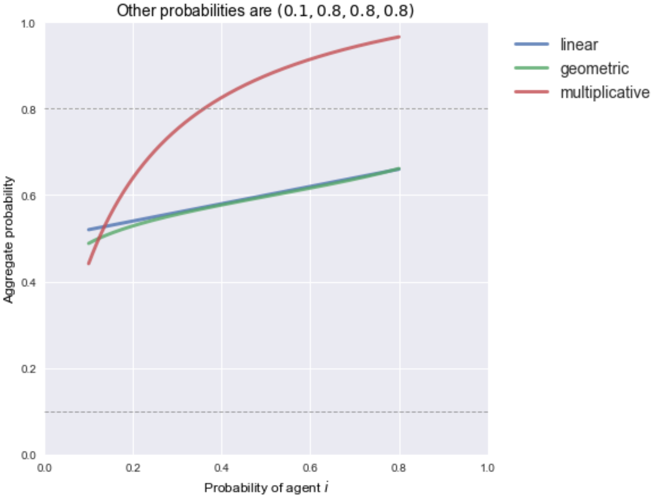

Other probabilities are (0.1, 0.8)
Aggregate probability
Probability of agent i
linear
geometric
multiplicative
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Example, III



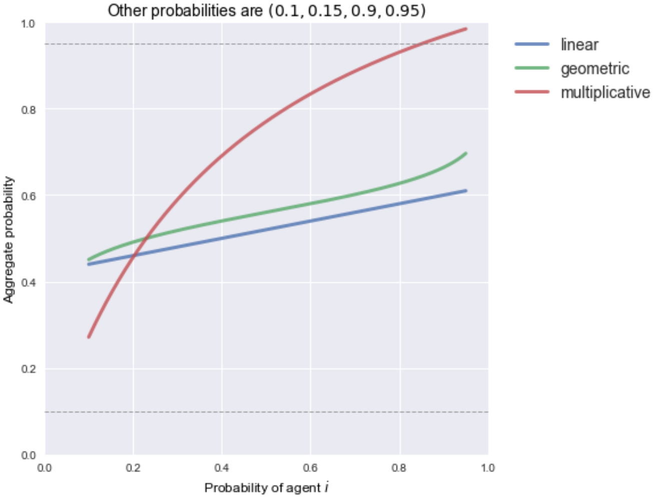
Other probabilities are (0.1, 0.2, 0.8, 0.9)
linear
geometric
multiplicative
Aggregate probability
Probability of agent i
0.0
0.2
0.4
0.6
0.8
1.0

Example, IV



Other probabilities are (0.1, 0.8, 0.8, 0.8)
1.0
0.8
0.6
0.4
0.2
0.0
Aggregate probability
0.0
0.2
0.4
0.6
0.8
1.0
Probability of agent i
linear
geometric
multiplicative

Example, V



Other probabilities are (0.1, 0.15, 0.9, 0.95)
linear
geometric
multiplicative
Aggregate probability
Probability of agent i
0.0
0.2
0.4
0.6
0.8
1.0

Linear Pooling

J. Aczel and C. Wagner. *A characterization of weighted arithmetic means*. SIAM Journal on Algebraic and Discrete Methods 1(3), pp. 259 - 260, 1980.

K. J. McConway. *Marginalization and Linear Opinion Pools*. Journal of the American Statistical Association, 76(374), pp. 410 - 414, 1981.

Eventwise Independence For each event $A \in \mathcal{E}$, there exists a function $D_A : [0, 1]^n \rightarrow [0, 1]$ such that for each $\mathbf{P} = (P_1, \ldots, P_n)$,

$$f(\mathbf{P})(A) = D_A(P_1(A), \ldots, P_n(A))$$

Unanimity preservation For every profile $\mathbf{P} = (P_1, \ldots, P_n)$ in the domain of the aggregation function f, if all P_i are identical, then $f(\mathbf{P})$ is identical to them.

Theorem (Aczel and Wagner 1980; McConway 1981) Suppose $|W| > 2$. The linear pooling functions are the only eventwise-independent and unanimity-preserving aggregation functions (with domain $\mathcal{P}^n$).

Conditioning and Linear Pooling

Suppose that there are two experts: $N = \{1, 2\}$.

Each expert has different information about a basket of fruit: $W = \{w_1, w_2, w_3\}$ where

w_1: there are precisely one apple and one banana in the basket

w_2: there is precisely one pear in the basket

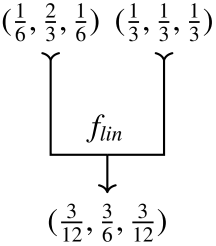
w_3: there is precisely one apple in the basket

$P_1(w_1) = \frac{1}{6}$, $P_1(w_2) = \frac{2}{3}$, $P_1(w_3) = \frac{1}{6}$

$P_2(w_1) = \frac{1}{3}$, $P_2(w_2) = \frac{1}{3}$, $P_2(w_3) = \frac{1}{3}$

Conditioning and Linear Pooling

$$P_1(w_1) = \frac{1}{6}, P_1(w_2) = \frac{2}{3}, P_1(w_3) = \frac{1}{6}$$
$$P_2(w_1) = \frac{1}{3}, P_2(w_2) = \frac{1}{3}, P_2(w_3) = \frac{1}{3}$$

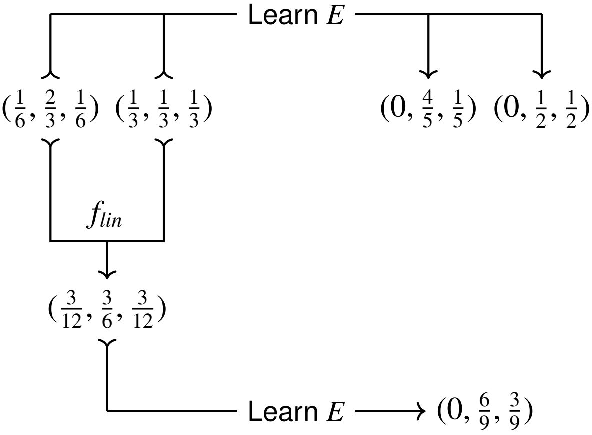


Conditioning and Linear Pooling

$P_1(w_1) = \frac{1}{6}, P_1(w_2) = \frac{2}{3}, P_1(w_3) = \frac{1}{6}$

$P_2(w_1) = \frac{1}{3}, P_2(w_2) = \frac{1}{3}, P_2(w_3) = \frac{1}{3}$

$E = \{w_2, w_3\}$

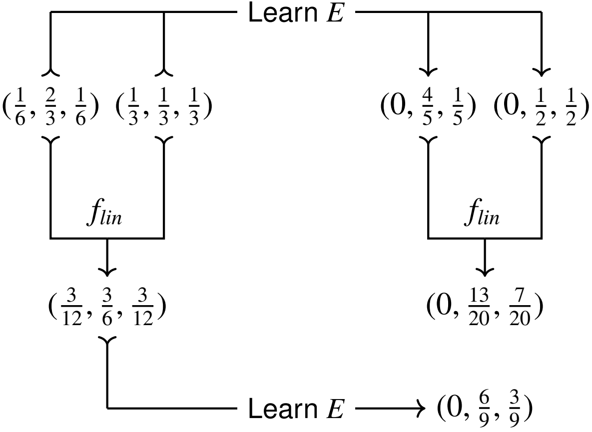


Conditioning and Linear Pooling

$P_1(w_1) = \frac{1}{6}, P_1(w_2) = \frac{2}{3}, P_1(w_3) = \frac{1}{6}$

$P_2(w_1) = \frac{1}{3}, P_2(w_2) = \frac{1}{3}, P_2(w_3) = \frac{1}{3}$

$E = \{w_2, w_3\}$

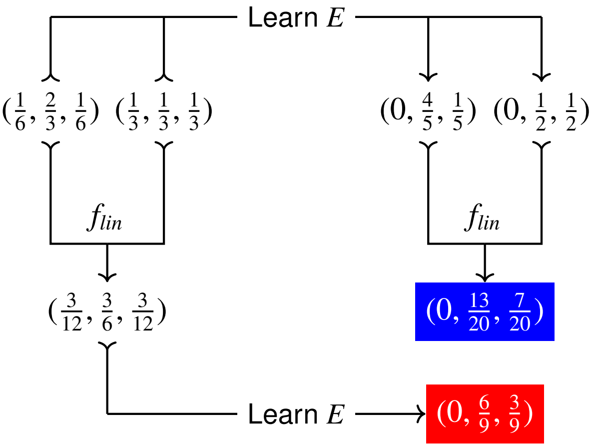


Conditioning and Linear Pooling

$P_1(w_1) = \frac{1}{6}, P_1(w_2) = \frac{2}{3}, P_1(w_3) = \frac{1}{6}$

$P_2(w_1) = \frac{1}{3}, P_2(w_2) = \frac{1}{3}, P_2(w_3) = \frac{1}{3}$

$E = \{w_2, w_3\}$



Conditioning and Linear Pooling

F. Dietrich. *Bayesian Group Belief*. Social Choice and Welfare, 35, pp. 595 - 626, 2010.

H. Leitgeb. *Imaging all the people*. Episteme, 14(4), pp. 463-479, 2017.

K. Steele. *Testimony as Evidence: More Problems for Linear Pooling*. Journal of Philosophical Logic, 41, pp. 983 - 999, 2012.

Independence and Linear Pooling

K. Lehrer and C. Wagner. *Probability amalgamation and the independence issue: a reply to Laddaga*. Synthese 55, pp. 339 - 346, 1983.

C. Wagner. *On the Formal Properties of Averaging as a Method of Aggregation*. Synthese, 62, pp. 97 - 108, 1985.

C. Wagner. *Aggregating subjective probabilities: some limitative theorems*. Notre Dame Journal of Formal Logic, 25(3), pp. 233 - 240, 1984.

RI (Respect for Individual Attributions of Independence) For any propositions E and F and profile $\mathbf{P} = (P_1, \ldots, P_n)$, if $P_i(E \cap F) = P_i(E)P_i(F)$ for all $i = 1, \ldots, n$, the $f(\mathbf{P})(E \cap F) = f(\mathbf{P})(E)f(\mathbf{P})(F)$

RI (Respect for Individual Attributions of Independence) For any propositions E and F and profile $\mathbf{P} = (P_1, \ldots, P_n)$, if $P_i(E \cap F) = P_i(E)P_i(F)$ for all $i = 1, \ldots, n$, the $f(\mathbf{P})(E \cap F) = f(\mathbf{P})(E)f(\mathbf{P})(F)$

Theorem (Wagner). Suppose that $f : \mathcal{P}^n \to \mathcal{P}$. Then, f satisfies eventwise-independence, unanimity-preservation and respect for individual attributions of independence if, and only if, f is a dictatorship.

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} \\
p_4 &= w_1 p_{14} + w_2 p_{24} + \cdots + w_d p_{d4} + \cdots + w_n p_{n4} \\
&\vdots \\
p_k &= w_1 p_{1k} + w_2 p_{2k} + \cdots + w_d p_{dk} + \cdots + w_n p_{nk}
\end{aligned}
$$

$$\begin{array}{c} \\ 1 \\ \vdots \\ d \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \mapsto (p_1, \ldots, p_k)$$

$$\begin{array}{lcllcl} p_1 & = & w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} & & \\ p_2 & = & w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} & & \\ p_3 & = & w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} & & \\ p_4 & = & w_1 p_{14} + w_2 p_{24} + \cdots + w_d p_{d4} + \cdots + w_n p_{n4} & = & 0 \\ & \vdots & & \vdots & \vdots \\ p_k & = & w_1 p_{1k} + w_2 p_{2k} + \cdots + w_d p_{dk} + \cdots + w_n p_{nk} & = & 0 \end{array}$$

$$
\begin{array}{c} \\ 1 \\ \vdots \\ d \\ \vdots \\ n \end{array}
\begin{array}{c} \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \mapsto (p_1, \ldots, p_k)
$$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2} \textstyle\sum_{j \neq d} w_j \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
 \\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c} \\ 1 \\ \vdots \\ d \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

For all i, $P_i(\{s_1, s_2\} \cap \{s_2, s_3\}) = P_i(\{s_1, s_2\})P_i(\{s_2, s_3\})$

$$
\begin{array}{lcl}
p_1 & = & w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} & = & \frac{1}{2}(1 - w_d) \\
p_2 & = & w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} & = & \frac{1}{2} \\
p_3 & = & w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} & = & \frac{1}{2} w_d
\end{array}
$$

$$
\begin{array}{c}
 \\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $f(\mathbf{P})(\{s_1, s_2\} \cap \{s_2, s_3\}) = f(\mathbf{P})(\{s_1, s_2\}) f(\mathbf{P})(\{s_2, s_3\})$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $p_2 = (p_1 + p_2)(p_2 + p_3)$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c} \\ 1 \\ \vdots \\ d \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $\frac{1}{2} = (\frac{1}{2}(1 - w_d) + \frac{1}{2})(\frac{1}{2} + \frac{1}{2}w_d)$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &&= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &&= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &&= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ \vdots \\ d \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI implies that $2 = ((1 - w_d) + 1)(1 + w_d)$

$$
\begin{aligned}
p_1 &= w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} &= \tfrac{1}{2}(1 - w_d) \\
p_2 &= w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} &= \tfrac{1}{2} \\
p_3 &= w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} &= \tfrac{1}{2} w_d
\end{aligned}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ d \\ \vdots \\ n \end{array} &
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix} \mapsto (p_1, \ldots, p_k)
\end{array}
$$

RI implies that $w_d = 0$ or $w_d = 1$

$$
\begin{aligned}
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\end{aligned}
$$

$$
\begin{array}{c} \\ 1 \\ \vdots \\ d \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & \cdots & s_k \end{array} \\
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & \vdots \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \ddots & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{array}
\mapsto (p_1, \ldots, p_k)
$$

RI and $w_d > 0$ implies that $w_d = 1$

$$
\begin{array}{lcl}
p_1 & = & w_1 p_{11} + w_2 p_{21} + \cdots + w_d p_{d1} + \cdots + w_n p_{n1} & = & \frac{1}{2}(1 - w_d) \\
p_2 & = & w_1 p_{12} + w_2 p_{22} + \cdots + w_d p_{d2} + \cdots + w_n p_{n2} & = & \frac{1}{2} \\
p_3 & = & w_1 p_{13} + w_2 p_{23} + \cdots + w_d p_{d3} + \cdots + w_n p_{n3} & = & \frac{1}{2} w_d
\end{array}
$$

Geometric Pooling

Geometric pooling: for all $w \in W, f(\mathbf{P})(w) = c \cdot [P_1(w)]^{w_1} \cdots [P_n(w)]^{w_n}$ with $\sum_i w_i = 1$ and $c = \frac{1}{\sum_{w' \in W} [P_i(w')]^{w_1} \cdots [P_i(w')]^{w_n}}$

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- Unanimity-preserving.

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- Unlike linear pooling, it is not eventwise independent.

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- Unanimity-preserving.
- Unlike linear pooling, it is not eventwise independent.
- However, it does satisfy **external Bayesianity**.

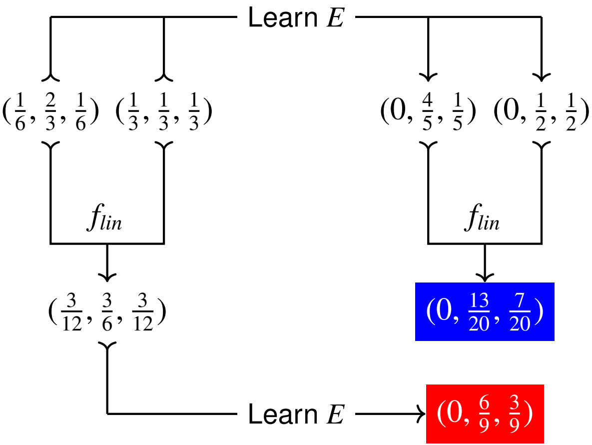
A. Madansky. *Externally Bayesian groups*. Technical Report RM-4141- PR, RAND Corporation, 1964.

Bayesian Externality

$$P_1(w_1) = \frac{1}{6}, P_1(w_2) = \frac{2}{3}, P_1(w_3) = \frac{1}{6}$$

$$P_2(w_1) = \frac{1}{3}, P_2(w_2) = \frac{1}{3}, P_2(w_3) = \frac{1}{3}$$

$$E = \{w_2, w_3\}$$



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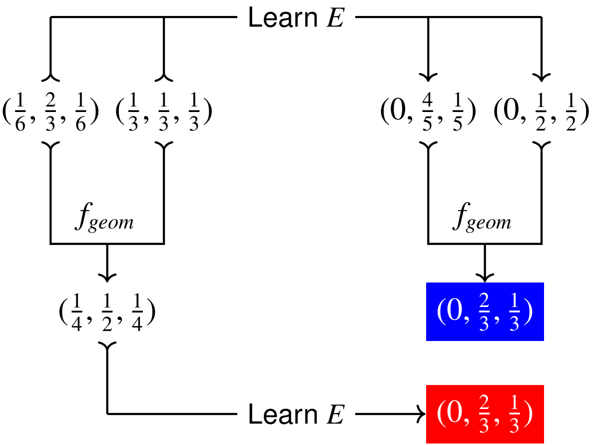

Bayesian Externality

Likelihood function: A function $L : W \to \mathbb{R}^+$.

Given a function $P : W \to [0, 1]$, $P^L : W \to [0, 1]$ where for all $w \in W$, $P^L(w) = \frac{P(w)L(w)}{\sum_{w' \in W} P(w')L(w')}$.

External Bayesianity. For every opinion profile $\mathbf{P} = (P_1, \ldots, P_n)$ and every likelihood function L, pooling and updating are commutative: $f(\mathbf{P})^L = f(\mathbf{P}^L)$, where $\mathbf{P}^L = (P_1^L, \ldots, P_n^L)$.

Bayesian Externality

Theorem (Genest). The geometric pooling functions are externally Bayesian and unanimity-preserving.

G. Genest. *A characterization theorem for externally Bayesian groups*. Annals of Statistics 12(3), pp. 1100-1105, 1984.

C. Genest, K. J. McConway and M. J. Schervish. *Characterization of externally Bayesian pooling operators*. Annals of Statistics 14(2), pp. 487-501, 1986.

F. Dietrich. *A Theory of Bayesian Groups*. Nous, 2017.

Theorem. Update by **general imaging** (with respect to fixed transfer function T) is the unique update mechanism that commutes with linear pooling with respect to arbitrary coefficients.

H. Leitgeb. *Imaging all the people*. Episteme, 14(4), pp. 463 - 479, 2017.

P. Gärdenfors. *Imaging and Conditionalization*. The Journal of Philosophy, 79(12), pp. 747 - 760, 1982.