

An Axiomatic Characterization of Split Cycle

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Abstract

A number of rules for resolving majority cycles in elections have been proposed in the literature. Recently, Holliday and Pacuit (Journal of Theoretical Politics 33 (2021) 475-524) axiomatically characterized one such cycle-resolving rule, dubbed Split Cycle: in each majority cycle, discard the majority preferences with the smallest majority margin. They showed that any rule satisfying five standard axioms, plus a weakening of Arrow’s Independence of Irrelevant Alternatives (IIA) called Coherent IIA, is refined by Split Cycle. In this paper, we go further and show that Split Cycle is the *only* rule satisfying the axioms of Holliday and Pacuit together with two additional axioms: Coherent Defeat and Positive Involvement in Defeat. Coherent Defeat states that any majority preference not occurring in a cycle is retained, while Positive Involvement in Defeat is closely related to the well-known axiom of Positive Involvement (as in J. Pérez, Social Choice and Welfare 18 (2001) 601-616). We characterize Split Cycle not only as a collective choice rule but also as a social choice correspondence, over both profiles of linear ballots and profiles of ballots allowing ties.

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1 Introduction

The possibility of cycles in the majority relation of an election—wherein for candidates c_1, \dots, c_n , a majority of voters prefer c_1 to c_2 , a majority prefer c_2 to c_3 , and so on, while a majority prefer c_n to c_1 —has been taken to show that “majority rule is fatally flawed by an internal inconsistency” (Wolff 1970, p. 59). Yet voting theorists have studied many collective choice rules based on pairwise majority comparisons of candidates designed to resolve majority cycles. One prominent family of such rules resolves cycles by paying attention to the size of majority victories, e.g., as measured by the majority *margin* between candidates x and y , defined as the number of voters who prefer x to y minus the number who prefer y to x . Examples include the Ranked Pairs (Tideman 1987; Zavist and Tideman 1989), River (Heitzig 2004b), Beat Path (Schulze 2011, 2022), Weighted Covering (Dutta and Laslier 1999; Pérez-Fernández and De Baets 2018), and Split Cycle (Holliday and Pacuit 2021a, Forthcoming) rules. For instance, according to Split Cycle,¹ majority cycles are resolved as follows:

1. For each majority cycle, identify the pairwise majority victories with the smallest margin in that cycle.
2. After completing step 1 for all cycles, discard the identified victories. All remaining victories count as *defeats* of the losing candidate.

The result is a binary relation of defeat between candidates that contains no cycles. Faced with such a rule for resolving cycles, the question becomes: why this rule and not something else? The question can be partially answered by identifying axioms that distinguish between known rules. A deeper answer comes from a complete *axiomatic characterization* of a rule as the unique rule satisfying some list of natural axioms. For example, such axiomatic characterizations exist for the collective choice rules that rank candidates by Copeland score (Rubinstein 1980) and Borda scores (Nitzan and Rubinstein 1981; Mihara 2017).²

Recently Holliday and Pacuit (2021a) have characterized the Split Cycle rule using six axioms, five of which are standard (see Section 3.1 below) and the sixth of which is a weakening of Arrow’s (1963) axiom of Independence of Irrelevant Alternatives (IIA). They call their new axiom Coherent IIA. Recall that IIA

¹Eppley’s (2000) “Beatpath Criterion Method” can be defined in the same way but using *winning votes* instead of *margins* as the measure of strength of majority preference. This distinction does not matter for our axiomatization in the context of linear ballots, but it does matter for our axiomatization in the context of ballots allowing ties (see Section 7). See Ding et al. 2022 for axioms that force a voting method based on head-to-head majority comparisons to measure the strength of majority preference in terms of margins.

²A conjectured axiomatization of Ranked Pairs can be found in Tideman 1987.

states that for any two profiles \mathbf{P} and \mathbf{P}' of voter preferences, if \mathbf{P} and \mathbf{P}' are the same with respect to how each voters ranks x vs. y , then if x defeats y in \mathbf{P} , x must also defeat y in \mathbf{P}' . The motivation for weakening IIA to Coherent IIA can be seen in a simple example. In the profile \mathbf{P} in Figure 1, displayed alongside its corresponding margin graph, arguably any sensible collective choice rule should judge that candidate a defeats candidate b . However, in the profile \mathbf{P}' in Figure 1, no sensible collective choice rule—technically, no anonymous, neutral, and acyclic collective choice rule—can judge that a defeats b . This is a counterexample to IIA. Holliday and Pacuit argue that the “Fallacy of IIA” is to ignore how the context of a full election can force us to suspend judgment on some relations of defeat that we could coherently accept in a different context. However, they accept the core intuition behind IIA when contextual incoherence does not interfere, leading to their axiom of Coherent IIA, which can be stated informally as follows:

- Coherent IIA (informally): if \mathbf{P} and \mathbf{P}' are the same with respect to how each voters ranks x vs. y , and \mathbf{P}' is not more incoherent than \mathbf{P} with respect to x and y , then if x defeats y in \mathbf{P} , x must also defeat y in \mathbf{P}' .

More formally, they use the following sufficient condition for \mathbf{P}' to be no more incoherent than \mathbf{P} with respect to x and y : the margin graph of \mathbf{P}' is obtained from that of \mathbf{P} by deleting zero or more candidates other than x or y or deleting or reducing the margins on zero or more edges not involving x or y . Holliday and Pacuit then prove that Split Cycle is the most resolute collective choice rule³ satisfying the five standard axioms plus Coherent IIA. Here “most resolute” means that for any other rule f that satisfies the six axioms, if x defeats y according to f , then x defeats y according to Split Cycle.

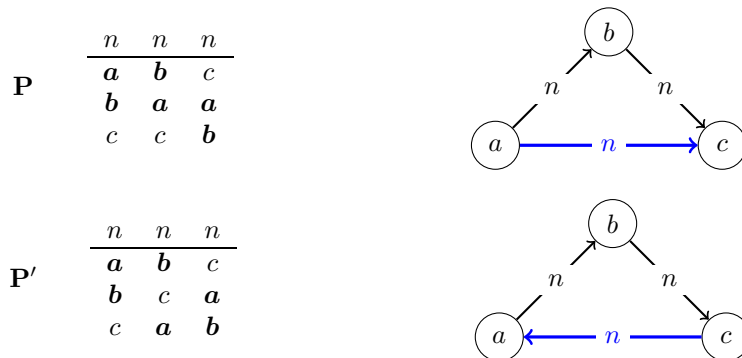


Figure 1: Profiles (left) and their margin graphs (right) illustrating the “Fallacy of IIA.”

Holliday and Pacuit’s theorem may be viewed as characterizing Split Cycle as the unique collective choice rule satisfying their six axioms plus a seventh axiom stating that the rule should be the most resolute rule satisfying the first six axioms. In a sense, however, this characterization using the notion of resoluteness is only half of a characterization of Split Cycle.⁴ We would like axioms on a collective choice rule such that for any rule f satisfying the axioms, x defeats y according to f if and only if x defeats y according to Split Cycle. Such an axiomatic characterization is the main result of the present paper.

Our new characterization of Split Cycle involves two natural axioms:

³Technically, they characterize Split Cycle as what they call a *variable-election* collective choice rule, whose domain contains profiles with different sets of candidates and voters (see Section 2.1 below). Until the end of this section, we use ‘collective choice rule’ to refer to the variable-election variety.

⁴As Holliday and Pacuit (2021a, p. 501) write, “A natural next step would be to obtain another axiomatic characterization of Split Cycle as the only VCCR satisfying some axioms without reference to resoluteness.”

- Coherent Defeat: if a majority of voters prefer x to y , and there is no majority cycle involving x and y , then x *defeats* y .
- Positive Involvement in Defeat: if y does *not* defeat x in profile \mathbf{P} , and \mathbf{P}' is obtained from \mathbf{P} by adding one new voter who ranks x above y , then y still does not defeat x in \mathbf{P}' .

Coherent Defeat is a point of common ground between Split Cycle, Ranked Pairs, River, Beat Path, and Weighted Covering: in the absence of cyclic incoherence, majority preference is sufficient for defeat. We will discuss other ways of motivating Coherent Defeat below, drawing on [Heitzig 2002](#).⁵ As for Positive Involvement in Defeat, perhaps surprisingly, many collective choice rules violate this axiom, including Ranked Pairs, River, Beat Path, and Covering (see [Duggan 2013](#) for many versions) viewed as collective choice rules. As we will explain, the name is borrowed from the related axiom of “Positive Involvement” (see [Saari 1995](#), [Pérez 2001](#), [Holliday and Pacuit 2021b](#)). Our main result in this paper is that Split Cycle is the unique collective choice rule satisfying the five standard axioms, Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat. The proof is graph-theoretic, using reasoning about minimal cuts in graphs.

So far we have discussed Split Cycle as a *collective choice rule* that outputs for a given profile a binary relation of defeat on the set of candidates—or more precisely, what we call a *variable-election* collective choice rule (VCCR), whose domain includes profiles with different set of candidates and voters. In this paper, we also characterize the associated variable-election *social choice correspondence* (VSCC) that outputs for a given profile the set of undefeated candidates. This involves a translation from VSCCs to VCCRs which induces a translation in the reverse direction from IIA (resp. Coherent IIA) for VCCRs to IIA (resp. Coherent IIA) for VSCCs. The additional axioms of Coherent Defeat and Positive Involvement in Defeat also have natural analogues for VSCCs. Interestingly, the VSCC analogue of the VCCR axiom of Positive Involvement in Defeat, which we call Tolerant Positive Involvement, strengthens the axiom of Positive Involvement from the prior literature mentioned above. We prove that the Split Cycle VSCC is the unique VSCC satisfying five standard axioms, Coherent IIA, Coherent Defeat, and Tolerant Positive Involvement.

The use in our axiomatizations of Coherent IIA, on the one hand, and variants of Positive Involvement, on the other, nicely matches the two main normative points made about the Split Cycle VSCC in [Holliday and Pacuit Forthcoming](#), which stresses Split Cycle’s ability to handle both (i) the “Problem of Spoilers” and (ii) the “Strong No Show Paradox” ([Pérez 2001](#)). In essence, Coherent IIA is a strengthening (assuming other uncontroversial axioms) of anti-spoiler axioms that mitigate (i), while Tolerant Positive Involvement is a strengthening of the Positive Involvement axiom that prevents (ii).

The rest of the paper is organized as follows. In Section 2, we formally define VCCRs and VSCCs in general and the Split Cycle VCCR and VSCC in particular. In Section 3, we review the axioms from [Holliday and Pacuit 2021a](#) (Sections 3.1-3.2) and discuss in more depth the additional axioms of Coherent Defeat (Section 3.3) and Positive Involvement in Defeat (Section 3.4). Using these axioms, we prove our characterization result for the Split Cycle VCCR in Section 4. We then turn to the Split Cycle VSCC. In Section 5, we define analogues for VSCCs of the axioms for VCCRs in Section 3. Using these analogous axioms, we prove our characterization result for the Split Cycle VSCC in Section 6.⁶ In Section 7, we adapt our characterization results to the setting in which ties are permitted in voters’ ballots. Finally, we conclude with some suggestions for related axiomatization problems in voting theory in Section 8.

⁵Heitzig’s (2004a) definition of an “immune” candidate is the same as that of a Split Cycle winner if we replace “stronger” with “at least as strong” in his definition.

⁶The Appendix proves that in this characterization of the Split Cycle VSCC, we need the axiom of Tolerant Positive Involvement instead of the weaker axiom of Positive Involvement.

2 Preliminaries

2.1 VCCRs and VSCCs

Fix infinite sets \mathcal{V} and \mathcal{X} of *voters* and *candidates*, respectively. We begin by assuming that voters submit linear orders of the candidates. For $X \subseteq \mathcal{X}$, let $\mathcal{L}(X)$ be the set of all strict linear orders on X . In Section 7, we will adapt our results to the setting in which voters may submit strict weak orders, allowing ties.

Definition 2.1. A (*linear*) *profile* is a function $\mathbf{P} : V \rightarrow \mathcal{L}(X)$ for some nonempty finite $V \subset \mathcal{V}$ and nonempty finite $X \subset \mathcal{X}$, which we denote by $V(\mathbf{P})$ (called the set of *voters in* \mathbf{P}) and $X(\mathbf{P})$ (called the set of *candidates in* \mathbf{P}), respectively. We call $\mathbf{P}(i)$ voter i 's *ballot*. When \mathbf{P} is clear from context, we write ' $x \succ_i y$ ' for $(x, y) \in \mathbf{P}(i)$, and when i is also clear from context, we write $x \succ y$ for $x \succ_i y$.

Given a set $Y \subseteq X(\mathbf{P})$ of candidates, the restriction $\mathbf{P}|_Y$ of \mathbf{P} to Y is the profile such that for each $i \in V(\mathbf{P})$, $\mathbf{P}|_Y(i)$ is the restriction of the ballot $\mathbf{P}(i)$ to Y .

Sometimes we will display profiles in their *anonymized* form, as in Figure 1, meaning that we only display how many voters have each type of ballot, rather than indicating which voters have which ballots.

Our main objects of study are (i) functions that assign to each profile a binary relation on the set of candidates and (ii) functions that assign to each profile a subset of the candidates.

Definition 2.2. A *variable-election collective choice rule* (VCCR) is a function f on the domain of all profiles such that for any profile \mathbf{P} , $f(\mathbf{P})$ is an asymmetric binary relation on $X(\mathbf{P})$, which we call *the defeat relation for* \mathbf{P} *according to* f . For $x, y \in X(\mathbf{P})$, we say that x *defeats* y *in* \mathbf{P} *according to* f when $(x, y) \in f(\mathbf{P})$.

Definition 2.3. A *variable-election social choice correspondence* (VSCC) is a function F on the domain of all profiles such that for any profile \mathbf{P} , we have $\emptyset \neq F(\mathbf{P}) \subseteq X(\mathbf{P})$.

The point of the adjective 'variable-election' is that different profiles in the domain of a function may have different sets of voters and candidates. By contrast, in the literature, collective choice rules (CCRs) and social choice correspondences (SCCs) are often defined such that all profiles in the domain of a given function have the same sets of voters and candidates, respectively.

A VCCR f is said to be *acyclic* if for any profile \mathbf{P} , the binary relation $f(\mathbf{P})$ is acyclic. Such a VCCR induces a VSCC \bar{f} that returns for a given profile \mathbf{P} the maximal elements of $f(\mathbf{P})$.

Lemma 2.4. Given any acyclic VCCR f , the function \bar{f} on the set of profiles defined by

$$\bar{f}(\mathbf{P}) = \{x \in X(\mathbf{P}) \mid \text{there is no } y \in X(\mathbf{P}) : y \text{ defeats } x \text{ in } \mathbf{P} \text{ according to } f\}$$

is a VSCC, as $\emptyset \neq \bar{f}(\mathbf{P}) \subseteq X(\mathbf{P})$.

Definition 2.5. For any VCCR f and VSCC F , if $F = \bar{f}$, we say that F is *defeat-rationalized* by f .

Our axiomatization proofs rely heavily on comparing the resoluteness, or in other words, refinedness, of VCCRs and VSCCs, so we define these notions here.

Definition 2.6. Let f and f' be VCCRs. We say that f *refines* f' , or that f is *at least as resolute as* f' , if for any profile \mathbf{P} , $f(\mathbf{P}) \supseteq f'(\mathbf{P})$; that is, f outputs all the defeats that f' does and possibly more.

Let F and F' be VSCCs. We say that F *refines* F' , or that F is *at least as resolute as* F' , if for any profile \mathbf{P} , $F(\mathbf{P}) \subseteq F'(\mathbf{P})$; that is, F always selects a subset of the candidates selected by F' .

2.2 Split Cycle

In this section, we define the Split Cycle VCCR. First we need the notion of the margin graph of a profile.

Definition 2.7. Let \mathbf{P} be a profile and $x, y \in X(\mathbf{P})$. The *margin of x over y in \mathbf{P}* , written $\text{Margin}_{\mathbf{P}}(x, y)$, is

$$|\{i \in V(\mathbf{P}) \mid (x, y) \in \mathbf{P}(i)\}| - |\{i \in V(\mathbf{P}) \mid (y, x) \in \mathbf{P}(i)\}|.$$

The *margin graph* of \mathbf{P} , denoted $\mathcal{M}(\mathbf{P})$, is the directed graph with weighted edges whose set of nodes is $X(\mathbf{P})$ with an edge from x to y iff $\text{Margin}_{\mathbf{P}}(x, y) > 0$, the weight of which is $\text{Margin}_{\mathbf{P}}(x, y)$.

A *majority path* in \mathbf{P} is a path in $\mathcal{M}(\mathbf{P})$, i.e., a sequence $\rho = \langle x_1, x_2, \dots, x_n \rangle$ of nodes of $\mathcal{M}(\mathbf{P})$ such that for each $i = 1 \dots n - 1$ there is an edge from x_i to x_{i+1} , i.e., $\text{Margin}_{\mathbf{P}}(x_i, x_{i+1}) > 0$. Given such a majority path ρ , we define its *strength*, written $\text{Strength}_{\mathbf{P}}(\rho)$ as

$$\min\{\text{Margin}_{\mathbf{P}}(x_i, x_{i+1}) \mid i = 1 \dots n - 1\}.$$

We also call ρ a majority path *from x_1 to x_n* . A *simple majority path* is a majority path in which no candidate is repeated. When $x_n = x_1$ in ρ , we call ρ a *majority cycle*, and in this case $\text{Strength}_{\mathbf{P}}(\rho)$ is also called the *splitting number* of ρ , denoted $\text{Split}\#\mathbf{P}(\rho)$. A *simple majority cycle* is a majority cycle in which no candidate is repeated except the first and last.

There are many equivalent ways to define the Split Cycle VCCR, four of which are based on following lemma (for proofs, see [Holliday and Pacuit Forthcoming](#) and [Holliday et al. 2021](#)).

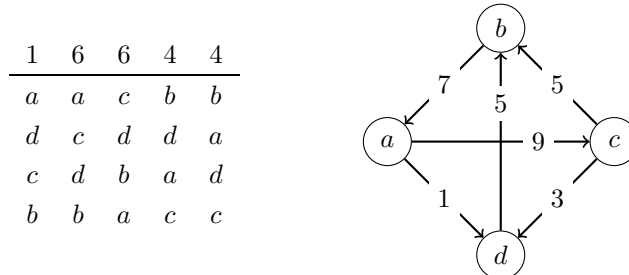
Lemma 2.8. For any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, the following are equivalent:

1. $\text{Margin}_{\mathbf{P}}(x, y) > 0$ and $\text{Margin}_{\mathbf{P}}(x, y) > \text{Split}\#\mathbf{P}(\rho)$ for every majority cycle ρ containing x and y ;
2. $\text{Margin}_{\mathbf{P}}(x, y) > 0$ and $\text{Margin}_{\mathbf{P}}(x, y) > \text{Split}\#\mathbf{P}(\rho)$ for every simple majority cycle ρ containing x and y ;
3. $\text{Margin}_{\mathbf{P}}(x, y) > 0$ and $\text{Margin}_{\mathbf{P}}(x, y) > \text{Split}\#\mathbf{P}(\rho)$ for every simple majority cycle ρ in which y directly follows x ;
4. $\text{Margin}_{\mathbf{P}}(x, y) > 0$ and $\text{Margin}_{\mathbf{P}}(x, y) > \text{Strength}_{\mathbf{P}}(\rho)$ for every simple majority path ρ from y to x .

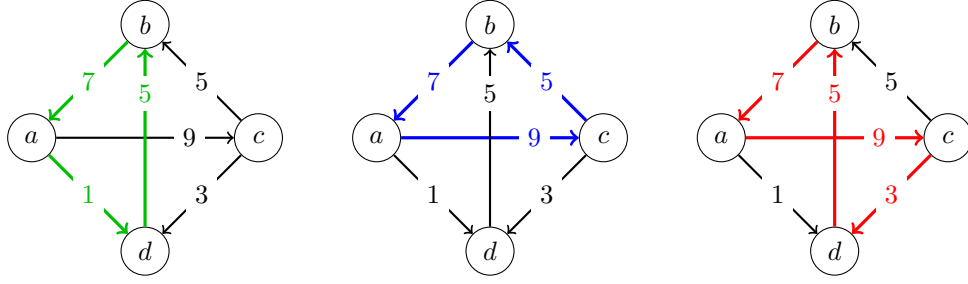
Definition 2.9. The Split Cycle VCCR, denoted sc , is defined as follows: for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, $(x, y) \in sc(\mathbf{P})$ iff the condition on x, y in Lemma 2.8 holds.

The two-step algorithm in Section 1 provides one way of calculating $sc(\mathbf{P})$.

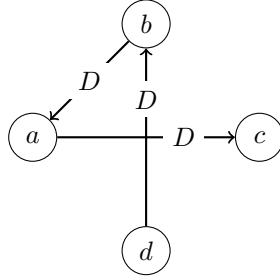
Example 2.10. Consider the following anonymized profile with its margin graph:



The simple cycles of the above margin graph are highlighted with thickened arrows below:



Thus, after removing the smallest edge in each simple cycle, what remains is the defeat graph:



The only undefeated candidate is d , and thus d is the only Split Cycle winner of the profile.

Finding all cycles may be computationally costly; for a more efficient algorithm based on the Floyd-Warshall algorithm, see the implementation of Split Cycle in the Preferential Voting Tools library as `split_cycle_faster` (<https://pref-voting.readthedocs.io>).

The definition of Split Cycle leads directly to the following observation used in our later proofs.

Definition 2.11. Let \mathcal{M} be a margin graph (the margin graph of some profile) and $k \in \mathbb{N}$. Then $\mathcal{M}|k$ is the result of keeping all and only the edges in \mathcal{M} with weight at least k .

Lemma 2.12. For any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, $(x, y) \in sc(\mathbf{P})$ iff $Margin_{\mathbf{P}}(x, y) > 0$ and there is no (simple) majority path from y to x in $\mathcal{M}(\mathbf{P})|Margin_{\mathbf{P}}(x, y)$.

Finally, since Split Cycle only cares about the margin graph of a profile (formally, if $\mathcal{M}(\mathbf{P}) = \mathcal{M}(\mathbf{P}')$, then $sc(\mathbf{P}) = sc(\mathbf{P}')$), the only way that the assumption that voters submit *linear* orders can matter is by affecting the types of margin graphs that can arise. This happens with the following parity constraint.

Lemma 2.13. If \mathbf{P} is a linear profile, then either all margins between distinct candidates are even or all margins between distinct candidates are odd.

Indeed, this parity constraint is the only consequence of assuming linear ballots.

Proposition 2.14 (Debord 1987). If \mathcal{M} is a weighted directed graph in which all edges have the same parity, and all weights are even if there are two vertices with no edge between them (representing a zero margin), then there is a linear profile \mathbf{P} such that $\mathcal{M} = \mathcal{M}(\mathbf{P})$.

In Section 7, we will drop the assumption of linear ballots and hence the parity constraint on margins.

3 Axioms on VCCRs

In this section, we present the axioms used in our characterization of the Split Cycle VCCR.

3.1 Standard axioms

First we recall what Holliday and Pacuit (2021a) consider the “standard axioms” in their characterization.

Definition 3.1. Let f be a VCCR.

1. f satisfies *Anonymity* if for any profile \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by swapping the ballots assigned to two voters, we have $f(\mathbf{P}) = f(\mathbf{P}')$, and f satisfies *Neutrality* if for any profile \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by swapping x and y in $X(\mathbf{P})$ on each voter’s ballot, we have $(x, y) \in \mathbf{P}$ iff $(y, x) \in \mathbf{P}'$.
2. f satisfies *Availability* if for any profile \mathbf{P} , $\bar{f}(\mathbf{P})$ is nonempty.
3. f satisfies *Homogeneity* (resp. *Upward Homogeneity*) if for any profile \mathbf{P} and $2\mathbf{P}$, where $2\mathbf{P}$ is the result of replacing each voter in \mathbf{P} by 2 copies of that voter, $f(\mathbf{P}) = f(2\mathbf{P})$ (resp. $f(\mathbf{P}) \subseteq f(2\mathbf{P})$).
4. f satisfies *Monotonicity* (resp. *Monotonicity for two-candidate profiles*) if for any profile (resp. two-candidate profile) \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by moving $xX(\in \mathbf{P})$ up one place in some voter’s ballot, we have $(x, y) \in f(\mathbf{P})$ only if $(x, y) \in f(\mathbf{P}')$.
5. f satisfies *Neutral Reversal* if for any profile \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by adding two voters whose ballots are converses of each other, we have $f(\mathbf{P}) = f(\mathbf{P}')$.

Axioms 1-4 are widely satisfied by VCCRs in the literature. While axioms 5 (from Saari 2003) is violated by some prominent VCCRs (e.g., the Plurality VCCR according to which x defeats y iff x receives more first-place votes than y , or the Pareto VCCR according to which x defeats y iff every voter strictly prefers x to y), it is satisfied by all VCCRs that depend only on the majority margins between candidates (including, e.g., the Borda CCR, which can be defined in terms of majority margins as in Zwicker 2016, p. 28).

3.2 Coherent IIA

The key axiom in Holliday and Pacuit 2021a is the axiom of Coherent IIA, already discussed informally in Section 1. First we recall Arrow’s (1963) IIA, of which Coherent IIA is a weakening.

Definition 3.2. A VCCR f satisfies Independence of Irrelevant Alternatives (IIA)⁷ if for any profiles \mathbf{P} and \mathbf{P}' , if x defeats y in \mathbf{P} according to f , and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in \mathbf{P}' according to f .

Coherent IIA strengthens the assumption of IIA on the relation of \mathbf{P} and \mathbf{P}' so that not only $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$ but also \mathbf{P}' is related to \mathbf{P} in such a way that \mathbf{P}' cannot be more incoherent, in terms of majority cycles and their strengths, than \mathbf{P} is with respect to x, y .

Definition 3.3. For any two profiles \mathbf{P} and \mathbf{P}' with $x, y \in X(\mathbf{P}) \cap X(\mathbf{P}')$, let

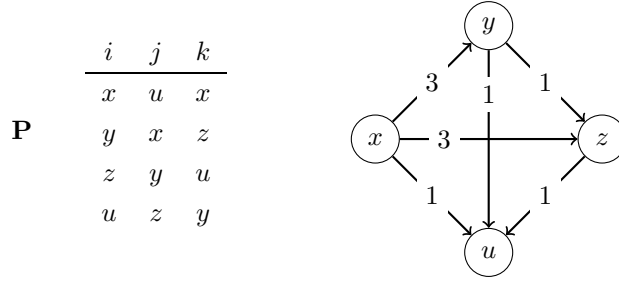
$$\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$$

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$ and $\mathcal{M}(\mathbf{P}')$ can be obtained from $\mathcal{M}(\mathbf{P})$ by deleting zero or more candidates other than x and y or deleting or reducing the margins on zero or more edges not involving x and y .

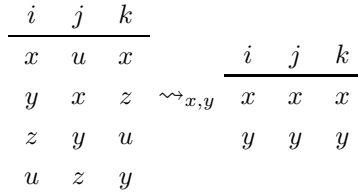
Note that trivially $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$ only if $V(\mathbf{P}') = V(\mathbf{P})$ and $X(\mathbf{P}') \subseteq X(\mathbf{P})$; that is, the profiles must have the same set of voters and \mathbf{P}' cannot have additional candidates.

⁷Holliday and Pacuit (2021a) call this principle *variable-election* IIA (VIIA), since it allows \mathbf{P} and \mathbf{P}' to have different sets of candidates, as opposed to *fixed-election* IIA (FIIA), which requires that \mathbf{P} and \mathbf{P}' have the same set of candidates.

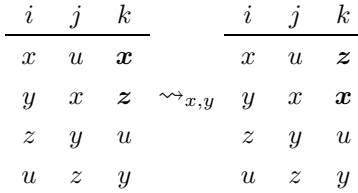
Example 3.4. Consider the following profile \mathbf{P} with three voters i, j, k and four candidates x, y, z, u , whose margin graph is shown on the right:



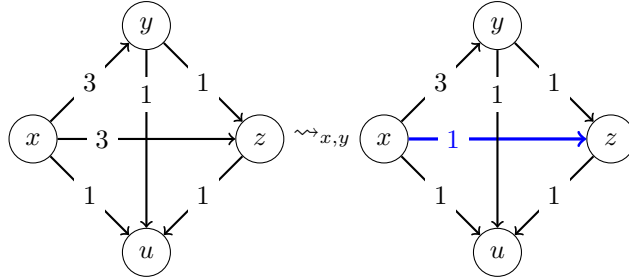
By definition, if \mathbf{Q} is the restriction of \mathbf{P} to just the candidates x and y , which is the profile where i, j , and k vote unanimously that x is better than y , then trivially $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{Q}$. That is,



If we consider only changing the relative position of one pair of candidates for one voter, then the only way to produce a different profile \mathbf{Q} such that $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{Q}$ is to switch x and z in the ballot of k . That is, we have



with their corresponding margin graphs:



Note that from left to right, no new edges are created, and the only change is that the margin of the edge from x to z is reduced from 3 to 1. The reason this swap of x and z in the ballot of k is the only allowed swap if we want a profile \mathbf{Q} such that $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{Q}$ is that (1) obviously we cannot swap x and y and (2) for any other pair of candidates, the margin between them in the original margin graph is 1 so that a swap would flip the edge. The edges are directed, so flipping edges means creating new majority edges, which is not allowed in the definition of $\rightsquigarrow_{x,y}$.

We now define Coherent IIA by replacing $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$ in the definition of IIA with $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$.

Definition 3.5. A VCCR f satisfies *Coherent IIA* if for any profiles \mathbf{P} and \mathbf{P}' , if x defeats y in \mathbf{P} according to f , and $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$, then x defeats y in \mathbf{P}' according to f .

Note a simple consequence of Coherent IIA, known as Weak IIA (cf. Baigent 1987): if $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$ and x defeats y in \mathbf{P} , then it is not the case that y defeats x in \mathbf{P}' . This follows from the fact that if x defeats y in \mathbf{P} (resp. y defeats x in \mathbf{P}'), then according to Coherent IIA, x defeats y in $\mathbf{P}_{\{x,y\}}$ (resp. y defeats x in $\mathbf{P}'_{\{x,y\}}$). For extensive discussion of Coherent IIA and its consequences, see Holliday and Pacuit 2021a, Section 4.3. Here we focus our attention on the two new axioms used in our main result.

3.3 Coherent Defeat

We come now to the first of our two new axioms.

Definition 3.6. A VCCR f satisfies *Coherent Defeat* if for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, if $\text{Margin}_{\mathbf{P}}(x, y) > 0$ and there is no majority cycle containing x and y in \mathbf{P} (or equivalently, there is no majority path from y to x), then $(x, y) \in f(\mathbf{P})$.

The key idea behind Coherent Defeat is simple: when there is no incoherence due to majority cycles, majority preference is sufficient for defeat. In other words, majority cycles are the only reason we deviate from majority preference for deciding defeat. One caveat is that we understand incoherence locally: when deciding whether x defeats y , only majority cycles involving x and y matter; thus, regardless of whether there are majority cycles involving other candidates, if there are no majority cycles involving x and y , then a majority preference for x over y is sufficient for x to defeat y . As mentioned in Section 1, a number of VCCRs (Ranked Pairs, River, Beat Path, Weighted Covering) together with Split Cycle share this core commitment and thus can all be seen as ways of resolving incoherence due to relevant majority cycles. To put the point in terms of the Advantage-Standard Model of Holliday and Kelley 2021, according to which x defeats y once the *advantage* of x over y (which depends only on how voter ranks x versus y) exceeds the *standard* for x to defeat y (which may depend on other information in the profile, including how voters rank x against non- y candidates and y against non- x candidates), Coherent Defeat follows assuming the advantage and standard functions satisfy the following weak constraints:

- the advantage of x over y is greater than 0 if x is majority preferred to y ;
- the standard for x to defeat y is 0 if there is no majority cycle involving x and y .

The second constraint is clearly necessary if we take the standard for x to defeat y to measure in some way local incoherence due to majority cycles involving x and y .

Coherent Defeat can also be viewed as a principle of “unchallenged defeat” that follows from three principles:

- for x to defeat y , it is sufficient to find one reason for x to defeat y and make sure that no reasons for x to defeat y can be challenged;
- a majority preference for x over y is a reason for x to defeat y ;
- any challenge to any reason for x to defeat y must be based on a majority preference path from y to x .

For a more concrete model, we can view the margin graph as providing *arguments* for and against propositions of the form “ x defeats y ” and their negations. There are two types of arguments. First, there is an argument for “ x defeats y ” whenever there is a majority preference for x over y . Second, when there is a majority path from x to y , there is an argument for “ y does not defeat x ”, since for each link a, b in the path we

have an argument for “ a defeats b ”, but the defeat relation must be acyclic, so taken together the steps along a majority path constitute an argument for “ y does not defeat x ”. It should be noted here that an argument based on a majority path from x to y to the conclusion “ x defeats y ” is not necessarily a good argument, since defeat relations are not obviously transitive. Now if we grant that arguments for and against candidates defeating each other can only be generated in the above two ways, then once we have an x who is majority preferred to y and there is no majority path from y to x , we have an argument for “ x defeats y ” but no counterargument to the contrary. Thus, we should accept that x defeats y . Similar ideas of treating majority preferences and paths as arguments appear in [Heitzig 2002](#), and our Coherent Defeat is essentially his Immunity to Binary Arguments (Im_A) applied to VCCRs, with A as the majority preference relation.

3.4 Positive Involvement in Defeat

Our second new axiom for VCCRs is based on the standard axiom of Positive Involvement for SCCs ([Saari 1995](#), [Pérez 2001](#), [Holliday and Pacuit 2021b](#)). Positive Involvement can be stated for VSCCs as follows.

Definition 3.7. A VSCC F satisfies *Positive Involvement* if for any profile \mathbf{P} , if $y \in F(\mathbf{P})$ and \mathbf{P}' is obtained from \mathbf{P} by adding one new voter who ranks y in first place, then $y \in F(\mathbf{P}')$.

Thus, Positive Involvement says that a candidate y ’s winning should be preserved under the addition of a voter who gives y maximum support by ranking y at the top of her ballot. As Perez (2001, p. 605) remarks, it can be seen as “the minimum to require concerning the coherence in the winning set when new voters are added.” To generalize the axiom to VCCRs, we need to ask what is the “minimum to require concerning the coherence” in the defeat relation when new voters are added.

First, there is a trivial way to generalize Positive Involvement to VCCRs.

Definition 3.8. An (acyclic) VCCR f satisfies *Positive Involvement* if the VSCC \bar{f} (recall Lemma 2.4) satisfies Positive Involvement: for any profile \mathbf{P} , candidate y undefeated in \mathbf{P} according to f , and \mathbf{P}' obtained from \mathbf{P} by adding one new voter who ranks y in first place, y is still undefeated in \mathbf{P}' according to f .

The problem with this generalization is not that it is unreasonable for a VCCR to satisfy it but rather that we find it reasonable to ask for more. Given a VCCR f , we have information not only about who is defeated or undefeated but also about who is defeated *by whom*. Thus, there is the question of how the defeat relation between two candidates x and y should react to the addition of a voter. A natural answer is this: adding a voter who ranks y above x should not lead to y ’s defeat by x .

Definition 3.9. A VCCR f satisfies *Positive Involvement in Defeat* if for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, if y is not defeated by x in \mathbf{P} according to f , and \mathbf{P}' is obtained from \mathbf{P} by adding one new voter who ranks y above x , then y is not defeated by x in \mathbf{P}' according to f .

This is of course not the only possible answer. For example, we could require not only that x is ranked above y in the new ballot but also that x is ranked at the top.

Definition 3.10. A VCCR f satisfies *First-place Involvement in Defeat* if for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, if y is not defeated by x in \mathbf{P} according to f , and \mathbf{P}' is obtained from \mathbf{P} by adding one new voter who ranks y in first place, then y is not defeated by x in \mathbf{P}' according to f .

Since the requirement on x and y is stronger in First-place Involvement in Defeat, it is entailed by Positive Involvement in Defeat: any VCCR satisfying Positive Involvement in Defeat also satisfies First-place Involvement in Defeat. Moreover, it is easy to see that First-place Involvement in Defeat entails Positive Involvement for VCCRs.

It turns out, as we show in the Appendix, that the stronger axiom of Positive Involvement in Defeat is necessary for our characterization of Split Cycle (compared to First-place Involvement in Defeat and hence Positive Involvement), fixing the other axioms we use. However, we do not see this as a problem for Split Cycle as an appealing VCCR, since we find the intuitive appeal of Positive Involvement in Defeat rather clear and perhaps even clearer than First-place Involvement in Defeat. If the only change to a profile in which y is already undefeated by x is the addition of a ballot ranking y above x , thereby lending support to y against x , then there is no reason to now say that x defeats y .

As an example of working with Positive Involvement in Defeat, we mention that Weighted Covering (Dutta and Laslier 1999, Pérez-Fernández and De Baets 2018) satisfies it.

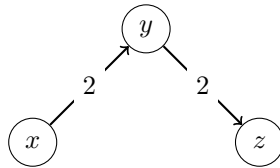
Definition 3.11. The Weighted Covering VCCR, denoted wc , is defined as follows: for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, $(x, y) \in wc(\mathbf{P})$ iff $Margin_{\mathbf{P}}(x, y) > 0$ and for all $z \in X(\mathbf{P})$, $Margin_{\mathbf{P}}(x, z) \geq Margin_{\mathbf{P}}(y, z)$.

Proposition 3.12. Weighted Covering satisfies Positive Involvement in Defeat.

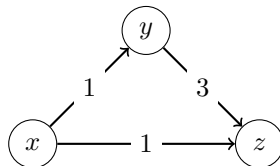
Proof. Let \mathbf{P} be any profile, $x, y \in X(\mathbf{P})$, and suppose y is undefeated by x . Then either (1) $Margin_{\mathbf{P}}(x, y) \leq 0$ or (2) there is $z \in X(\mathbf{P})$ such that $Margin_{\mathbf{P}}(x, z) < Margin_{\mathbf{P}}(y, z)$.

Now let \mathbf{P}' be the result of adding to \mathbf{P} one ballot where y is above x . In case (1), $Margin_{\mathbf{P}'}(x, y) < 0$ since the new ballot decreases the margin from x to y by 1, and thus y is still undefeated by x . In case (2), since in the new ballot, $y \succ x$, there are only three possible places for z relative to y and x : $z \succ y \succ x$, $y \succ z \succ x$, and $y \succ x \succ z$. In the first and the third case, $\Delta_{x,z} = Margin_{\mathbf{P}'}(x, z) - Margin_{\mathbf{P}}(x, z)$ and $\Delta_{y,z} = Margin_{\mathbf{P}'}(y, z) - Margin_{\mathbf{P}}(y, z)$ are the same (-1 and 1 , respectively), and in the second case, $\Delta_{x,z} = -1 < \Delta_{y,z} = 1$. Thus, in all three cases, since $\Delta_{x,z} \leq \Delta_{y,z}$, we still have $Margin_{\mathbf{P}'}(x, z) < Margin_{\mathbf{P}'}(y, z)$, and y is still undefeated by x in \mathbf{P}' . \square

On the other hand, the many non-weighted versions of Covering (see Gillies 1959, Fishburn 1977, Miller 1980) do not satisfy Positive Involvement in Defeat. For example, for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, we say that x defeats y according to the Right Covering VCCR when $Margin_{\mathbf{P}}(x, y) > 0$ and for all $z \in X(\mathbf{P})$, if $Margin_{\mathbf{P}}(y, z) > 0$ then $Margin_{\mathbf{P}}(x, z) > 0$. Then to see that the Right Covering VCCR fails Positive Involvement in Defeat, consider any profile \mathbf{P} whose margin graph is:



By definition, x does not defeat y in \mathbf{P} according to Right Covering because of z . However, once we add a ballot $y \succ x \succ z$ to \mathbf{P} , the margin graph becomes



and x now defeats y according to Right Covering, since x now also has a positive margin over z .

One may naturally wonder why we focus solely on Positive Involvement, given that there is also the axiom of Negative Involvement in the literature (again see Pérez 2001).

Definition 3.13. A VSCC F satisfies *Negative Involvement* if for any profile \mathbf{P} , if $y \notin F(\mathbf{P})$ and \mathbf{P}' is obtained by adding one new voter who ranks y in last place, then $y \notin F(\mathbf{P}')$.

An acyclic VCCR f satisfies *Negative Involvement* if \bar{f} satisfies Negative Involvement. We say that a VCCR f satisfies *Negative Involvement in Defeat* if for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, if $(x, y) \in f(\mathbf{P})$, and \mathbf{P}' is obtained from \mathbf{P} by adding one new voter who ranks x above y , then $(x, y) \in f(\mathbf{P}')$.

The reason we can focus solely on Positive Involvement is given by the following proposition.

Proposition 3.14. An acyclic VCCR (resp. VCCR) f satisfying Neutral Reversal satisfies Positive Involvement (resp. Positive Involvement in Defeat) iff it satisfies Negative Involvement (resp. Negative Involvement in Defeat).

Proof. Let f be an acyclic VCCR f satisfying Neutral Reversal. Suppose f fails Positive Involvement. Then we have a profile \mathbf{P} , $x \in \bar{f}(\mathbf{P})$, and \mathbf{P}' that adds to \mathbf{P} a ballot L ranking x in first place and yet $x \notin \bar{f}(\mathbf{P}')$. Now let \mathbf{P}'' be any profile that further adds to \mathbf{P}' the converse of the ballot L , which ranks x in last place. Then \mathbf{P}'' is the result of adding to \mathbf{P} a pair of ballots in full reversal. So by Neutral Reversal, $x \in \bar{f}(\mathbf{P}')$. But then the pair \mathbf{P}' and \mathbf{P}'' witness the failure of Negative Involvement for \bar{f} and also f . Clearly, the same strategy of adding the converse of L works for the other direction and also works for showing that a VCCR f satisfies Positive Involvement in Defeat iff it satisfies Negative Involvement in Defeat. \square

Thus, as long as we are focusing on VCCRs satisfying Neutral Reversal, there is no loss of generality in focusing just on the positive axioms.

4 Characterization of the Split Cycle VCCR

We are now ready to prove our first main result, an axiomatic characterization of the Split Cycle VCCR.

Theorem 4.1. The Split Cycle VCCR is the unique VCCR satisfying Anonymity, Neutrality, Availability, (Upward) Homogeneity, Monotonicity (for two-candidate profiles), Neutral Reversal, Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat.

Holliday and Pacuit have already proved half of what we need.

Theorem 4.2 (Holliday and Pacuit 2021a). If f is a VCCR satisfying Anonymity, Neutrality, Availability, (Upward) Homogeneity, Monotonicity (for two-candidate profiles), Neutral Reversal, and Coherent IIA, then the Split Cycle VCCR is at least as resolute as f : for any profile \mathbf{P} , $sc(\mathbf{P}) \supseteq f(\mathbf{P})$.

It remains to prove that the axioms in Theorem 4.1 force f to be at least as resolute as Split Cycle, in which case $f = sc$. In fact, we will prove that just Coherent Defeat and Positive Involvement in Defeat together do so. First we recall the following well-known extension lemma.

Lemma 4.3. Any acyclic binary relation can be extended to a strict linear order.

Proof. Take the transitive closure and apply the Szpilrajn extension theorem (Szpilrajn 1930). \square

Next comes the key lemma. The formulation is slightly cumbersome, and it actually proves more than we need for Theorem 4.1. However, it is precisely what we need for characterizing Split Cycle as a VSCC in Section 6.

Lemma 4.4. For any profile \mathbf{P} and $(x, y) \in sc(\mathbf{P})$, if $Margin_{\mathbf{P}}(x, y) > 2$, then there is a ballot $L \in \mathcal{L}(X(\mathbf{P}))$ such that

- for any $z \in X(\mathbf{P}) \setminus \{y\}$ with $Margin_{\mathbf{P}}(y, z) \leq 0$, we have $(y, z) \in L$ (in particular, $(y, x) \in L$), and
- $(x, y) \in sc(\mathbf{P} + L)$.

In other words, if x defeats y according to Split Cycle by a sufficient margin (more than 2), then x can still defeat y after the addition of a specifically chosen ballot in which y is ranked very high in the sense that y is ranked above all candidates that it does not beat head-to-head, including x since y is defeated by x according to Split Cycle.

For the proof of Lemma 4.4, recall that in a graph containing vertices y and x , a *cut from y to x* is a set of edges such that every path from y to x contains an edge from the set. Equivalently, a cut from y to x is a set of edges whose removal results in the loss of reachability of x from y . The main idea of the proof is illustrated in Figure 2.

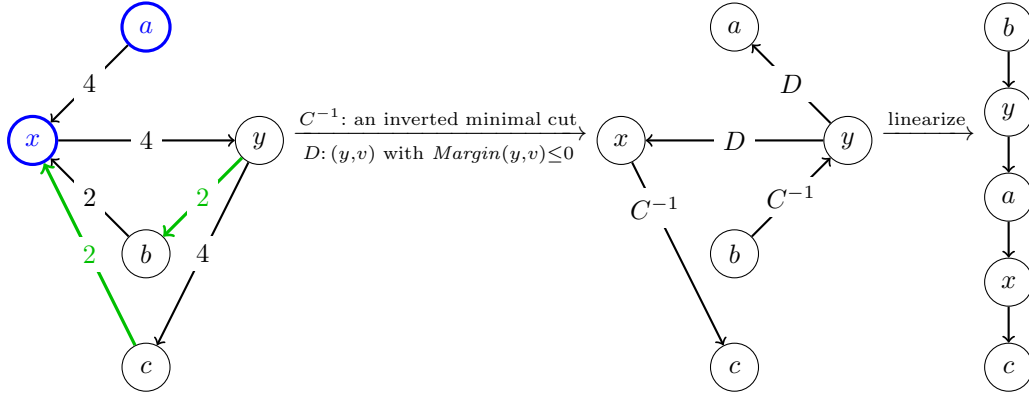


Figure 2: An example of generating a linear ballot L as in Lemma 4.4. Given x, y , we first identify (a) a minimal cut in the margin graph using only edges with margin smaller than $Margin(x, y)$ and (b) the nodes that are not reachable from y in one step (all highlighted on the left). Then we form the graph with the cut inverted and with arrows from y to those thickened nodes. This graph must be acyclic as a minimal cut must be bipartite. Then we linearize this acyclic graph to obtain L .

Proof. Let \mathcal{M} be the margin graph of \mathbf{P} and $k = Margin_{\mathbf{P}}(x, y)$. Since $(x, y) \in sc(\mathbf{P})$, by Lemma 2.12 we know that x is not reachable from y in $\mathcal{M} \setminus k$. This means that the set of edges in \mathcal{M} of weight at most $k - 1$ is a cut from y to x . Thus, there is a minimal cut C (minimal in the subset ordering) from y to x consisting only of edges of weight at most $k - 1$, since the graph is finite. By the minimality of C , in the graph $\mathcal{M} \setminus C$ resulting from removing the edges in C from \mathcal{M} , adding back any edge of C reestablishes the reachability from y to x . In other words,

for any edge $(u, v) \in C$, there is a path from y to u disjoint from C , and there is a path from v to x disjoint from C .

From this, it follows that there are no connecting pairs of edges from C : there are no u, v, w such that (u, v) and (v, w) are both in C , since otherwise there is a path from v to x and a path from y to v , both disjoint from C , forming a path from y to x disjoint from C , contradicting that C is a cut from y to x .

Now let $D = \{(y, z) \in \{y\} \times (X \setminus \{y\}) \mid \text{Margin}_{\mathbf{P}}(y, z) \leq 0\}$ (for the particular y given above). Note that D is also the set $\{(y, z) \in \{y\} \times (X \setminus \{y\}) \mid (y, z) \notin \mathcal{M}(\mathbf{P})\}$. Now we show that $C^{-1} \cup D$ is acyclic. Clearly there is no reflexive loop, and also there cannot be any cycle completely inside C^{-1} since we have shown that C and hence also C^{-1} do not have connecting pairs of edges. Thus, if there is a cycle ρ , it must contain an edge (y, z) from D . Let the next edge in ρ be (z, u) . Since z is not y , this (z, u) is not in D and must be in C^{-1} . Now there are two cases. First, if $u = y$, then we have $(z, y) \in C^{-1}$ and thus $(y, z) \in C$. But (y, z) is also in D , and by definition, $(y, z) \notin \mathcal{M}(\mathbf{P})$. All edges in C are in $\mathcal{M}(\mathbf{P})$, however, so we have a contradiction in this case. Second, if $u \neq y$, then consider the next edge (u, v) in ρ . Since $u \neq y$, $(u, v) \notin D$, so $(u, v) \in C^{-1}$. But then we have both (z, u) and (u, v) in C^{-1} , which is impossible since C does not have connecting pairs of edges. Thus, $C^{-1} \cup D$ is acyclic. Hence let L be any strict linear order in $\mathcal{L}(X(\mathbf{P}))$ extending $C^{-1} \cup D$ by Lemma 4.3. This is the ballot required by the lemma. Since L extends D , it satisfies the first requirement, and in particular, $(y, x) \in L$. Let $\mathbf{P}' = \mathbf{P} + L$ and $\mathcal{M}' = \mathcal{M}(\mathbf{P}')$. Now we only need to show that $(x, y) \in sc(\mathbf{P}')$. Since $(y, x) \in L$, $\text{Margin}_{\mathbf{P}'}(x, y) = k - 1$. Since for each $(u, v) \in C$, $\text{Margin}_{\mathbf{P}}(u, v) \leq k - 1$ and $(v, u) \in L$, $\text{Margin}_{\mathbf{P}'}(u, v) \leq k - 2$. Let $E_{\leq k-2}$ be the set of edges in \mathcal{M}' of weight at most $k - 2$. Then $C \subseteq E_{\leq k-2}$. Note also that there may be edges in \mathcal{M}' but not \mathcal{M} . However, if (u, v) is in \mathcal{M}' but not in \mathcal{M} , then since from \mathbf{P} to \mathbf{P}' only one ballot is added, $\text{Margin}_{\mathbf{P}'}(u, v) = 1$. Hence every edge in \mathcal{M}' but not in \mathcal{M} is also in $E_{\leq k-2}$ since $k > 2$ and hence $k - 2 \geq 1$. Now it is easy to see that $E_{\leq k-2}$ is a cut from y to x in \mathcal{M}' : for any path ρ from y to x in \mathcal{M}' , if it uses only edges in \mathcal{M} , then ρ intersects C and hence $E_{\leq k-2}$; if ρ uses a new edge in \mathcal{M}' but not \mathcal{M} , then since the new edges are all in $E_{\leq k-2}$, ρ intersects $E_{\leq k-2}$. Thus, there is no path from y to x in $\mathcal{M}' \setminus E_{\leq k-2}$, and so $(x, y) \in sc(\mathbf{P}')$. \square

Using Lemma 4.4, we can now prove that just two of our axioms force f to be at least as resolute as Split Cycle. The idea is to use Lemma 4.4 repeatedly so that defeat can be decided merely by Coherent Defeat. An example is shown in Figure 3.

Theorem 4.5. *If f is a VCCR satisfying Coherent Defeat and Positive Involvement in Defeat, then f is at least as resolute as the Split Cycle VCCR: for any profile \mathbf{P} , $f(\mathbf{P}) \supseteq sc(\mathbf{P})$.*

Proof. Pick an arbitrary profile \mathbf{P} and $x, y \in X(\mathbf{P})$ such that $(x, y) \in sc(\mathbf{P})$. We only need to show that $(x, y) \in f(\mathbf{P})$. Let $k = \text{Margin}_{\mathbf{P}}(x, y)$ and $\mathcal{M} = \mathcal{M}(\mathbf{P})$. Now if $k \leq 2$, then it is easy to see that there cannot be a majority path from y to x , as then by the parity constraint from Lemma 2.13, any majority path has strength at least k and hence x cannot defeat y according to Split Cycle. So if $k \leq 2$, we already have $(x, y) \in f(\mathbf{P})$ by Coherent Defeat. If $k > 2$, then we inductively define $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_{k-2}$ and L_1, L_2, \dots, L_{k-2} where $\mathbf{P}_0 = \mathbf{P}$, $\mathbf{P}_{i+1} = \mathbf{P}_i + L_{i+1}$, and L_{i+1} is obtained by applying Lemma 4.4 to \mathbf{P}_i . By the lemma, (1) (y, x) is in each L_i , and (2) in each \mathbf{P}_i , $(x, y) \in sc(\mathbf{P}_i)$. As the margin from x to y decreases by $k - 2$ times in this sequence, $(x, y) \in sc(\mathbf{P}_{k-2})$ but $\text{Margin}_{\mathbf{P}_{k-2}}(x, y) = 2$. This means that there is no majority path from y to x . Thus, by Coherent Defeat, $(x, y) \in f(\mathbf{P}_{k-2})$. Finally, by Positive Involvement in Defeat in its contrapositive form, if $(x, y) \in f(\mathbf{P}_{i+1})$, then $(x, y) \in f(\mathbf{P}_i)$, since x is ranked below y in L_{i+1} . Thus, by an induction from $k - 2$ back to 0, $(x, y) \in f(\mathbf{P}) = f(\mathbf{P}_0)$. \square

Combining Theorems 4.5 and 4.2, any VCCR satisfying the axioms in Theorem 4.1 refines and is refined by Split Cycle, so it is equal to Split Cycle. This completes the proof of Theorem 4.1.

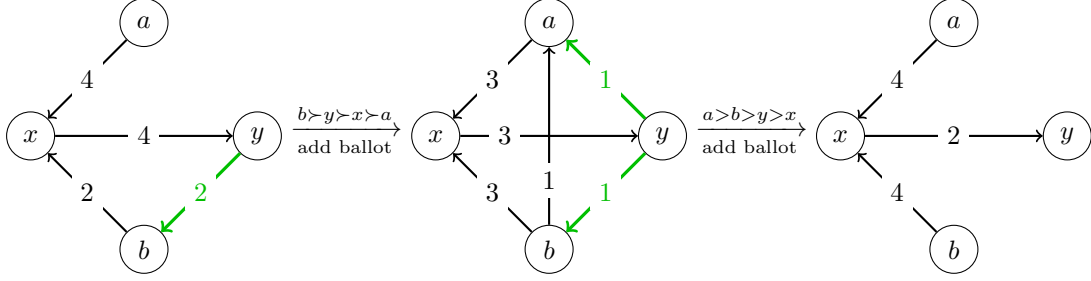


Figure 3: An example of eliminating all paths from y to x by adding ballots that put y in a sufficiently high position. For each of the first two margin graphs, a minimal cut from y to x is highlighted by thickened arrows. There are many ways to eliminate all paths from y to x , and the key is to do this quickly before the positive margin from x to y runs out.

5 Axioms on VSCCs

In this section, we characterize Split Cycle as a VSCC. As before, we first introduce the standard axioms (5.1). Then we define Coherent IIA for VSCCs (5.2). Care must be taken here as existing IIA-like axioms for SCCs have implicit commitments irrelevant to the spirit of IIA. Next we define Coherent Defeat for VSCCs (5.3), the obvious analogue of Coherent Defeat for VCCRs, and finally a new axiom of Tolerant Positive Involvement (5.4), a proper strengthening of Positive Involvement necessary for our characterization.

5.1 Standard axioms

Adapting the standard axioms for VCCRs from Section 3.1 to VSCCs, we obtain the following.

Definition 5.1. Let F be a VSCC.

1. F satisfies *Anonymity* if for any profile \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by swapping the ballots assigned to two voters, we have $F(\mathbf{P}) = F(\mathbf{P}')$, and F satisfies *Neutrality* if for any profile \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by swapping x and y in $X(\mathbf{P})$ on each voter's ballot, we have $x \in F(\mathbf{P})$ iff $y \in F(\mathbf{P}')$.
2. F satisfies *Availability* if for any profile \mathbf{P} , $F(\mathbf{P})$ is non-empty.
3. F satisfies *Homogeneity* (resp. *Upward Homogeneity*) if for any profile \mathbf{P} and $2\mathbf{P}$, where $2\mathbf{P}$ is the result of replacing each voter in \mathbf{P} by 2 copies of that voter, we have $F(\mathbf{P}) = F(2\mathbf{P})$ (resp. $F(\mathbf{P}) \supseteq F(2\mathbf{P})$ ⁸).
4. F satisfies *Monotonicity* (resp. *Monotonicity for two-candidate profiles*) if for any profile (resp. two-candidate profile) \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by moving $x \in X(\mathbf{P})$ up one place in some voter's ballot, we have $x \in F(\mathbf{P})$ only if $x \in F(\mathbf{P}')$.
5. F satisfies *Neutral Reversal* if for any profile \mathbf{P} and \mathbf{P}' obtained from \mathbf{P} by adding two voters whose ballots are converses of each other, we have $F(\mathbf{P}) = F(\mathbf{P}')$.

5.2 Coherent IIA

We now turn to the crucial question of how to formulate the analogue for VSCCs of Coherent IIA for VCCRs. First, we must ask: what is the analogue for VSCCs of Arrow's IIA for VCCRs? One answer is the following, adopting terminology of Denicolo (2000), based on Hansson 1969.

⁸Equivalently: $X(\mathbf{P}) \setminus F(\mathbf{P}) \subseteq X(\mathbf{P}) \setminus F(2\mathbf{P})$. In short, Upward Homogeneity for VSCCs means losing is preserved under doubling the profile.

Definition 5.2. A VSCC F satisfies *Hansson’s Pairwise Independence* (HPI) if for any profiles \mathbf{P} and \mathbf{P}' with $x, y \in X(\mathbf{P})$, if $x \in F(\mathbf{P})$, $y \notin F(\mathbf{P})$, and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then $y \notin F(\mathbf{P}')$.

The problem with this proposal, simply put, is this: who said x is the candidate who defeats y in \mathbf{P} ? If we knew on the basis of $x \in F(\mathbf{P})$ and $y \notin F(\mathbf{P})$ that x defeats y in \mathbf{P} , then we could indeed conclude that x still defeats y in \mathbf{P}' , so $y \notin F(\mathbf{P}')$. But it does not follow (without assumptions beyond IIA) from $x \in F(\mathbf{P})$ and $y \notin F(\mathbf{P})$ that x in particular is the candidate who defeats y in \mathbf{P} ; all that follows is that x is undefeated and y is defeated by someone or other. HPI in effect assumes that F is rationalized by a variable-election *social welfare function* (see [Denicolò 1993](#), Theorem 1), i.e., a VCCR f for which $f(\mathbf{P})$ is always a strict weak order, in which case $x \in F(\mathbf{P})$ and $y \notin F(\mathbf{P})$ do imply that x defeats y . Thus, HPI smuggles in an additional “social rationality” assumption, which should not be part of a pure independence condition. But we can fix this problem with the following weaker definition.

Definition 5.3. A VSCC F satisfies *Pure IIA* if for any profile \mathbf{P} and $y \in X(\mathbf{P})$, if $y \notin F(\mathbf{P})$, then there is an $x \in X(\mathbf{P})$ such that for any profile \mathbf{P}' with $x \in X(\mathbf{P}')$ and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, we have $y \notin F(\mathbf{P}')$.

The following proposition verifies that Pure IIA is the correct analogue of IIA for VSCCs.

Proposition 5.4. For any VSCC F , the following are equivalent:

1. F satisfies Pure IIA;
2. F is defeat-rationalized by a VCCR satisfying IIA.

Proof. From 1 to 2, suppose F satisfies Pure IIA. Define f as follows: for $x, y \in X(\mathbf{P})$, x defeats y in \mathbf{P} according to f if for every \mathbf{P}' with $x \in X(\mathbf{P}')$ and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, we have $y \notin F(\mathbf{P}')$. Thus, y is undefeated in \mathbf{P} according to f if and only if for every $x \in X(\mathbf{P})$, there is a \mathbf{P}' with $x \in X(\mathbf{P}')$, $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, and $y \in F(\mathbf{P}')$; this implies, by Pure IIA, that $y \in F(\mathbf{P})$. Conversely, $y \in F(\mathbf{P})$ implies that y is undefeated in \mathbf{P} according to f by taking $\mathbf{P}' = \mathbf{P}$. Thus, F is defeat-rationalized by f .

From 2 to 1, suppose F is defeat-rationalized by a VCCR f satisfying IIA. To show that F satisfies Pure IIA, suppose \mathbf{P} is a profile with $y \notin F(\mathbf{P})$. Then since f defeat-rationalizes F , there is an $x \in X(\mathbf{P})$ such that x defeats y in \mathbf{P} according to f . Then since f satisfies IIA, for any profile \mathbf{P}' with $x \in X(\mathbf{P}')$ and $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, we have that x defeats y in \mathbf{P}' according to f , which by defeat-rationalization implies that $y \notin F(\mathbf{P}')$. This shows that F satisfies Pure IIA. \square

Now just as we translated IIA for VCCRs to Pure IIA for VSCCs, we define Coherent IIA for VSCCs as follows, using the notation $\rightsquigarrow_{x,y}$ from [Definition 3.3](#).

Definition 5.5. A VSCC F satisfies *Coherent IIA* if for any profile \mathbf{P} and $y \in X(\mathbf{P})$, if $y \notin F(\mathbf{P})$, then there is $x \in X(\mathbf{P})$ such that for any profile \mathbf{P}' with $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$, we have $y \notin F(\mathbf{P}')$.

5.3 Coherent Defeat

Translating Coherent Defeat from VCCR to VSCC is straightforward.

Definition 5.6. A VSCC F satisfies *Coherent Defeat* if for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, if $\text{Margin}_{\mathbf{P}}(x, y) > 0$ and there is no majority path from y to x , then $y \notin F(\mathbf{P})$.

In fact, this is precisely Immunity to Binary Arguments (Im_A) in [Heitzig 2002](#) with A being the majority preference relation, only phrased in our variable-election setting.

5.4 Tolerant Positive Involvement

As we noted when we first introduced Positive Involvement in Section 3.4, a strengthening is required for characterizing Split Cycle. For the Split Cycle VCCR, we used the strengthening of Positive Involvement in Defeat. For the Split Cycle VSCC, we use the following.

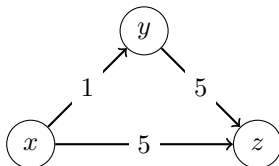
Definition 5.7. A VSCC F satisfies *Tolerant Positive Involvement* if for any profile \mathbf{P} , if $x \in F(\mathbf{P})$ and \mathbf{P}' is obtained from \mathbf{P} by adding one new voter who ranks x above all candidate y such that x is not majority preferred to y in \mathbf{P} , then $x \in F(\mathbf{P}')$.

We use the word ‘tolerant’ since Tolerant Positive Involvement is applicable more broadly than Positive Involvement; it says that for a winner x ’s winning to be preserved under the addition of a ballot L , x can tolerate being ranked below some candidates as long as x is ranked high enough: it is above all those y to which it is not majority preferred before adding L . In other words, if L weakens all potential weaknesses of x in the sense that L decreases all non-negative margins over x , then adding L will not cause x to lose. In contrast, Positive Involvement guarantees the preservation of x ’s winning only for ballots that weaken all margins over x (by putting x at the top), not just the non-negative margins.

Example 5.8. To illustrate the difference between Positive Involvement and Tolerant Positive Involvement, let us consider Instant Runoff Voting—in particular, the version (as in Taylor and Pacelli 2008, p. 7) where at each stage, all candidates with the fewest first-place votes are eliminated, unless all candidates would thereby be eliminated. Consider the following anonymized profile \mathbf{P} :

	3	4	2
\mathbf{P}	x	y	z
	y	x	x
	z	z	y

According to Instant Runoff, x is the winner in \mathbf{P} . It is also not hard to see that Instant Runoff satisfies Positive Involvement, and in this particular case, if we add a voter whose ballot puts x at the top, then x will remain the sole winner as z and y will still be eliminated consecutively. Now note that the margin graph of \mathbf{P} is



This shows that x is the Condorcet winner, the candidate who is majority preferred to every other candidate. In other words, there are no other candidates to whom x is not majority preferred. This makes the precondition for Tolerant Positive Involvement trivial, and for Tolerant Positive Involvement to hold for Instant Runoff, x should remain a winner no matter what ballot is added to \mathbf{P} . But this is not the case. Let \mathbf{Q} be the result of adding to \mathbf{P} a new voter whose ballot is $z \succ x \succ y$:

	3	4	3
	x	y	z
	y	x	x
	z	z	y

According to Instant Runoff, x is eliminated in the first round and therefore does not win in \mathbf{Q} . Thus, Instant Runoff satisfies Positive Involvement but not Tolerant Positive Involvement.

The following proposition relates Tolerant Positive Involvement to Positive Involvement in Defeat. For this we recall the axiom dubbed Majority Defeat in [Holliday and Pacuit 2021a](#): for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, if $(x, y) \in f(\mathbf{P})$, then $\text{Margin}_{\mathbf{P}}(x, y) > 0$, i.e., x defeats y only if x is majority preferred to y .

Proposition 5.9. If an acyclic VCCR f satisfies Positive Involvement in Defeat and Majority Defeat, then \bar{f} satisfies Tolerant Positive Involvement. In particular, the Split Cycle VSCC satisfies Tolerant Positive Involvement since the Split Cycle VCCR satisfies Positive Involvement and Majority Defeat.

Proof. Let f be an acyclic VCCR satisfying Positive Involvement in Defeat and Majority Defeat. Let \mathbf{P} be a profile, $x \in F(\mathbf{P})$, and \mathbf{P}' a profile obtained by adding to \mathbf{P} a new voter with ballot L . Moreover, assume that for any $y \in X(\mathbf{P}) \setminus \{x\}$, if $\text{Margin}_{\mathbf{P}}(y, x) \geq 0$, then xLy . Our goal is to show that $x \in \bar{f}(\mathbf{P}')$, i.e., that for any $y \in X(\mathbf{P}) \setminus \{x\}$, we have $(y, x) \notin f(\mathbf{P}')$. Now there are two cases: either $\text{Margin}_{\mathbf{P}}(y, x) \geq 0$ or $\text{Margin}_{\mathbf{P}}(y, x) < 0$. In the former case, by the assumption on L , xLy . Also, since $x \in \bar{f}(\mathbf{P})$, we have $(y, x) \notin f(\mathbf{P})$. So $(y, x) \notin f(\mathbf{P}')$ since f satisfies Positive Involvement in Defeat. In the latter case, since only one new voter is added, $\text{Margin}_{\mathbf{P}'}(y, x) \leq 0$. Then $(y, x) \notin f(\mathbf{P}')$ by Majority Defeat. \square

6 Characterization of the Split Cycle VSCC

We are now ready to axiomatically characterize Split Cycle as a VSCC.

Theorem 6.1. The Split Cycle VSCC is the unique VSCC satisfying Anonymity, Neutrality, Availability, (Upward) Homogeneity, Neutral Reversal, Monotonicity (for two-candidate profiles), Coherent IIA, Coherent Defeat, and Tolerant Positive Involvement.

Unlike the situation with the Split Cycle VCCR where we can directly use the theorem in [Holliday and Pacuit \(2021a\)](#) to finish one direction of the theorem, we need to prove both directions specifically for the VSCC. On the face of it, this can be explained by the fact that the standard axioms, when stated for VSCCs, are weaker than their counterparts for VCCRs since a VSCC carries less information than a VCCR, and the standard axioms involve preservation of information over profiles. For example, Upward Homogeneity for VSCCs says that losing is preserved under doubling profiles. However, one could be losing for different reasons, i.e., by being defeated by different candidates. Upward Homogeneity for VCCRs preserves also the defeaters of the losers by preserving the whole defeat relation, while Upward Homogeneity for VSCCs cannot track the defeaters since a VSCC does not carry the full information of defeat. It is an interesting question whether we can recover from a VSCC a VCCR that defeat rationalizes the VSCC, and in particular, whether we can recover a VCCR satisfying the standard axioms and Coherent IIA from a VSCC also satisfying these axioms where the VCCR defeat rationalized the VSCC. We leave this question for future work.

We start with the direction that uses the standard axioms. First, by a standard argument using symmetries, Anonymity, Neutrality, Monotonicity for two-candidate profiles, and Availability together ensure that on two-candidate profiles, the loser (if there is one) is majority dispreferred. This will be used a few times later and is already used in [Holliday and Pacuit \(2021a\)](#), so we formally state it here.

Definition 6.2. A VCCR f satisfies *Majority Defeat for two-candidate profiles* if for any profile \mathbf{P} with $X(\mathbf{P}) = \{x, y\}$ and $x \neq y$, if $xf(\mathbf{P})y$ then $\text{Margin}_{\mathbf{P}}(x, y) > 0$.

$m/2$	$m/2$	$m/2$	$m/2$	$m/2$	$m/2$	\dots	$m/2$	$m/2$
\mathbf{x}	z_n	\mathbf{y}	x	z_1	y	\dots	z_n	z_{n-1}
\mathbf{y}	\vdots	z_1	z_n	z_2	x	\dots	\mathbf{x}	\vdots
z_1	\vdots	z_2	\vdots	\vdots	z_n	\dots	y	z_1
\vdots	z_1	\vdots	z_2	z_n	\vdots	\dots	z_1	y
\vdots	\mathbf{x}	z_n	\mathbf{y}	x	z_1	\dots	\vdots	z_n
z_n	\mathbf{y}	x	z_1	y	z_2	\dots	z_{n-1}	\mathbf{x}

Figure 4: the profile \mathbf{Q} .

A VSCC F satisfies *Majority Defeat for two-candidate profiles* if for any profile \mathbf{P} with $X(\mathbf{P}) = \{x, y\}$ and $x \neq y$, if $y \notin F(\mathbf{P})$ then $\text{Margin}_{\mathbf{P}}(x, y) > 0$.

Lemma 6.3. Any VCCR or VSCC f satisfying Anonymity, Neutrality, Monotonicity for two-candidate profiles, and Availability satisfies Majority Defeat for two-candidate profiles.

Now we prove the analogue for VSCCs of Theorem 4.2.

Theorem 6.4. Let SC be the Split Cycle VSCC and F a VSCC satisfying Anonymity, Neutrality, Availability, (Upward) Homogeneity, Neutral Reversal, Monotonicity (for two-candidate profiles), and Coherent IIA. Then SC refines F : for any profile \mathbf{P} , $SC(\mathbf{P}) \subseteq F(\mathbf{P})$.

Proof. The proof is only slightly different than the corresponding one in Holliday and Pacuit (2021a). Let \mathbf{P} be any profile and $y \in SC(\mathbf{P})$. Now we need to show that $y \in F(\mathbf{P})$. Using Upward Homogeneity, $F(\mathbf{P}) \supseteq F(2\mathbf{P})$. Hence we only need to show that $y \in F(2\mathbf{P})$. Since $SC(\mathbf{P}) = SC(2\mathbf{P})$, for notational convenience, let us assume from now on that \mathbf{P} has an even number of voters and $y \in SC(\mathbf{P})$.

Let M be the largest margin of \mathbf{P} . Since \mathbf{P} is a linear profile, M is even. Let \mathbf{P}' be the result of adding to \mathbf{P} enough pairs of voters with converse ballots so that $|V(\mathbf{P}')| \geq M \cdot |X(\mathbf{P}')|$. Since adding these pairs of voters does not change either M or the number of candidates, the inequality can always be fulfilled.

By Neutral Reversal, it is enough to show that $y \in F(\mathbf{P}')$. Toward a contradiction, suppose that $y \notin F(\mathbf{P}')$. Then by Coherent IIA, there is an $x \in X(\mathbf{P}') = X(\mathbf{P})$ such that for any profile \mathbf{Q}' with $\mathbf{P}' \rightsquigarrow_{x,y} \mathbf{Q}'$, $y \notin F(\mathbf{Q}')$. Consider the two-candidate profile $\mathbf{Q}' = \mathbf{P}'|_{\{x,y\}}$. Clearly $\mathbf{P}' \rightsquigarrow \mathbf{Q}'$, and by assumption $y \notin F(\mathbf{Q}')$. Thus, by Lemma 6.3, $\text{Margin}_{\mathbf{P}'}(x, y) = \text{Margin}_{\mathbf{Q}'}(x, y) > 0$.

Let $m = \text{Margin}_{\mathbf{P}'}(x, y) > 0$, which is even since $|V(\mathbf{P}')|$ is even. Since $y \in SC(\mathbf{P}')$, there must be a majority cycle $\langle x, y, z_1, z_2, \dots, z_n, x \rangle$ such that all the margins of the consecutive edges are at least m . Now construct \mathbf{Q} and \mathbf{Q}' as in Holliday and Pacuit (2021a):

- First, take m voters in \mathbf{P}' that rank x above y , and let half of them vote $x \succ y \succ z_1 \succ z_2 \succ \dots \succ z_n$, and the other half vote $z_n \succ z_{n-1} \succ \dots \succ z_1 \succ x \succ y$. Since $m = \text{Margin}_{\mathbf{P}'}(x, y) = \text{Margin}_{\mathbf{P}}(x, y)$, the required m voters can be found.
- The rest of the voters can then be split evenly according to whether they rank x above y or y above x . So take $m/2$ fresh voters in \mathbf{P}' ranking y above x and let them vote $y \succ z_1 \succ z_2 \succ \dots \succ z_n \succ x$. Then take $m/2$ fresh voters in \mathbf{P}' ranking x above y and let them vote $x \succ z_n \dots \succ z_2 \succ y \succ z_1$.

$m/2$	$m/2$	$m/2$	$m/2$	$m/2$	$m/2$	\dots	$m/2$	$m/2$
\mathbf{y}	x	\mathbf{z}_1	y	\mathbf{z}_2	z_1	\dots	\mathbf{x}	z_n
\mathbf{z}_1	z_n	\mathbf{z}_2	x	\mathbf{z}_3	y	\dots	\mathbf{y}	\vdots
z_2	\vdots	\vdots	z_n	\vdots	x	\dots	z_1	\vdots
\vdots	z_2	z_n	\vdots	z_n	\vdots	\dots	\vdots	z_1
z_n	\mathbf{y}	x	\mathbf{z}_1	y	\mathbf{z}_2	\dots	\vdots	\mathbf{x}
x	\mathbf{z}_1	y	\mathbf{z}_2	z_1	\mathbf{z}_3	\dots	z_n	\mathbf{y}

Figure 5: the profile $\sigma(\mathbf{Q})$.

- For any $i = 1, 2, \dots, n-1$, first take $m/2$ fresh voters in \mathbf{P}' ranking x above y and let them vote $z_i \succ z_{i+1} \succ z_{i+2} \succ \dots \succ z_n \succ x \succ y \succ z_1 \succ \dots \succ z_{i-1}$. Then take $m/2$ fresh voters in \mathbf{P}' ranking y above x and let them vote $z_{i-1} \succ \dots \succ z_1 \succ y \succ x \succ z_n \succ \dots \succ z_{i+2} \succ z_i \succ z_{i+1}$.
- Take $m/2$ fresh voters in \mathbf{P}' ranking x above y and let them vote $z_n \succ x \succ y \succ z_1 \succ \dots \succ z_{n-1}$. Then take $m/2$ fresh voters in \mathbf{P}' ranking y above x and let them vote $z_{n-1} \succ \dots \succ z_1 \succ y \succ z_n \succ x$.
- The above uses $(n+2) \cdot m$ many voters in \mathbf{P}' . Since $n+2 \leq X(\mathbf{P}')$, $m \leq M$, and $V(\mathbf{P}') \geq M \cdot |X(\mathbf{P}')|$, we have enough voters for the above construction. Thus, let \mathbf{Q} be the profile using the voters used above with their ballots also specified as above. Figure 4 gives the anonymized summary of profile \mathbf{Q} . There may be unused voters in \mathbf{P}' . That is, $V(\mathbf{P}') \setminus V(\mathbf{Q})$ may be non-empty. However, they must come in pairs with respect to whether they rank x above y or y above x , since $\text{Margin}_{\mathbf{Q}}(x, y) = \text{Margin}_{\mathbf{P}'}$. Thus pick a ballot $L \in \mathcal{L}(\{x, y, z_1, \dots, z_n\})$ with xLy , and construct \mathbf{Q}' by adding to \mathbf{Q} the voters in $X(\mathbf{P}') \setminus X(\mathbf{Q})$ and let them vote L if they rank x above y in \mathbf{P}' and L^{-1} otherwise.

Observe now that $\mathbf{P}' \rightsquigarrow_{x,y} \mathbf{Q}'$ since the margin graph of \mathbf{Q}' is a pure cycle with $x, y, z_1, z_2, \dots, z_n$ with each consecutive edge's margin being m , which is no greater than their original margin in \mathbf{P}' . Thus, by Coherent IIA, $y \notin F(\mathbf{Q}')$. Since \mathbf{Q}' is the result of adding (possibly zero) reversal pairs to \mathbf{Q} , $y \notin F(\mathbf{Q})$. However, using the symmetries of the profile \mathbf{Q} , by Anonymity and Neutrality, x and the z_i 's are also not in $F(\mathbf{Q})$, contradicting Availability. (This is the same argument as in [Holliday and Pacuit 2021a](#); if we consider the rotation σ of candidates with $\sigma(x) = y, \sigma(y) = z_1, \sigma(z_i) = z_{i+1}$, and $\sigma(z_n) = x$, then applying σ to \mathbf{Q} results in the same anonymized profile. See Figure 5.) \square

Now we prove the other direction.

Theorem 6.5. Let SC be the Split Cycle VSCC and F any VSCC satisfying Coherent Defeat and Tolerant Positive Involvement. Then F refines SC : for any profile \mathbf{P} , $F(\mathbf{P}) \subseteq SC(\mathbf{P})$.

Proof. We prove the contrapositive. Suppose that $y \notin SC(\mathbf{P})$. Then there is an $x \in X(\mathbf{P})$ such that $(x, y) \in sc(\mathbf{P})$, where sc is the Split Cycle VCCR. Let $k = \text{Margin}_{\mathbf{P}}(x, y)$ and $\mathcal{M} = \mathcal{M}(\mathbf{P})$. If $k \leq 2$, then $k = 2$ and there cannot be a majority path from y to x . Then by Coherent Defeat, $y \notin F(\mathbf{P})$, and we are done. If $k > 2$, then we use Lemma 4.4 again. Inductively construct $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_{k-2}$ and L_1, L_2, \dots, L_{k-2} where $\mathbf{P}_{i+1} = \mathbf{P}_i + L_{i+1}$ and L_{i+1} is obtained by Lemma 4.4 applied to \mathbf{P}_i . Then according to the lemma and a simple induction:

- y is ranked by L_i above any candidate whom y does not beat head-to-head in \mathbf{P}_{i-1} ;

- then by Tolerant Positive Involvement, if $y \notin F(\mathbf{P}_{i+1})$, then $y \notin F(\mathbf{P}_i)$ for each $i = 0, \dots, k-3$;
- (x, y) is in each $sc(\mathbf{P}_i)$, and in particular, $(x, y) \in sc(\mathbf{P}_{k-2})$;
- $\text{Margin}_{\mathbf{P}_{k-2}}(x, y) = 2$.

Since $\text{Margin}_{\mathbf{P}_{k-2}}(x, y) = 2$ and $(x, y) \in sc(\mathbf{P}_{k-2})$, there cannot be a majority path from y to x . Then by Coherent Defeat, $y \notin F(\mathbf{P}_{k-2})$. Then, by backward induction from $k-2$ to 0, $y \notin F(\mathbf{P}_0) = F(\mathbf{P})$. \square

Combining Theorems 6.4 and 6.5, the main theorem Theorem 6.1 follows.

7 Ballots with Ties

In many applications of voting methods, it is unreasonable to expect voters to submit a linear ordering of all candidates, since the voter may lack information or willingness to make strict comparison between all pairs of candidates. Mathematically, this can be accommodated by using weak orders instead of linear orders, where candidates are first put into equivalence classes of ‘equally good’ candidates, and then the equivalence classes are linearly ordered. In this section, we show that a slight modification of the above axioms also axiomatizes Split Cycle in this generalized framework.

For any $X \subseteq \mathcal{X}$, let $\mathcal{W}(X)$ be the set of all strict weak orders on X . Recall that \succ is a strict weak order on X iff \succ is irreflexive, transitive, and *negatively transitive*: for all $x, y, z \in X$, if $x \not\succeq y$ and $y \not\succeq z$, then $x \not\succeq z$. Given a strict weak order \succ on X , we say that $x, y \in X$ are in a *tie*, written $x \sim y$, if $x \not\succeq y$ and $y \not\succeq x$. The relation \succ being a strict weak order guarantees that being in a tie is an equivalence relation, and note that the empty set \emptyset is a strict linear order where all candidates are in a big tie. Now a *profile* is a function $\mathbf{P} : V \rightarrow \mathcal{W}(X)$ for some finite $V \subset \mathcal{V}$ and $X \subset \mathcal{X}$. Given a finite set X , we typically specify a strict weak order $\succ \in \mathcal{W}(X)$ by the notation

$$P_1 \succ P_2 \succ \dots \succ P_m$$

where $\{P_1, P_2, \dots, P_m\}$ is a partition of X ; e.g., where $X = \{a, b, c, d, e\}$, we may write $\{c, e\} \succ \{b\} \succ \{a, d\}$. This means that $\{P_1, P_2, \dots, P_m\}$ is the set of all \sim -equivalence classes (groups that are tied internally) and for any $i < j$, $x \in P_i$, and $y \in P_j$, $x \succ y$; these two conditions uniquely determine a strict weak ordering of X . When some P_i is a singleton $\{x\}$, we may write ‘ $\dots \succ x \succ \dots$ ’ instead of ‘ $\dots \succ \{x\} \succ \dots$ ’.

In this section, all notions previously defined on linear profiles are now defined relative to all profiles, including the definitions of VCCR and VSCC. Thus, for example, a VCCR must return an asymmetric relation for any profile of strict weak orders. In some previous definitions, we used the word ‘above’ applied to places in a ballot. Those uses are intended for strict preference. Thus, when understanding previous definitions in the context of profiles of strict weak orders, ‘ x is above y in a ballot \succ ’ is understood as $x \succ y$, even though in this section we will typically say ‘strictly above’. Moreover, the concept of margin is now relative to strict weak orders and is still counting and comparing only strict preferences; the margin of x over y does not depend directly on how many voters put x or y in a tie. As before, we say ‘ x is majority preferred to y ’ when the margin of x over y is greater than 0, but because of the possibility of ties, the number of voters putting x strictly above y may not really be a majority among all voters. But we take this as only a minor verbal inconvenience, as it can be understood that ‘majority’ here is relative to those voters who express a strict preference between x and y .

When defining the axiom of Monotonicity, we used the notion of ‘moving a candidate up one place in a ballot’. To make this precise for strict weak orders, we use the following definition.

Definition 7.1. Let X be a finite set, $x \in X$, and \succ, \succ' two strict weak orders on X where x is not the greatest element in \succ , i.e., there is $x' \in X \setminus \{x\}$ such that $x \not\succeq x'$. Since X is finite, we can find a minimal (relative to \succ) $x' \neq x$ such that $x \not\succeq x'$. We say that \succ' is the result of *moving x up one place in \succ* if

- for any $y, z \in X \setminus \{x\}$, $y \succ z$ iff $y \succ' z$;
- if x is in a tie with x' in \succ , then in \succ' , x is not in a tie with any other element, $x \succ x'$, and for any $y \in X \setminus \{x\}$, if $y \succ x$ then $y \succ' x$;
- if x is not in a tie (relative to \succ) with any other element, then x is in a tie with x' relative to \succ' .

Thus, intuitively, to move x up one place, we either break x from a tie so that x is immediately above those with whom x previously tied, or we merge x into the tied group that was immediately above x if x was not in a tie. Then Lemma 6.3 holds under this definition of ‘moving up one place’.

Still, the generalization to strict weak orders requires a new axiom. As we noted, the empty set that puts every candidate in a single tie is a legal ballot, and intuitively such a ballot should not affect the outcome of an election. We codify this requirement as an axiom.

Definition 7.2. A VCCR or VSCC f satisfies *Neutral Participation* if for any profile \mathbf{P} and \mathbf{P}' obtained by adding to \mathbf{P} a voter whose ballot is the empty set, we have $f(\mathbf{P}) = f(\mathbf{P}')$.

While Neutral Participation is similar to Neutral Reversal, it does not follow from Neutral Reversal without enough help from other axioms. For example, we can easily devise VSCCs satisfying Anonymity, Neutrality, and Neutral Reversal by running different rules depending on the parity of the number of voters. Full Homogeneity is one such axiom that can reduce Neutral Participation to Neutral Reversal.

Proposition 7.3. Any VCCR or VSCC f satisfying Neutral Reversal and Homogeneity satisfies Neutral Participation.

Proof. Let $\mathbf{P} : V \rightarrow \mathcal{W}(X)$ be any profile, $\mathbf{P} + \emptyset$ a profile obtained by adding a voter v with \emptyset as her ballot to \mathbf{P} , and f a VCCR or VSCC satisfying Neutral Reversal and Homogeneity. Note that we can first double \mathbf{P} to $2\mathbf{P}$ (not using v) and obtain $2\mathbf{P} + \emptyset + \emptyset$ by adding v and another v' with \emptyset as both of their ballots. Now by Homogeneity, $f(\mathbf{P}) = f(2\mathbf{P})$. By Neutral Reversal and noting that $\emptyset = \emptyset^{-1}$, $f(2\mathbf{P}) = f(2\mathbf{P} + \emptyset + \emptyset)$. But $2\mathbf{P} + \emptyset + \emptyset$ is a doubling of $\mathbf{P} + \emptyset$. So by Homogeneity and chaining the previous equalities, $f(\mathbf{P}) = f(\mathbf{P} + \emptyset)$. \square

We are now prepared to characterize the Split Cycle VCCR over profiles of strict weak orders. We replace (Upward) Homogeneity from Theorem 4.1 with Neutral Participation.

Theorem 7.4. Let f be a VCCR satisfying Anonymity, Neutrality, Availability, Neutral Participation, Monotonicity (for two-candidate profiles), Neutral Reversal, and Coherent IIA. Then Split Cycle is at least as resolute as f : for any profile \mathbf{P} , $sc(\mathbf{P}) \supseteq f(\mathbf{P})$.

Proof. Let f be a VCCR satisfying the stated axioms. Toward a contradiction, let \mathbf{P} be a profile and $x, y \in X(\mathbf{P})$ such that $(x, y) \in f(\mathbf{P}) \setminus sc(\mathbf{P})$. By Lemma 6.3 applied to $\mathbf{P}|_{\{x, y\}}$ and Coherent IIA, $m := \text{Margin}_{\mathbf{P}}(x, y) = \text{Margin}_{\mathbf{P}|_{x, y}}(x, y) > 0$. By the definition of Split Cycle, there are $z_1, z_2, \dots, z_n \in X(\mathbf{P})$

m	m	m	m	m	m	\cdots	m	m
\mathbf{x}	z_n	\mathbf{y}	x	z_1	y	\cdots	z_n	z_{n-1}
\mathbf{y}	\vdots	z_1	z_n	z_2	x	\cdots	\mathbf{x}	\vdots
z_1	\vdots	z_2	\vdots	\vdots	z_n	\cdots	y	z_1
\vdots	z_1	\vdots	z_2	z_n	\vdots	\cdots	z_1	y
\vdots	\mathbf{x}	z_n	\mathbf{y}	x	z_1	\cdots	\vdots	z_n
z_n	\mathbf{y}	x	z_1	y	z_2	\cdots	z_{n-1}	\mathbf{x}

Figure 6: the profile \mathbf{Q} . Candidates inside the same gray area are tied.

such that $\langle x, y, z_1, z_2, \dots, z_n, x \rangle$ is a majority cycle and $\text{Margin}_{\mathbf{P}}(x, y)$ is smallest among the margins of the consecutive edges.

Now we add voters to \mathbf{P} without affecting the result of f . Let $N_{\mathbf{P}}(x \succ y)$ be the number of voters in \mathbf{P} who rank x above y , $N_{\mathbf{P}}(x \prec y)$ the number of voters who rank x below y , and $N_{\mathbf{P}}(x \sim y)$ the number of voters who put x and y in a tie. Then let \mathbf{P}' be the result of adding to \mathbf{P}

- $\max(3m - N_{\mathbf{P}}(x \succ y), 0)$ many pairs of voters whose ballots are in complete reversal with one of the voters in each pair ranking x above y , and
- $\max((2n - 1)m - N_{\mathbf{P}}(x \sim y), 0)$ many voters whose ballots have only ties.

Note first that by Neutral Reversal and Neutral Participation, (x, y) is still in $f(\mathbf{P}')$. Also, by the number of pairs of voters and voters added, $\text{Margin}_{\mathbf{P}'}(x, y)$ is still m , $N_{\mathbf{P}'}(x \succ y) \geq 3m$, $N_{\mathbf{P}'}(x \prec y) = N_{\mathbf{P}}(x \prec y) - m \geq 2m$, and $N_{\mathbf{P}'}(x \sim y) \geq (2n - 1)m$.

Next we construct a profile \mathbf{Q} , depicted in Figure 6, using some voters in \mathbf{P}' such that $\mathcal{M}(\mathbf{Q})$ consists of precisely a majority cycle $\langle x, y, z_1, \dots, z_n, x \rangle$ with each edge's weight being precisely m ; the tallying at the end of the last paragraph ensures that we have enough voters with desired types.

- Take m voters in \mathbf{P}' ranking x above y , and let their ballots in \mathbf{Q} be $x \succ y \succ \{z_1, \dots, z_n\}$. Then take m voters in \mathbf{P}' tying x with y , and let their ballots in \mathbf{Q} be $\{z_1, \dots, z_n\} \succ \{x, y\}$.
- Take another m voters ranking y above x in \mathbf{P}' , and let their ballots in \mathbf{Q} be $y \succ z_1 \succ \{x, z_2, \dots, z_n\}$. Then take another m voters tying x with y in \mathbf{P}' , and let their ballots in \mathbf{Q} be $\{z_2, \dots, z_n, x\} \succ \{y, z_1\}$.
- For each $i = 1, \dots, n - 1$, first take another m voters tying x with y in \mathbf{P}' and let their ballots in \mathbf{Q} be $z_i \succ z_{i+1} \succ \{z_1, \dots, z_{i-1}, z_{i+2}, \dots, z_n, x, y\}$. Then take another m voters tying x with y in \mathbf{P}' and let their ballots in \mathbf{Q} be $\{z_1, \dots, z_{i-1}, z_{i+2}, \dots, z_n, x, y\} \succ \{z_i, z_{i+1}\}$.
- Finally, take another m voters ranking x above y in \mathbf{P}' and let their ballots in \mathbf{Q} be $z_n \succ x \succ \{y, z_1, \dots, z_{n-1}\}$. Then take another m voters ranking y above x in \mathbf{P}' and let their ballots in \mathbf{Q} be $\{y, z_1, \dots, z_{n-1}\} \succ \{x, z_n\}$.

By a standard argument with permutations, using Anonymity, Neutrality, and Availability, $(x, y) \notin f(\mathbf{Q})$. Now let L be an arbitrary strict weak order on x, y, z_1, \dots, z_n with xLy . Then let \mathbf{Q}' be the result of adding the remaining voters in $V(\mathbf{P}') \setminus V(\mathbf{Q})$ to \mathbf{Q} with their ballots given by:

- if the voter ranks x above y in \mathbf{P}' , her ballot in \mathbf{Q}' is L ;
- if the voter ranks y above x in \mathbf{P}' , her ballot in \mathbf{Q}' is L^{-1} ;
- if the voter ties x with y , then her ballot in \mathbf{Q}' is \emptyset .

Two immediate observations follow. First, it is easy to check that $\mathbf{P}'|_{\{x,y\}} = \mathbf{Q}'|_{\{x,y\}}$. Second, since $\text{Margin}_{\mathbf{Q}}(x, y)$ is easily seen to be m , the number of voters from $V(\mathbf{P}') \setminus V(\mathbf{Q})$ ranking x above y in \mathbf{P}' must be equal to the number of voters from $V(\mathbf{P}') \setminus V(\mathbf{Q})$ ranking y above x . Thus, \mathbf{Q}' is the result of adding to \mathbf{Q} some (possibly zero) pairs of voters with converse ballots and some (possibly zero) voters with the fully tied ballot. Thus:

- $\mathcal{M}(\mathbf{Q}') = \mathcal{M}(\mathbf{Q})$. Since the latter is just a majority cycle with each edge's weight being m , $\mathcal{M}(\mathbf{Q}')$ can be obtained from $\mathcal{M}(\mathbf{P}') = \mathcal{M}(\mathbf{P})$ by deleting candidates and edges and lowering weights not involving the edge between x and y (the weights of the edges in the cycle x, y, z_1, \dots, z_n, x are originally all at least m). So Coherent IIA applies, and $(x, y) \in f(\mathbf{Q}')$.
- Since $(x, y) \notin f(\mathbf{Q})$, and \mathbf{Q}' is the result adding pairs of converse ballots and ballots with one big tie, by Neutral Reversal and Neutral Participation, $(x, y) \notin f(\mathbf{Q}')$.

So we have a contradiction. \square

Appeal to Neutral Participation cannot be eliminated in the proof since we may have to add an odd number of big tie ballots at some point. In particular, $\max((2n - 1)M - N_{\mathbf{P}}(x \sim y), 0)$ may be odd.

We now prove the analogous theorem for the Split Cycle VSCC over profiles of strict weak orders.

Theorem 7.5. Let F be a VSCC satisfying Anonymity, Neutrality, Availability, Neutral Participation, Monotonicity (for two-candidate profiles), Neutral Reversal, and Coherent IIA. Then SC refines F : for any profile \mathbf{P} , $SC(\mathbf{P}) \subseteq F(\mathbf{P})$.

Proof. The proof essentially combines the strategies in Theorem 6.4 and 7.4. Let F be a VSCC satisfying the axioms and \mathbf{P} a profile (allowing ties in ballots). Since we are dealing with a VSCC, as in the proof for Proposition 6.4, we need to first add voters to prepare for a later construction. Let M be the largest margin in the margin graph of \mathbf{P} . Then fix a linear order L^* on $X(\mathbf{P})$, and let \mathbf{P}' be the result of adding to \mathbf{P}

- $3M$ pairs of voters in which one voter's ballot is L^* and the other voter's ballot is $(L^*)^{-1}$ and
- $2|X(\mathbf{P})| \cdot M$ many voters whose ballot is the empty set.

Clearly \mathbf{P} and \mathbf{P}' have the same margin graph, and by the assumed axioms, $F(\mathbf{P}') = F(\mathbf{P})$. So to show that $SC(\mathbf{P}) \subseteq F(\mathbf{P})$, it is enough to show that $SC(\mathbf{P}') \subseteq F(\mathbf{P}')$.

Toward a contradiction, suppose that $y \in SC(\mathbf{P}')$ but $y \notin F(\mathbf{P}')$. Then by Coherent IIA there is $x \in X(\mathbf{P}')$ such that for any profile \mathbf{Q}' with $\mathbf{P}' \rightsquigarrow_{x,y} \mathbf{Q}'$, we have $y \notin F(\mathbf{Q}')$. Using Lemma 6.3 and Coherent IIA, $m := \text{Margin}_{\mathbf{P}'}(x, y) = \text{Margin}_{\mathbf{P}}(x, y) > 0$. Since $y \in SC(\mathbf{P}')$, there is a majority cycle $\langle x, y, z_1, \dots, z_n, x \rangle$ with (x, y) being an edge with the smallest margin. Moreover, by the construction of \mathbf{P}' , using the notation in Theorem 7.4, $N_{\mathbf{P}'}(x \succ y) \geq 3m$, $N_{\mathbf{P}'}(y \succ x) \geq 2m$, and $M_{\mathbf{P}'}(x \sim y) \geq (2n - 1)m$ since $M \geq m$ and $2|X(\mathbf{P})| \geq n$. Thus, the construction of \mathbf{Q} and \mathbf{Q}' in the proof of Proposition 7.4 can be applied in exactly the same way. Using the symmetries of \mathbf{Q} and Anonymity, Neutrality, and Availability, $y \in F(\mathbf{Q})$, and hence by Neutral Reversal and Neutral Participation, $y \in F(\mathbf{Q}')$. On the other hand, by construction $\mathbf{P}' \rightsquigarrow_{x,y} \mathbf{Q}'$, so $y \notin F(\mathbf{Q}')$. So we obtain a contradiction as desired. \square

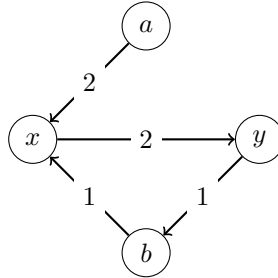
Now we consider the other direction for axiomatization where, in the case of a VCCR f , we seek to show that $(x, y) \in f(\mathbf{P})$ given that $(x, y) \in sc(\mathbf{P})$. The strategy is to use Lemma 4.4 repeatedly until we can resort to Coherent Defeat since $Margin_{\mathbf{P}}(x, y)$ is still positive but there is no majority path back from y to x . Note that Lemma 4.4 can only be used to lower the margin from x to y to 2, and to finish the argument, we used the parity constraint for the margin graphs obtained from profiles using only linear ballots (Lemma 2.13): the margins must share the same parity, and thus if $Margin_{\mathbf{P}}(x, y) = 2$, then there cannot be a majority path from y back to x using only margins smaller than 2 since the number 1 is not available. However, now that we are allowing ballots with ties, the parity constraint is no longer there to help.

Note first that Lemma 4.4 works also for profiles allowing ties in ballots, since nothing in the proof depends on the parity constraint for linear profiles; the proof is exactly the same, and the constructed ballot L is still linear. We record the result here.

Lemma 7.6. For any profile $\mathbf{P} : V \rightarrow \mathcal{W}(X)$ and any $(x, y) \in sc(\mathbf{P})$, if $Margin_{\mathbf{P}}(x, y) > 2$, then there is a linear ballot $L \in \mathcal{L}(X)$ such that

- for any $z \in X \setminus \{y\}$ with $Margin_{\mathbf{P}}(y, z) \leq 0$, we have $(y, z) \in L$ (in particular, $(y, x) \in L$), and
- $(x, y) \in sc(\mathbf{P} + L)$.

To apply Lemma 7.6 directly to our axiomatization for all profiles allowing ties, one may hope to relax the assumption that $Margin_{\mathbf{P}}(x, y) > 2$ to $Margin_{\mathbf{P}}(x, y) > 1$. But this cannot be done as shown by the following example:



For any ballot L in which y is ranked above x and a , adding L to any profile \mathbf{P} generating the above margin graph would render (x, y) no longer a defeat edge according to Split Cycle, since there will be a majority cycle (x, y, a, x) in which (x, y) is a weakest edge with margin 1.

For this reason, we resort to the axiom of Downward Homogeneity to restore the parity constraint.

Definition 7.7. A VCCR f (resp. VSCC F) satisfies *Downward Homogeneity* if for any profile \mathbf{P} and $2\mathbf{P}$, where $2\mathbf{P}$ is the result of replacing each voter in \mathbf{P} by 2 copies of that voter, $f(\mathbf{P}) \supseteq f(2\mathbf{P})$ (resp. $F(\mathbf{P}) \supseteq F(2\mathbf{P})$).

Theorem 7.8. Let f be a VCCR satisfying Downward Homogeneity, Coherent Defeat, and Positive Involvement in Defeat. Then f is at least as resolute as the Split Cycle VCCR: for any profile \mathbf{P} , $f(\mathbf{P}) \supseteq sc(\mathbf{P})$.

Proof. Pick an arbitrary profile $\mathbf{P} : V \rightarrow \mathcal{W}(X)$ and $x, y \in X$ such that $(x, y) \in sc(\mathbf{P})$. Using Downward Homogeneity, to show that $(x, y) \in f(\mathbf{P})$, it is enough to show that $(x, y) \in f(2\mathbf{P})$. Let $k = Margin_{2\mathbf{P}}(x, y)$ (which must be at least 2) and inductively define $\mathbf{P}_0, \dots, \mathbf{P}_{k-2}$ where $\mathbf{P}_0 = 2\mathbf{P}$ and $\mathbf{P}_{i+1} = \mathbf{P}_i + L_i$ with L_i obtained by applying Lemma 7.6 to \mathbf{P}_i . An easy induction shows that $(x, y) \in sc(\mathbf{P}_{k-2})$ and $Margin_{\mathbf{P}_{k-2}}(x, y) = 2$. But more importantly, since each L_i is linear, the parity constraint is preserved, and

in each \mathbf{P}_i inductively constructed, the margins in \mathbf{P}_i 's margin graph share the same parity. This means that there is no majority edge in \mathbf{P}_{k-2} with margin 1. Thus, there is no majority path from y back to x in \mathbf{P}_{k-2} . By Coherent Defeat, $(x, y) \in f(\mathbf{P}_{k-2})$, and by backward induction using Positive Involvement in Defeat, $(x, y) \in f(2\mathbf{P})$. \square

Theorem 7.9. Let F be a VSCC satisfying Downward Homogeneity, Coherent Defeat, and Tolerant Positive Involvement. Then F refines the Split Cycle VSCC: for any profile \mathbf{P} , $F(\mathbf{P}) \subseteq SC(\mathbf{P})$.

Proof. By combining the parity argument in the proof of Theorem 7.8 and the inductive argument using Tolerant Positive Involvement in the proof of Theorem 6.5. \square

To sum up, now we have the following axiomatizations of Split Cycle when ties are allowed in ballots.

Theorem 7.10. Allowing ties in ballots, the Split Cycle VCCR is the unique VCCR satisfying Anonymity, Neutrality, Availability, Downward Homogeneity, Neutral Participation, Neutral Reversal, Monotonicity (for two-candidate profiles), Coherent IIA, Coherent Defeat, and Positive Involvement in Defeat. Downward Homogeneity and Neutral Participation can be replaced by Homogeneity.

Theorem 7.11. Allowing ties in ballots, the Split Cycle VSCC is the unique VSCC satisfying Anonymity, Neutrality, Availability, Downward Homogeneity, Neutral Participation, Neutral Reversal, Monotonicity (for two-candidate profiles), Coherent IIA, Coherent Defeat, and Tolerant Positive Involvement. Downward Homogeneity and Neutral Participation can be replaced by Homogeneity.

8 Conclusion

In this paper, we provided axiomatizations for Split Cycle both as a VCCR and as a VSCC, and both over profiles prohibiting ties and over profiles allowing ties. Most of the axioms are largely uncontroversial, while the special axioms—Coherent IIA, Coherent Defeat, Positive Involvement in Defeat, and Tolerant Positive Involvement—have direct intuitive appeal. The axiomatizations also show where and how Split Cycle diverges from similar margin-based voting methods such as Ranked Pairs and Beat Path: Coherent Defeat is a shared starting point; then Positive Involvement in Defeat (resp. Tolerant Positive Involvement for VSCCs) establishes a lower bound for resoluteness; and Coherent IIA forces an upper bound for resoluteness.

We conclude with several directions for possible future investigation. First, while our method for the axiomatization of Split Cycle cannot be applied directly to Beat Path and Ranked Pairs or other unaxiomatized margin-based voting methods, our results at least suggest that the axiomatization question for margin-based voting methods could be amenable to analysis of graphs by well-known graph theoretical concepts. In particular, given the similarity between the definitions of Beat Path and Split Cycle, we believe an axiomatization of Beat Path is within reach.

Second, as we show in the Appendix, within the set of all the margin-based VCCRs satisfying Monotonicity, Coherent IIA, Coherent Defeat, and First-place Involvement in Defeat, what we call *Ignore-source Split Cycle* is the least resolute while Split Cycle is the most resolute. It seems to be an interesting problem to classify all VCCRs in this set and also study their ordering by resoluteness. This could help us better understand the axioms (especially First-place Involvement in Defeat).

Third, as we mentioned at the beginning of Section 6, in general a VSCC carries less information than a VCCR, but when we focus on VCCRs and VSCCs satisfying certain axioms, recovering canonically a rationalizing VCCR from a VSCC may be possible. In fact, the axiom of Coherent IIA itself suggest a

method of recovery: given a VSCC F that is rationalized by some VCCR f satisfying Coherent IIA and a losing candidate y in a profile \mathbf{P} , any candidate x that defeats y according to f should be such that for any \mathbf{P}' with $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$, x should still defeat y in \mathbf{P}' according to f , making y still lose according to F in \mathbf{P}' . Viewing this statement about x as a property of a potential defeater of y , we can take as the defeaters of y according to the VSCC F all candidates $x \in X(\mathbf{P})$ such that for any \mathbf{P}' with $\mathbf{P} \rightsquigarrow_{x,y} \mathbf{P}'$, $y \notin F(\mathbf{P}')$. Some of these candidates may not really defeat y according to f , but nevertheless we can use this method to recover a VCCR f' from F . It remains to be seen what axioms can be preserved by this method of constructing a VCCR from a VSCC and what other interesting properties this method may exhibit.

Finally, there are open questions concerning the relation of Split Cycle to other voting methods. For example, Holliday and Pacuit (2022) introduce a VSCC they call Simple Stable Voting, whose definition says nothing about cycles but which appears to be deeply related to Split Cycle. In particular, it is unknown whether there is any *uniquely-weighted profile*—that is, profile in which no margins between distinct pairs of candidates are the same—in which Simple Stable Voting fails to select from among the Split Cycle winners. Extensive computer searches have produced no such example, but a proof that Simple Stable Voting refines Split Cycle on uniquely-weighted profiles has so far proven elusive. More generally, we hope to better understand the class of VSCCs that serve as tiebreaking methods for Split Cycle, by selecting in any uniquely-weighted profile a unique tiebreak winner from among the Split Cycle winners.

Acknowledgements

We thank the organizers and audience at the 2022 Meeting of the Society for Social Choice and Welfare, where this work was presented.

A Appendix

In this appendix, we show that Positive Involvement is not sufficient for axiomatizing Split Cycle if we keep the other axioms the same.

As a convention, we define the minimum of the empty set to be ∞ (an object greater than every natural number) and the maximum of the empty set to be 0.

Definition A.1. For any positively weighted directed graph \mathcal{M} and simple path ρ from x to y in \mathcal{M} , define the *ignore-source strength* $Strength_{\mathcal{M}}^{is}(\rho)$ of ρ to be the minimum of the weights of the consecutive edges in ρ that do not start from x . Then the ignore-source strength $Strength_{\mathcal{M}}^{is}(x, y)$ from x to y is the maximum of $Strength_{\mathcal{M}}^{is}(\rho)$ for all paths ρ from x to y . Then define the VCCR *Ignore-source Split Cycle* isc by: for any profile \mathbf{P} and $x, y \in X(\mathbf{P})$, $(x, y) \in isc(\mathbf{P})$ iff $Margin_{\mathbf{P}}(x, y) > Strength_{\mathcal{M}(\mathbf{P})}^{is}(y, x)$. The Ignore-source Split Cycle VSCC ISC is defined as \overline{isc} (recall Lemma 2.4).

From the definition, the following observations are immediate.

Lemma A.2. Let \mathcal{M} be a margin graph and x, y vertices in \mathcal{M} .

1. $Strength_{\mathcal{M}}^{is}(x, y) = \infty$ iff (x, y) is an edge in \mathcal{M} . $Strength_{\mathcal{M}}^{is}(x, y) = 0$ iff y is not reachable from x .
2. $Strength_{\mathcal{M}}^{is}(x, y) = Strength_{\mathcal{M}'}(x, y)$ where \mathcal{M}' is the result of raising the weights to ∞ for each edge whose source is x . Here $Strength_{\mathcal{M}'}$ is calculated in the standard way: it is the maximum of the strengths of all paths from x to y , where the strength of a path is the minimum of the weights of the consecutive edges in that path. Thus ‘ignore source’ can be viewed also as ‘infinity source’.

3. Suppose $Strength_{\mathcal{M}}^{is}(x, y) < \infty$. Then if we delete all edges in \mathcal{M} that (1) do not start from x and (2) have weights no greater than $Strength_{\mathcal{M}}^{is}(x, y)$, then y is no longer reachable from x . In other words, when $Strength_{\mathcal{M}}^{is}(x, y) < \infty$, the set of all edges not starting from x and with weights no greater than $Strength_{\mathcal{M}}^{is}(x, y)$ is a cut from x to y .
4. For any cut C from x to y that does not use any edge starting from x , $Strength_{\mathcal{M}}^{is}(x, y)$ is at most the maximal weight of the edges in C .

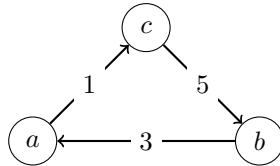
Proposition A.3. *isc* satisfies Anonymity, Neutrality, Availability, Homogeneity, Monotonicity, Neutral Reversal, Coherent IIA, Coherent Defeat, and First-place Involvement in Defeat.

Proof. We only verify First-place Involvement in Defeat here; the rest can be verified in the same way as they are verified for Split Cycle. Suppose x does not defeat y according to *isc* in \mathbf{P} , and let \mathbf{P}' be the result of adding a ballot that ranks y at the top. Then $Margin_{\mathbf{P}}(x, y) \leq Strength_{\mathcal{M}(\mathbf{P})}^{is}(y, x)$, and we want to show the same inequality for \mathbf{P}' . Since the new ballot ranks y at the top, $Margin_{\mathbf{P}'}(x, y) = Margin_{\mathbf{P}}(x, y) - 1$. So we only need to show that $Strength_{\mathcal{M}(\mathbf{P}')}^{is}(y, x) \geq Strength_{\mathcal{M}(\mathbf{P})}^{is}(y, x) - 1$. For the standard strength, this is trivial since we only added a single ballot, so the margin changes can be at most 1. However, by the example in the proof of the next proposition, extra care must be taken for $Strength^{is}$. Since the added ballot ranks y at the top, all the edges starting from y in $\mathcal{M}(\mathbf{P})$ are strengthened in weight, and in particular every edge in $\mathcal{M}(\mathbf{P})$ starting from y is still in $\mathcal{M}(\mathbf{P}')$. Thus, letting $\mathcal{M}(\mathbf{P})^+$ and $\mathcal{M}(\mathbf{P}')^+$ be the result of raising the weight of all edges from y to ∞ in $\mathcal{M}(\mathbf{P})$ and $\mathcal{M}(\mathbf{P}')$, respectively, the weight *decrease* of the edges from $\mathcal{M}(\mathbf{P})^+$ to $\mathcal{M}(\mathbf{P}')^+$ is at most 1 (including eliminating edges with weight 1); the weight *increase* could be infinite due to new edges starting from y , but this is irrelevant for the direction of the inequality we are trying to show. Hence $Strength_{\mathcal{M}(\mathbf{P}')^+}^{is}(y, x) \geq Strength_{\mathcal{M}(\mathbf{P})^+}^{is}(y, x) - 1$. Then using the second item of Lemma A.2, we are done. \square

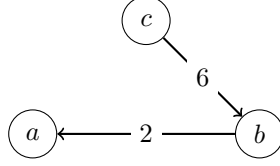
As we have shown that First-place Involvement in Defeat entails Positive Involvement, *isc* satisfies Positive Involvement as well.

Proposition A.4. *isc* is not as resolute as Split Cycle, and it fails Positive Involvement in Defeat. Moreover, *ISC* fails Tolerant Positive Involvement.

Proof. Consider a profile \mathbf{P} whose margin graph is



According to Split Cycle, b defeats a . But according to *isc*, b does not defeat a , since $Strength^{is}(a, b) = 5$ as we need to ignore the edge from a to c . Since clearly c does not defeat a either according to *isc*, c is undefeated and a winner according to *ISC*. To see the failure of Positive Involvement in Defeat and Tolerant Positive Involvement, let \mathbf{P}' be the result of adding a ballot \succ such that $c > a > b$ (where a is above b and thus above all candidates to whom a is not majority preferred) to \mathbf{P} . Then $\mathcal{M}(\mathbf{P}')$ is



Now b defeats a according to isc since there is no path from a to b and the ignore-source strength from a to b decreased from 5 above to 0. \square

Thus, when axiomatizing Split Cycle, we cannot replace Positive Involvement in Defeat by First-place Involvement in Defeat for the VCCR or replace Tolerant Positive Involvement by Positive Involvement for the VSCC. On the other hand, we can show that Coherent Defeat and First-place Involvement in Defeat provide a one-sided axiomatization for isc in the direction opposite to what [Holliday and Pacuit \(2021a\)](#) did for the Split Cycle VCCR.

Theorem A.5. For any VCCR f that satisfies Coherent Defeat and First-place Involvement in Defeat, f is at least as resolute as isc .

Proof. We repeat the same cut argument in Lemma 4.4 and the inductive argument in Theorem 4.5. The counterpart of Lemma 4.4 for isc takes the following form: for any profile $\mathbf{P} : V \rightarrow \mathcal{L}(X)$ and any $(x, y) \in isc(\mathbf{P})$, if $Margin_{\mathbf{P}}(x, y) > 2$, then there is a ballot $L \in \mathcal{L}(X)$ such that

- y is at the top of L , and
- $(x, y) \in isc(\mathbf{P} + L)$.

The proof is almost the same as that of Lemma 4.4: since we are using $Strength_{\mathcal{M}}^{is}$, we can guarantee that the cut does not include any edge starting from the defeated candidate according to isc . Hence the union of the converse of the cut and the set of all pairs starting from y is acyclic, and the rest of the proof is exactly the same. Then we repeatedly use this counterpart of Lemma 4.4 to show that if $(x, y) \in isc(\mathbf{P})$, then there is a sequence of ballots L_1, \dots, L_n , all putting y at the top, such that $(x, y) \in isc(\mathbf{P} + L_1 + \dots + L_n)$, and in $\mathbf{P} + L_1 + \dots + L_n$, x defeats y merely by Coherent Defeat. Then by First-place Involvement in Defeat, x defeats y in \mathbf{P} according to f . \square

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