Logics for Social Choice Theory

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Lecture 4

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Social choice correspondence

A **voting method** is a function F on the domain of all profiles such that for any profile P, $\varnothing \neq F(\mathrm{P}) \subseteq X(\mathrm{P})$ (also called a **variable social choice correspondence** VSCC).

- A (V, X)-SCC is a social choice correspondence defined on (V, X)-profiles.
- A voting method F is **resolute** if for all P, $|F(\mathrm{P})| = 1$. Resolute SCCs are called **social choice functions**.

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There are many examples of voting methods.

See `https://pref_voting.readthedocs.io` for a Python package that provides computational tools to study different voting methods.

Positional Scoring Rules

Borda:	$\mathcal{S}(n) = \langle n-1, n-2, \ldots, 1, 0 \rangle$
Plurality:	$\mathcal{S}(n) = \langle 1, 0, \ldots, 0 \rangle$
Anti-Plurality:	$\mathcal{S}(n) = \langle 1, 1, \ldots, 1, 0 \rangle$

1	3	2	4
a	b	b	c
c	a	c	a
b	c	a	b

Borda winner	c
Plurality winner	b
Anti-Plurality winner	a

Condorcet criteria

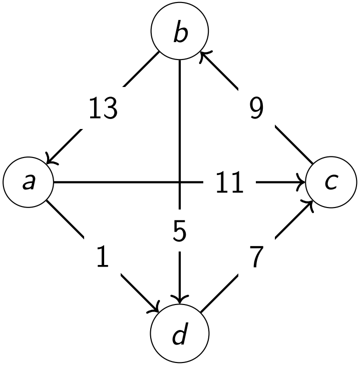
The **Condorcet winner** in a profile P is a candidate $x \in X(\mathrm{P})$ that is the maximum of the majority ordering, i.e., for all $y \in X(\mathrm{P})$, if $x \neq y$, then $Margin_{\mathrm{P}}(x, y) > 0$.

The **Condorcet loser** in a profile P is a candidate $x \in X(\mathrm{P})$ that is the minimum of the majority ordering, i.e., for all $y \in X(\mathrm{P})$, if $x \neq y$, then $Margin_{\mathrm{P}}(y, x) < 0$.

A voting method F is **Condorcet consistent**, if for all P, if x is a Condorcet winner in P, then $F(\mathrm{P}) = \{x\}$.

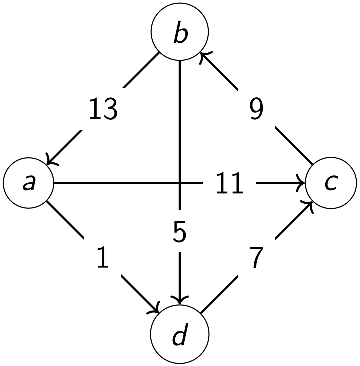
A voting method F is susceptible to the **Condorcet loser paradox** (also known as *Borda's paradox*) if there is some P such that x is a Condorcet loser in P and $x \in F(\mathrm{P})$.

Condorcet consistent methods



Minimax:	$\{d\}$
Copeland:	$\{a, b\}$
Beat Path:	$\{d\}$
Ranked Pairs:	$\{b\}$
Split Cycle:	$\{b, d\}$

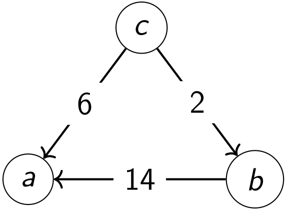
Condorcet consistent methods



Minimax:	$\{d\}$
Copeland:	$\{a, b\}$
Beat Path:	$\{d\}$
Ranked Pairs:	$\{b\}$
Split Cycle:	$\{b, d\}$

Proposition. Both Ranked Pairs and Beat Path refine Split Cycle (i.e., in all profiles, any Ranked Pairs (resp. Beat Path) winner is also a Split Cycle winner.

20	13	21	14	22	10
a	a	c	b	b	c
c	b	b	a	c	a
b	c	a	c	a	b



Condorcet winner: c

Instant Runoff winner: b

Plurality winner: b

Borda winner: b

Theorem (Smith 1973, Young 1974)

A voting method satisfies Anonymity, Neutrality and **Reinforcement** *if and only if F is a scoring rule.*

Saari's argument, Balinski and Laraki (2010, pg. 77); Zwicker (2016, Proposition 2.5): Multiple districts paradox, *f cancels properly*.

2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c

- no Condorcet winner in the left profile
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

The Gibbard-Satterthwaite Theorem

Gibbard-Satterthwaite Theorem. Assume that there are more than 3 candidates. Any resolute voting method satisfying non-imposition and strategyproofness is dictatorial.

M. A. Satterthwaite. *Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions*. Journal of Economic Theory, 10(2):187-217, 1975.

A. Gibbard. *Manipulation of voting schemes: A general result*. Econometrica, 41(4):587-601, 1973.

Theorem 13.1 For $n = 3$ voters and $m > 3$ alternatives, no (resolute) voting rule satisfies both strategyproofness and the majority criterion.

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Lemma 1. Let $m = 3$ and $n = 3$. There is no resolute voting rule F satisfying strategyproofness and the majority criterion

Lemma 2. Let $m \geq 3$ and $n = 3$. If F is a resolute voting rule satisfying strategyproofness and the majority criterion for $m + 1$ alternatives, then there exists a voting rule F' for m alternatives with the same properties.

Christian Geist and Dominik Peters. *Computer-aided Methods for Social Choice Theory*. Trends in Computational Social Choice, chapter 13, pages 249–267. AI Access, 2017.

Stregyrpoofness

Suppose that F is a resolute voting method
F is **strategyproof** provided there is no voter i and no profiles

$$\mathrm{P} = (P_1, \ldots, P_i, \ldots, P_n) \text{ and } \mathrm{P}' = (P'_1, \ldots, P'_i, \ldots, P'_n)$$

such that

$P_j = P'_j$ for all $j \neq i$, and

i strictly prefers the winner under P' to the winner under P:
aP_ib where $F(\mathrm{P}') = \{a\}$ and $F(\mathrm{P}) = \{b\}$.

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If $F(\mathrm{P})$ and $F(\mathrm{P}')$ are singletons, then "i **prefers** $F(\mathrm{P}')$ **to** $F(\mathrm{P})$" means $F(\mathrm{P}')\ P_i\ F(\mathrm{P})$

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What happens if $F(\mathrm{P})$ and $F(\mathrm{P}')$ are not singletons?

A. Taylor. *Social Choice and the Mathematics of Manipulation*. Cambridge University Press, 2005.

Suppose that $F(\mathrm{P}) = Y$ and $F(\mathrm{P}') = Z$ are not singletons

- Z **weakly dominates** Y for i provided

for all $z \in Z$ and $y \in Y$ z is weakly preferred to y by i and

there exists $z \in Z$ and $y \in Y$ such that $z \; P_i \; y$

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- Z is preferred by an **optimist** to Y: $\max_i(Z) \; P_i \; \max_i(Y)$

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- Z is preferred by an **optimist** to Y: $\max_i(Z) \; P_i \; \max_i(Y)$

- Z is preferred by a **pessimist** to Y: $\min_i(Z) \; P_i \; \min_i(Y)$

Fact. Plurality rules is weak dominance manipulable, but is never single-winner manipulable.

1	2	1
a	c	b
b	a	a
c	b	c

Plurality Winner: c

1	2	1
b	c	b
a	a	a
c	b	c

Plurality Winners: $\{b, c\}$

Fact. Condorcet rule is manipulable by optimists (and also by pessimists), but is never weak dominance manipulable.

1	1	1
a	b	c
c	c	a
b	a	b

Condorcet Winner: c

1	1	1
a	b	c
b	c	a
c	a	b

Condorcet Winners: $\{a, b, c\}$

The Duggan-Schwartz Theorem

i is a **nominator** if for all profiles P, then $Top(P_i) \in F(\mathrm{P})$.

Non-Imposed: For all $a \in X$ there exists a profile P such that $F(\mathrm{P}) = \{a\}$.

Manipulated by Optimist/Pessimist

F can be manipulated by an **optimist** if there is a profile P an ordering $Q_i \in O(X)$ such that $Q_i \neq P_i$ and

$$\exists x \in F(\mathrm{P}[P_i/Q_i]), \forall y \in F(\mathrm{P}), xP_iy$$

F can be manipulated by a **pessimist** if there is a profile P an ordering $Q_i \in O(X)$ such that $Q_i \neq P_i$ and

$$\forall x \in F(\mathrm{P}[P_i/Q_i]), \exists y \in F(\mathrm{P}), xP_iy$$

The Duggan-Schwartz Theorem

Theorem. Suppose that X has at least three elements. Any voting method $F : \mathcal{L}(X)^V \rightarrow (\wp(X) - \emptyset)$ that is non-imposed and cannot be manipulated by an optimist or a pessimist has a nominator

J. Duggan and T. Schwartz. *Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized*. Social Choice and Welfare, 17, pp. 85 - 93, 2000.

Fishburn set extension

Suppose i is a voter with a preference ordering R_i that expects the ties in the voting rule to be broken according to some linear tie-breaking order; however, i does not know which order will be used:

For sets of candidates X and Y, we have $X \mathrm{R_i}^F Y$ provided that

1. xR_iy for all $x \in X \setminus Y$ and $y \in X \cap Y$
2. yR_iz for all $y \in X \cap Y$ and $z \in Y \setminus X$
3. xR_iz for all $x \in X \setminus Y$ and $z \in Y \setminus X$

F is a C1 voting method provided that for all profiles P and P', if $M(\mathrm{P}) = M(\mathrm{P}')$, then $F(\mathrm{P}) = F(\mathrm{P}')$

Theorem (Brandt and Geist, 2016). There is no C1 voting method that satisfying Neutrality, Pareto and Fishburn-strategyproofness for $m \geq 5$ candidates and $n \geq 7$ voters.

candidates	unlabeled weak tournaments	6-voter ANECs	6-voter anonymous profiles	6-voter profiles
3	7	83	462	46,656
4	42	19,941	475,020	191,102,976
5	582	39,096,565	4,690,625,500	2,985,984,000,000

Definition

A *weak tournament solution* (resp. *tournament solution*) is a function F on a set of weak tournaments (resp. tournaments) such that for all $T \in \mathrm{dom}(F)$, we have $\varnothing \neq F(T) \subseteq X(T)$.

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Definition

Given $T, T' \in \mathcal{WT}$, let $T \cong T'$ if T and T' are isomorphic. For each $T \in \mathcal{WT}$, define the equivalence class $[T] = \{T' \in \mathcal{WT} \mid T \cong T'\}$ and pick a *canonical representative* $T_C \in [T]$. Let $\mathcal{WT}_C = \{T_C \mid T \in \mathcal{WT}\}$ and $\mathcal{T}_C = \{T_C \mid T \in \mathcal{T}\}$ be the sets of *canonical weak tournaments* and *canonical tournaments*, respectively. A *canonical weak tournament solution* (resp. *canonical tournament solution*) is a function F on a subset of $\mathcal{WT}_C$ (resp. $\mathcal{T}_C$) such that for all $T \in \mathrm{dom}(F)$, we have $\varnothing \neq F(T) \subseteq X(T)$.

Neutrality

There is a one-to-one correspondence between neutral weak tournament solutions and canonical weak tournament solutions satisfying what Brandt and Geist call the *orbit condition*.

Orbit Condition

Given $T \in \mathcal{WT}_C$ and $a, b \in X(T)$, let $a \sim_T b$ if there exists an automorphism h of T such that $h(a) = b$. The *orbit of a in T* is $\{b \in T \mid a \sim_T b\}$. Let $\mathcal{O}_T$ be the set of all orbits of elements of T.

Definition

Given a canonical weak tournament T and $Y \subseteq X(T)$, we say that (T, Y) satisfies the *orbit condition* if for all $O \in \mathcal{O}_T$, we have $O \subseteq Y$ or $O \cap Y = \varnothing$. A canonical weak tournament solution F satisfies the *orbit condition* if for all $T \in \mathrm{dom}(F)$, $(T, F(T))$ satisfies the orbit condition. Let

$$\mathcal{O}(T) = \{Y \subseteq X(T) \mid Y \neq \varnothing \text{ and } (T, Y) \text{ satisfies the orbit condition}\}.$$

Lemma

1. *Given a canonical weak tournament solution F satisfying the orbit condition, the function F^* on $\{T \in \mathcal{WT} \mid T_C \in \mathrm{dom}(F)\}$ defined by $F^*(T) = h_T[F(T_C)]$ for an isomorphism $h_T : T_C \to T$ is a neutral weak tournament solution.*
2. *Given a neutral weak tournament solution F, the function F_* on $\{T_C \mid T \in \mathrm{dom}(F)\}$ defined by $F_*(T_C) = F(T_C)$ is a canonical weak tournament solution satisfying the orbit condition.*
3. *For any canonical weak tournament solution F satisfying the orbit condition, $(F^*)_* = F$; and for any neutral weak tournament solution F, $(F_*)^* = F$.*

Key idea: Unequivocal increase in support for a candidate should not result in that candidate going from being a winner to being a loser.

1. *monotonicity*: if a candidate x is a winner given a preference profile P, and P′ is obtained from P by one voter moving x up in their ranking, then x should still be a winner given P′.
   (fixed-electorate axiom)

2. *positive involvement*: if a candidate x is a winner given P, and P^* is obtained from P by adding a new voter who ranks x in first place, then x should still be a winner given P^*.
   (variable-electorate axiom)

Violating Positive Involvement: Coombs

2	2	1	1	2	1	1
c	b	d	d	c	a	b
a	a	c	a	b	d	d
b	c	b	c	d	b	a
d	d	a	b	a	c	c

Coombs winner: $\{b\}$

(the order of elimination is d, c)

2	2	1	1	2	1	1	1
c	b	d	d	c	a	b	b
a	a	c	a	b	d	d	d
b	c	b	c	d	b	a	c
d	d	a	b	a	c	c	a

Coombs winner: $\{c\}$

(a and d are tied for the most last place votes)

Breaking Ties

There are many tiebreaking rules: non-anonymous, non-neutral, random

Parallel universe tiebreaking: x is a winner if x wins according to some tiebreaking rule.

S. Obraztsova, E. Elkind and N. Hazon. *Ties Matter: Complexity of Voting Manipulation Revisited*. Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence.

J. Wang, S. Sikdar, T. Shepherd, Z. Zhao, C. Jiang and L. Xia. *Practical Algorithms for Multi-Stage Voting Rules with Parallel Universes Tiebreaking*. Proceedings of AAAI, 2019.

Violating Positive Involvement: Coombs PUT

1	1	1	1	1
a	c	b	c	d
c	d	d	a	b
b	b	a	b	a
d	a	c	d	c

Coombs winner: $\{a, b\}$

1	1	1	1	1	1
a	c	b	c	d	a
c	d	d	a	b	d
b	b	a	b	a	b
d	a	c	d	c	c

Coombs winner: $\{b, d\}$

No Show Paradox

The term "No Show Paradox" was introduced by Fishburn and Brams for violations of what is now called *negative involvement*: Adding a new voter who ranks a candidate last should not result in the candidate going from being a loser to a winner.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine, 56(4), pp. 207 - 214, 1983.

D. Saari. *Basic Geometry of Voting*. Springer, 1995.

No Show Paradox

Moulin changed the meaning of "No Show Paradox" to refer to violations of participation: A resolute voting method satisfies participation if adding a new voter who ranks x above y cannot result in a change from x being the unique winner to y being the unique winner.

H. Moulin. *Condorcet's Principle Implies the No Show Paradox*. Journal of Economic Theory 45(1), pp. 53 - 64, 1988.

No Show Paradox

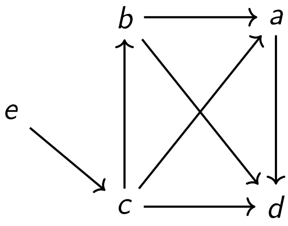
Peréz concludes that the Strong No Show Paradox is a common flaw of many *Condorcet consistent* voting methods, which are methods that always pick a Condorcet winner—a candidate who is majority preferred to every other candidate—if one exists.

J. Pérez. *The Strong No Show Paradoxes are a common flaw in Condorcet voting correspondences*. Social Choice and Welfare 18(3), pp. 601 - 616, 2001.

Theorem (Brandl et al., 2015). There is no majoritarian and Pareto optimal voting method that satisfies Fishburn-participation if $m \geq 4$ and $n \geq 6$.

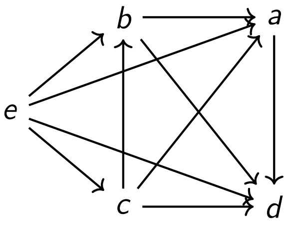
Violating Positive Involvement: Copeland

2	1	1
e	c	a
c	b	d
b	a	b
a	d	e
d	e	c



Copeland winners: $\{c\}$

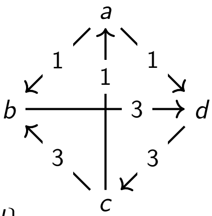
2	1	1	
e	c	a	c
c	b	d	e
b	a	b	d
a	d	e	c
d	e	c	a



Copeland winners: $\{e\}$

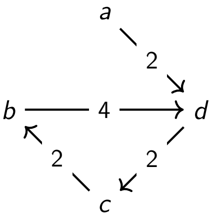
Violating Positive Involvement: Beat Path

1	1	1	1	2	1	1	1	1	1
a	d	c	c	a	b	a	d	d	b
d	c	a	b	c	d	b	c	b	a
c	b	b	d	b	c	d	a	c	d
b	a	d	a	d	a	c	b	a	c



Beat Path winners: $\{a, b, c, d\}$

1	1	1	1	2	1	1	1	1	1	1
a	d	c	c	a	b	a	d	d	b	b
d	c	a	b	c	d	b	c	b	a	a
c	b	b	d	b	c	d	a	c	d	c
b	a	d	a	d	a	c	b	a	c	d



Beat Path winners: $\{a\}$

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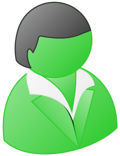
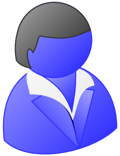
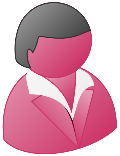
If  wins   and  wins 

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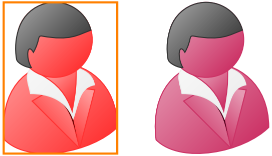
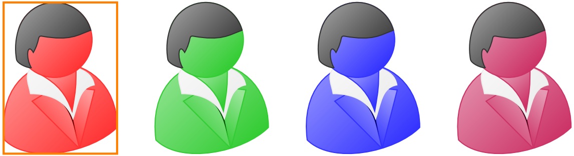
If  wins and  wins

then  wins .

Stability for Winners

Definition

A VCCR satisfies **Stability for Winners** if for any profile P and $a, b \in X(\mathrm{P})$, if a is undefeated in P_{-b} and $\mathit{Margin}_{\mathrm{P}}(a, b) > 0$, then a is undefeated in P.

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Example violations:

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Choice Consistency

Suppose that C is a choice function on X: for all $\emptyset \neq A \subseteq X$, $\emptyset \neq C(A) \subseteq A$.

Sen's α condition: if $A' \subseteq A$, then $C(A) \cap A' \subseteq C(A')$

Sen's γ condition (expansion): $C(A) \cap C(A') \subseteq C(A \cup A')$

Theorem (Sen 1971)

Let C be a choice function on a nonempty finite set X. TFAE:

1. *C satisfies α and γ*
2. *There exists a binary relation P on X such that for all $A \subseteq X$,*

$$C(A) = \{x \in A \mid \textit{there is no } y \in A \textit{ such that } y\, P\, x\}$$

A. Sen. *Choice Functions and Revealed Preference*. The Review of Economic Studies, 38:3, pp. 307-317, 1971.

Expansion in Voting

A **voting method** is a function F on the domain of all profiles such that for any profile $\mathbf{P}$, $\varnothing \neq F(\mathbf{P}) \subseteq X(\mathbf{P})$.

Expansion in Voting

A **voting method** is a function F on the domain of all profiles such that for any profile P, $\varnothing \neq F(\mathrm{P}) \subseteq X(\mathrm{P})$.

A voting method F satisfies Expansion if for all profiles P and Y, Y' with $Y \cup Y' = X(\mathrm{P})$,

$$F(\mathrm{P}_{|Y}) \cap F(\mathrm{P}_{|Y'}) \subseteq F(\mathrm{P}).$$

Warning: Functional Collective Choice Rules

A **collective choice rule** (CCR) is a function f on the domain of all profiles such that for any profile P, $f(\mathrm{P})$ is an asymmetric binary relation on $X(\mathrm{P})$.

A **functional collective choice rule** (FCCR) is a function F that assigns to each profile P a choice function $F(\mathrm{P})$ on $X(\mathrm{P})$.

An FCCR F satisfies Expansion if for all profiles P, $F(P)$ satisfies Expansion.

Warning: Global vs. Local Choice FCCR

Given an acyclic CCR f, there are two ways to induce an FCCR:

1. the global-choice FCCR $\mathcal{G}_f$: for any profile P and nonempty $Y \subseteq X(\mathrm{P})$,

$$\mathcal{G}_f(\mathrm{P})(Y) = \{y \in Y \mid \text{there is no } z \in Y \text{ such that } y\ f(\mathrm{P})\ x\}.$$

2. the local-choice FCCR $\mathcal{L}_f$: for any profile P and nonempty $Y \subseteq X(\mathrm{P})$,

$$\mathcal{L}_f(\mathrm{P})(Y) = \{y \in Y \mid \text{there is no } z \in Y \text{ that such that } y\ f(\mathrm{P}_{|Y})\ x\}.$$

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Warning: Global/Local Borda

Let $Y = \{x, a, y\}$

1	1	2
x	y	y
a	x	x
b	a	c
c	b	b
y	c	a

1	1	2
x	y	y
a	x	x
y	a	a

Global Borda: $\mathcal{G}_{Borda}(\text{P})(Y) = \{x\}$

Local Borda: $\mathcal{L}_{Borda}(\text{P})(Y) = \{y\}$.

Warning: Expansion for FCCRs

Global-choice FCCRs always satisfy Expansion (and α).

Local-choice FCCRs may not satisfy Expansion. (There are no reasonable CCRs whose local-choice FCCR satisfies α).

Our definition of Expansion for voting methods is similar to the local-choice version.

Expansion in Voting

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A voting method F satisfies Expansion if for all profiles P and Y, Y' with $Y \cup Y' = X(\mathrm{P})$,

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Expansion in Voting

> First, it seems "intuitively right" that if x is a "winner" in both A and A', then it should stay a winner in $A \cup A'$. Second, it limits the manipulability of the SCF in that it implies that if x is a winner in A, and if B is formed by adding to A new alternatives (no matter whether they are winning or losing) such that x is a winner in some subset of B that contains the new alternatives, then x is still a winner in B. In particular, this means that one cannot turn x into a loser by introducing new alternatives to which x does not lose in duels. (p. 125)

G. Bordes. *On the Possibility of Reasonable Consistent Majoritarian Choice: Some Positive Results*. Journal of Economic Theory, 31:1, pp. 122 - 132, 1983.

Binary Expansion

Expansion: For all $A, A' \subseteq X$, $C(A) \cap C(A') \subseteq C(A \cup A')$.

Binary Expansion: For all $A, A' \subseteq X$ such that $|A'| = 2$, $C(A) \cap C(A') \subseteq C(A \cup A')$.

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Modulo α, Expansion is equivalent to Binary Expansion. Thus, we can replace Expansion by Binary Expansion in Sen's representation theorem.

Binary Expansion for Voting Methods

Expansion: For all profiles P and Y, Y' with $Y \cup Y' = X(\mathrm{P})$, $F(\mathrm{P}_{|Y}) \cap F(\mathrm{P}_{|Y'}) \subseteq F(\mathrm{P})$.

Binary Expansion: For all profiles P and Y, Y' with $Y \cup Y' = X(\mathrm{P})$, if $|Y'| = 2$, then $F(\mathrm{P}_{|Y}) \cap F(\mathrm{P}_{|Y'}) \subseteq F(\mathrm{P})$.

Binary Expansion for Voting Methods

Expansion: For all profiles P and Y, Y' with $Y \cup Y' = X(\mathrm{P})$, $F(\mathrm{P}_{|Y}) \cap F(\mathrm{P}_{|Y'}) \subseteq F(\mathrm{P})$.

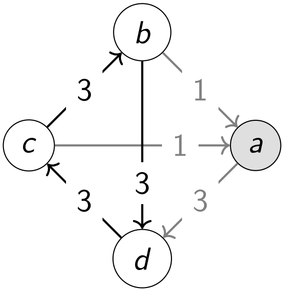
Binary Expansion: For all profiles P and Y, Y' with $Y \cup Y' = X(\mathrm{P})$, if $|Y'| = 2$, then $F(\mathrm{P}_{|Y}) \cap F(\mathrm{P}_{|Y'}) \subseteq F(\mathrm{P})$.

Strong Stability for Winners: For all profiles P and $a, b \in X(\mathrm{P})$, if $a \in F(\mathrm{P}_{-b})$ and $Margin_{\mathrm{P}}(a, b) \geq 0$, then $a \in F(\mathrm{P})$.

W. Holliday and EP. *Split Cycle: A New Condorcet Consistent Voting Method Independent of Clones and Immune to Spoilers*. https://arxiv.org/abs/2004.02350, 2021.

Beat Path and Minimax Violate Binary Expansion

2	1	1	1	3	1	1	1
b	a	d	a	c	b	c	d
a	b	a	d	b	d	d	c
d	d	c	c	a	c	a	a
c	c	b	b	d	a	b	b



Beat Path and Minimax both violate Binary Expansion: $F(\mathrm{P}_{-a}) = \{b, c, d\}$, $Margin_{\mathrm{P}}(b, a) > 0$, and $b \notin F(\mathrm{P})$.

Methods that satisfy Expansion: Top Cycle, Uncovered Set, Split Cycle

Methods that satisfy Binary Expansion but violate Expansion: Banks

Methods that violate Binary Expansion: Plurality, Borda, Instant Runoff, Copeland, Minimax, Ranked Pairs, Beat Path, . . .

Spoilers

Binary Expansion rules out *spoilers*.

37	29	34
d	d	p
p	p	d

IR Winner: d

37	29	34
r	d	p
d	p	d
p	r	r

IR Winner: p

Immunity to Spoilers: For all profiles P and $a, b \in X(\mathrm{P})$,
if $a \in F(\mathrm{P}_{-b})$, $Margin_{\mathrm{P}}(a, b) > 0$ and $b \notin F(\mathrm{P})$, then $a \in F(\mathrm{P})$

Minimax, Copeland, and GOCHA all satisfy Immunity to Spoilers, but not Binary Expansion

Quasi-Resoluteness

Several of the methods violating Binary Expansion satisfy:

Asymptotic Resolvability: in the limit as the number of voters goes to infinity, the proportion of profiles with a unique winner before any tiebreaking (e.g., runoff election, lottery, etc.) goes to 1.

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For margin-based methods, asymptotic resolvability is equivalent to:

Quasi-Resoluteness: for any profile P, if there are no ties in the margins of any candidates in P, then $|F(\mathrm{P})| = 1$.

The equivalence follows from results in

M. Harrison Trainor. *An Analysis of Random Elections with Large Numbers of Voters*. arXiv:2009.02979.

Violations of Quasi-Resoluteness

The known methods that satisfy Binary Expansion violate Asymptotic Resolvability/Quasi-Resoluteness.

Voting Method	3	4	5	6	7	8	9	10	20	30
Split Cycle	1	1.01	1.03	1.06	1.08	1.11	1.14	1.16	1.42	1.62
Uncovered Set	1.17	1.35	1.53	1.71	1.9	2.09	2.26	2.46	4.56	6.82
Top Cycle	1.17	1.44	1.8	2.21	2.72	3.31	3.94	4.68	13.55	22.94

Figure: Estimated **average sizes of winning sets** for profiles with a given number of candidates (top row) in the limit as the number of voters goes to infinity, obtained using the Monte Carlo simulation technique in M. Harrison Trainor, "An Analysis of Random Elections with Large Numbers of Voters," arXiv:2009.02979.