# Computational Game Theory in Julia

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Lecture 1

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# Plan

- A brief introduction to game theory: Games in strategic form, Symmetric games, Nash equilibrium, Correlated equilibrium
- A brief Introduction to the Julia programming language
- The package GameTheory.jl
- Agent based modeling in Julia: Agents.jl

# The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

https://epacuit-guessavg.streamlit.app/?round=1

# The Guessing Game, again

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

# The Guessing Game, again

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

https://epacuit-guessavg.streamlit.app/?round=2

# The Guessing Game

Guess a number between 1 & 100.
The closest to 2/3 of the average wins.

# The Guessing Game

Guess a number between 1 & 100.
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What number should you guess?

# The Guessing Game

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What number should you guess? 100

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What number should you guess? ~~100~~, 99

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What number should you guess? ~~100~~, ~~99~~, . . . , ~~67~~, . . . , 2, 1

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What number should you guess? ~~100~~, ~~99~~, . . . , ~~67~~, . . . , ~~2~~, (1)

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Suppose that you are randomly paired with another person from class. What number would you write down?

# Decision Problems

Economists distinguish between choice under:

- *certainty*: highly confident about the relationship between actions and outcomes
- *risk*: clear sense of possibilities and their likelihoods
- *uncertainty*: the relationship between actions and outcomes is so imprecise that it is not possible to assign likelihoods

# Game Situations

1. a group of *self-interested* agents (players) involved in some interdependent decision problem

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Bob

$L$ $R$

1 0

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# Game Situations

| | Bob | |
|---|---|---|
| | $L$ | $R$ |
| | 1 | 0 |
| | 0 | 1 |

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# Game Situations

| | | Bob | |
| --- | --- | --- | --- |
| | | $L$ | $R$ |
| Ann | $U$ | 1 1 | 0 0 |
| | $D$ | 0 0 | 1 1 |

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# Just Enough Game Theory

A **game** is a mathematical model of a strategic interaction that includes

- the actions the players *can* take
- the players' interests (i.e., preferences),
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*It does not specify the actions that the players do take.*

# Games

| | | *B* | |
|---|---|---|---|
| | | l | r |
| *A* | u | 3, 1 | 0, 0 |
| | d | 0, 0 | 1, 3 |

# Games

| | | B | |
|---|---|---|---|
| | | l | r |
| A | u | 3, 1 | 0, 0 |
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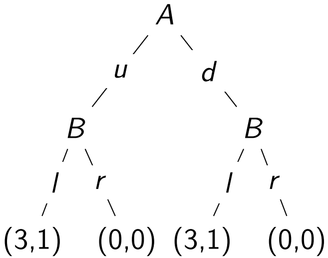

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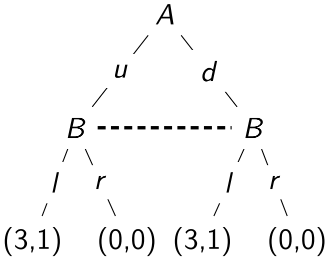

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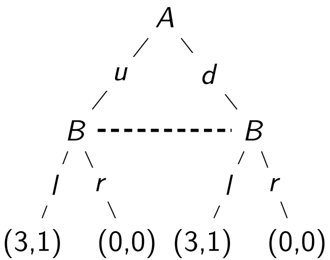


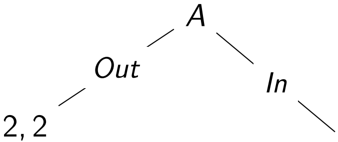


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|---|---|---|
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# From Decisions to Games, II

"*[T]he* fundamental insight of game theory [is] that a rational player must take into account that the players reason about each other in deciding how to play."

R. Aumann and J. Dreze. *Rational Expectations in Games*. American Economic Review, 98, pp. 72-86, 2008.

# Solution Concept

A **solution concept** is a systematic description of the outcomes that may emerge in a family of games.

This is the starting point for most of game theory and includes many variants: Nash equilibrium, backwards induction, or iterated dominance of various kinds.

These are usually thought of as the embodiment of “rational behavior” in some way and used to analyze game situations.

# Prisoner's Dilemma

Bob

| Ann | $c$ | $d$ |
|---|---|---|
| $c$ | 3,3 | 0,4 |
| $d$ | 4,0 | 1,1 |

# Symmetric Games

| | | Player 2 | |
|---|---|---|---|
| | | $c$ | $d$ |
| Player 1 | $c$ | $R, R$ | $S, T$ |
| | $d$ | $T, S$ | $P, P$ |

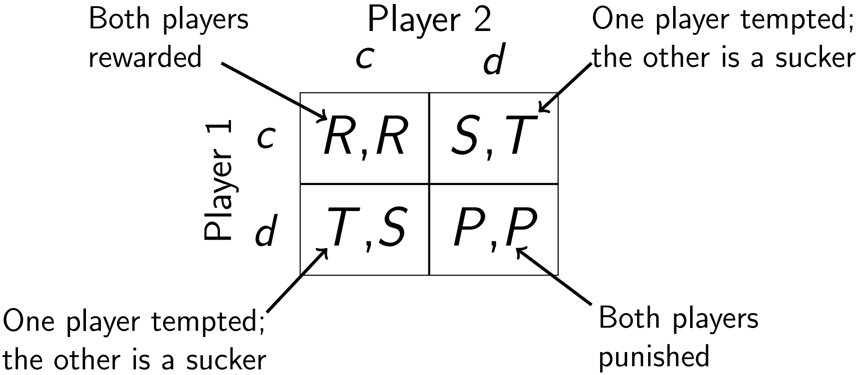
Both players rewarded ($R, R$)

One player tempted; the other is a sucker ($S, T$)

One player tempted; the other is a sucker ($T, S$)

Both players punished ($P, P$)

# Symmetric Games




Symmetric games are classified in terms of the relationship between $R$ (reward), $T$ (temptation), $S$ (sucker) and $P$ (punishment):

# Prisoner's Dilemma

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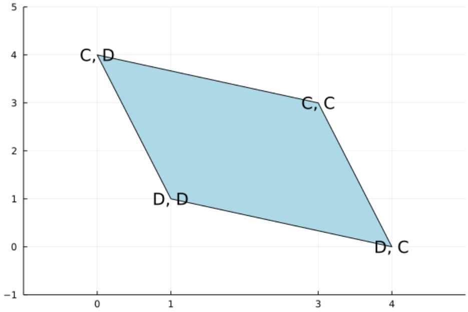



If $T > R > P > S$, then the game is a Prisoner's Dilemma.

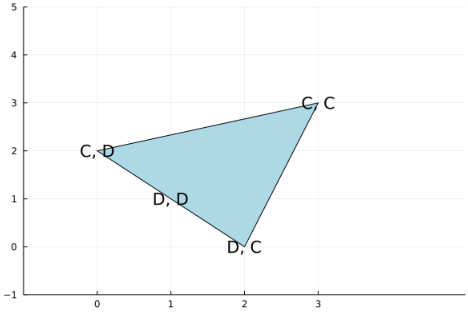
$d$ strictly dominates $c$

$(c, c)$ Pareto dominates $(d, d)$

$(d, d)$ is the unique Nash equilibrium

# Stag Hunt

| Ann \ Bob | $c$ | $d$ |
|---|---|---|
| $c$ | 3,3 | 0,2 |
| $d$ | 2,0 | 1,1 |




If $R > T$ and $P > S$, then the game is called Stag Hunt.

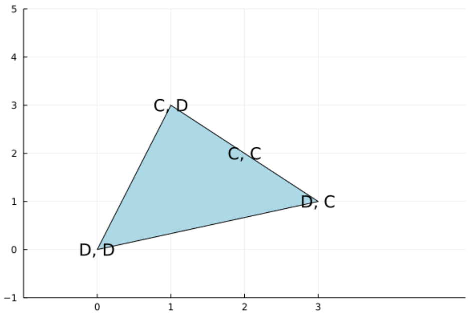
$d$ is a less "risky" option than $c$

$(c, c)$ Pareto dominates $(d, d)$

$(c, c)$ and $(d, d)$ are both Nash equilibria

# Chicken

| Ann \ Bob | $c$ | $d$ |
|---|---|---|
| $c$ | 2,2 | 1,3 |
| $d$ | 3,1 | 0,0 |




If $T > R$ and $S > P$, then the game is called Chicken (or Hawk-Dove).

$c$ is a less "risky" option than $d$

$(c, c)$ Pareto dominates $(d, d)$

$(c, d)$ and $(d, c)$ are both Nash equilibria

# Game in Normal Form

A **game in normal form** is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where:

- $N$ is a finite set of players.
- For each $i \in N$, $S_i$ is a (finite) set of actions, or strategies, for player $i$.
- For each $i \in N$, $u_i : \Pi_{i \in N} S_i \to \mathbb{R}$

# Notation

Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game in strategic form.

- For $s \in \Pi_{i \in N} S_i$, $s_i$ is the $i$th component of $s$ and $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$ is the tuple of all strategies except $s_i$
- For $i \in N$, let $S = \Pi_{i \in N} S_i$ be the set of **strategy profiles**, also called the outcomes of $G$.
- For $i \in N$, let $S_{-i} = \Pi_{j \in N, j \neq i} S_j$.

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- For $i \in N$, let $S = \Pi_{i \in N} S_i$ be the set of **strategy profiles**, also called the outcomes of $G$.
- For $i \in N$, let $S_{-i} = \Pi_{j \in N, j \neq i} S_j$.
- For a set $X$, let $\Delta(X)$ be the set of probability measures on $X$.
- $m \in \Delta(S_i)$ is called a **mixed strategy** for player $i$.
- A mixed strategy profile is an element of $\Pi_{i \in N} \Delta(S_i)$.

# Expected Utility, Best Response

Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game in strategic form.

For $a \in S_i$ and $p \in \Delta(S_{-i})$ the **expected utility of** $a$ **with respect to** $p$ is

$$EU_i(a, p) = \sum_{t \in S_{-i}} p(t) u_i(a, t)$$

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For $X \subseteq \Delta(S_{-i})$, the **best response set for player $i$**, $BR_i : X \to \wp(S_i)$, is defined as follows: for $p \in X$,

$$BR_i(p) = \{a \mid a \in S_i \text{ and } \forall a' \in S_i : EU_i(a, p) \geqslant EU_i(a', p)\}$$

Identify $S_{-i}$ with the set $\{p \mid p \in \Delta(S_{-i}), p(s) = 1 \text{ for some } s \in S_{-i}\}$,

A strategy profile $s \in \Pi_{i \in N} S_i$ is a (pure strategy) **Nash equilibrium** provided that for all $i \in N$, $s_i \in BR_i(s_{-i})$

# Mixed Extension

Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game in strategic form.

The **mixed extension of** $G$ is the tuple $\langle N, (\Delta(S_i))_{i \in N}, (U_i)_{i \in N} \rangle$, where for $m \in \Pi_{i \in N} \Delta(S_i)$

$$U_i(m) = \sum_{s \in S} u_i(s) \Pi_{i \in N} m_i(s_i)$$

A **mixed strategy Nash equilibrium** in $G$ is a tuple $m \in \Pi_{i \in N} \Delta(S_i)$ that is a Nash equilibrium in the mixed extension of $G$.