Logics for Social Choice Theory

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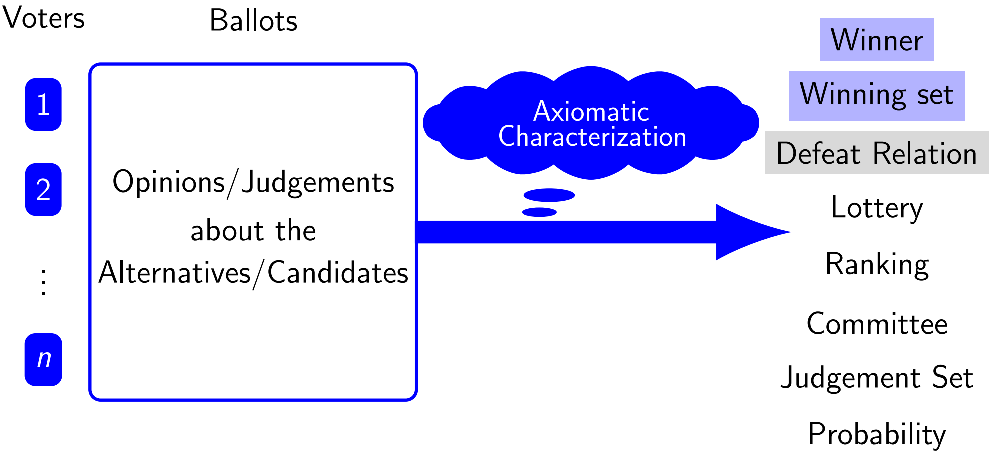
Lecture 2

ESSLLI 2022

Social Choice Theory



Anonymous profiles

40	35	25
t	r	k
k	k	r
r	t	t

Margin

Let P be a profile and $a, b \in X(\mathrm{P})$. Then the **margin of** a **over** b is:

$$Margin_{\mathrm{P}}(a, b) = |\{i \in V(\mathrm{P}) \mid a\mathrm{P}_i b\}| - |\{i \in V(\mathrm{P}) \mid b\mathrm{P}_i a\}|.$$

We say that a is **majority preferred** to b in P when $Margin_{\mathrm{P}}(a, b) > 0$.

Margin Graph

The **margin graph of** P, $\mathcal{M}(\mathrm{P})$, is the weighted directed graph whose set of nodes is $X(\mathrm{P})$ with an edge from a to b weighted by $Margin(a,b)$ when $Margin(a,b) > 0$. We write

$$a \xrightarrow{\alpha}_{\mathrm{P}} b \text{ if } \alpha = Margin_{\mathrm{P}}(a,b) > 0.$$

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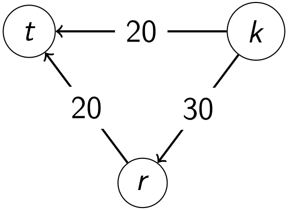
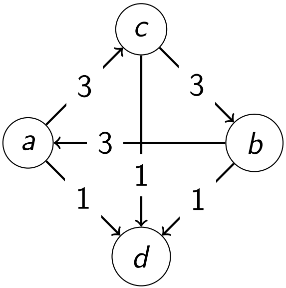
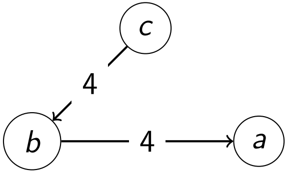


Margin Graph

A **margin graph** is a weighted directed graph $\mathcal{M}$ where all the weights have the same parity.





Theorem (Debord, 1987)

For any margin graph $\mathcal{M}$, there is a profile P *such that $\mathcal{M}$ is the margin graph of* P*.*

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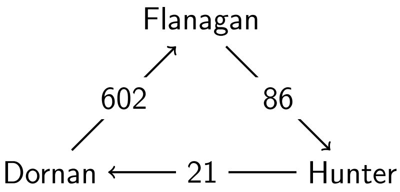

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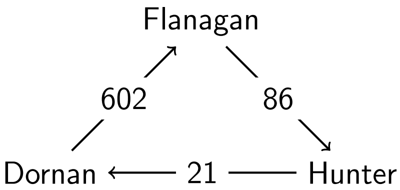

The numbers are **margins of victory**. Given a profile P and $x, y \in X(\mathrm{P})$, let

$$Margin_{\mathrm{P}}(x, y) = |\{i \in V(\mathrm{P}) : x\mathrm{P}_i y\}| - |\{i \in V(\mathrm{P}) : y\mathrm{P}_i x\}|.$$

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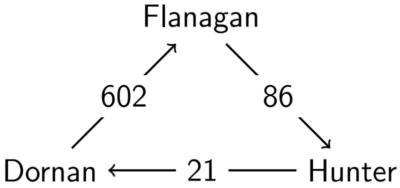

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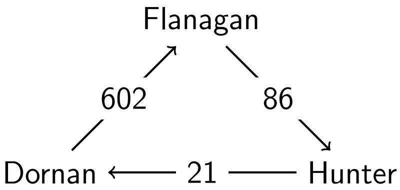

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Incoherence and inconsistency

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Here we have some help without a typical analogue in logic: the strength of majority preferences as measured by **margins**.

Examples of CCRs

- Copeland
- Uncovered Set
- Beat Path
- Split Cycle
- Borda defeat

Arrow's Theorem

Theorem (Arrow, 1951) Assume that $|X| \geq 3$ and V is finite. Then any (V, X)-CCR satisfying UD, IIA, FR, and P is a dictatorship.

K. Arrow. *Social Choice and Individual Values*. Yale University Press (1951, 2nd ed., 1963, 3rd ed., 2012).

Escaping impossibility

Key assumptions in Arrow's Theorem:

- The number of voters is finite

  P. Fishburn (1970). *Arrow's impossibility theorem: concise proof and infinitely many voters*. Journal of Economic Theory, 2, pp. 103 - 106.

- Universal domain

  W. Gaertner (2001). *Domain Conditions in Social Choice Theory*. Cambridge University Press.

  E. Elkind, M. Lackner, and D. Peters (2022). *Preference Restrictions in Computational Social Choice: A Survey*. `https://arxiv.org/abs/2205.09092`.

- There are at least 3 alternatives

- IIA

Anonymity and Neutrality

Anonymity: if x defeats y in P, and P′ is obtained from P by swapping the ballots assigned to two voters, then x still defeats y in P′.

Neutrality: if x defeats y in P, and P′ is obtained from P by swapping x and y on each voter's ballot, then y defeats x in P′.

Availability: for all profiles P, there is some undefeated candidate.

Monotonicity

Monotonicity (resp. Monotonicity for two-candidate profiles): if x defeats y in a profile (resp. two-candidate profile) P, and P' is obtained from P by some voter i moving x above the candidate that i ranked immediately above x in P, then x defeats y in P'.

Lemma

If f satisfies Anonymity, Neutrality, and Monotonicity with respect to two-candidate profiles, then f satisfies Special Majority Defeat: for any two-candidate profile P, *x defeats y in* P *according to f only if x is majority preferred to y.*

Other rules satisfying Anonymity, Neutrality and Monotonicity: The completely indecisive method; Unanimity; Quota rules (cf. Fishburn 1974, Section 1)

Pareto: if for all profiles P and $x, y \in X(\mathrm{P})$, if $x\mathrm{P}_i y$ for all $i \in V(\mathrm{P})$, then x defeats y in P.

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Neutral Reversal: if P′ is obtained from P *by adding two voters with reversed ballots*, then x defeats y in P if and only if x defeats y in P′.

Variable candidate/voter profiles

Fix infinite sets $\mathcal{V}$ and $\mathcal{X}$ of *voters* and *candidates*, respectively.

For $X \subseteq \mathcal{X}$, let $\mathcal{L}(X)$ be the set of all strict linear orders on X.

A *profile* is a function $\mathsf{P} : V(\mathsf{P}) \to \mathcal{L}(X(\mathsf{P}))$ for some nonempty finite $V(\mathsf{P}) \subseteq \mathcal{V}$ and nonempty finite $X(\mathsf{P}) \subseteq \mathcal{X}$.

We call $V(\mathsf{P})$ and $X(\mathsf{P})$ the sets of *voters in* P and *candidates in* P, respectively.

We call $\mathsf{P}(i)$ voter i's *ranking*, and we write '$x\mathsf{P}_i y$' for $(x, y) \in \mathsf{P}(i)$. As usual, we take $x\mathsf{P}_i y$ to mean that voter i strictly prefers candidate x to candidate y.

A **variable-election collective choice rule** (VCCR) is a function f on the domain of all profiles such that for any profile $\mathbf{P}$, $f(\mathbf{P})$ is an asymmetric binary relation on $X(\mathbf{P})$, which we call *the defeat relation for* $\mathbf{P}$ *under* f.

For $x, y \in X(\mathbf{P})$, we say that x *defeats* y *in* $\mathbf{P}$ *according to* f when $(x, y) \in f(\mathbf{P})$.

Characterizing Majority Rule

Proposition

For any VCCR f on two-candidate profiles, the following are equivalent:

1. *f coincides with majority rule;*
2. *f satisfies the following axioms: Anonymity, Neutrality, Monotonicity, Pareto, and Neutral Reversal.*

W. Holliday and EP (2021). *Axioms for Defeat in Democratic Elections*. Journal of Theoretical Politics, `https://arxiv.org/pdf/2008.08451.pdf`.

Characterizing Majority Rule

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Fixed vs. Variable-Candidate Axioms

f satisfies **fixed-candidate IIA** (FIIA) if for any profiles $\mathbf{P}$ and $\mathbf{P}'$ with $X(\mathbf{P}) = X(\mathbf{P}')$,

if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then x defeats y in $\mathbf{P}$ according to f if and only if x defeats y in $\mathbf{P}'$ according to f;

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Arrow's Theorem

Theorem (Arrow, 1952) Assume that $|X| \geq 3$ and V is finite. Then any collective choice rule for $\langle X, V \rangle$ satisfying universal domain, FIIA, full rationality, and Pareto is a dictatorship.

Pareto: if for all $i \in V(\mathrm{P})$, $x\mathrm{P}_i y$, then x defeats y according to $f(\mathrm{P})$.

full rationality: the defeat relation is a strict weak order.

dictatorship: there is an $i \in V$ such that for all P and $x, y \in X(\mathrm{P})$, if $x\mathrm{P}_i y$, then $xf(\mathrm{P})y$.

Escaping impossibility

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2. Weaken the IIA axiom

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   Weak IIA: For all profiles P and P′, if x defeats y in P according to f and $\mathrm{P}_{|\{x,y\}} = \mathrm{P}'_{|\{x,y\}}$, then y does *not* defeat x in P′ according to f.

3. Weaken properties of the defeat relation

   $x_1, \ldots, x_n$ is a **cycle** in B if $x_1 = x_n$ and for all $i = 1, \ldots, n-1$, $x_i B x_{i+1}$.
   A relation B is **acyclic** if there is no cycle in B.

Baigent's Theorem

Weak IIA: For all profiles P and P$'$, if x defeats y in P according to f and $\mathrm{P}_{|\{x,y\}} = \mathrm{P}'_{|\{x,y\}}$, then y does *not* defeat x in P$'$ according to f.

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Weak IIA: For all profiles P and P′, if x defeats y in P according to f and $\mathrm{P}_{|\{x,y\}} = \mathrm{P}'_{|\{x,y\}}$, then y does *not* defeat x in P′ according to f.

Given a (V, X)-CCR f, a voter $i \in V$ is a *vetoer for f* if for all (V, X)-profiles P and $x, y \in X(\mathrm{P})$, if $x\mathrm{P}_i y$, then y does not defeat x in P according to f.

Theorem (Baigent, 1987)

Assume V is finite and $|X| \geq 4$. Any (V, X)-SWF satisfying Weak IIA and Pareto has a vetoer.

Blau and Deb Theorem

A coalition $C \subseteq V$ of voters has *veto power for* f if for any (V, X)-profile $\mathbf{P}$ and $x, y \in X$, if $x\mathbf{P}_i y$ for all $i \in C$, then y does not defeat x in $\mathbf{P}$ according to f

Theorem (Blau and Deb, 1977)

Let f be an acyclic (V, X)-CCR satisfying IIA, Neutrality, and Monotonicity.

1. *For any partition of V into at least $|X|$-many coalitions, at least one of the coalitions has veto power.*
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Theorem (Holliday and P, 2021)

If f is an acyclic VCCR satisfying VIIA, Neutrality, and Monotonicity, then for any finite $V \subseteq \mathcal{V}$, f has a V-vetoer.

The Fallacy of IIA

Suppose x defeats y in a profile P, and a profile P′ is exactly like P with respect to how every voter ranks x vs. y. Should it follow that x defeats y in P′?

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Suppose x defeats y in a profile P, and a profile P$'$ is exactly like P with respect to how every voter ranks x vs. y. Should it follow that x defeats y in P$'$?

Arrow's Independence of Irrelevant Alternatives (IIA) says 'yes'.

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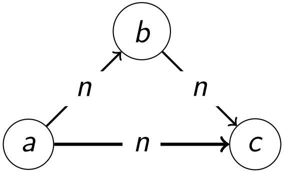
Arrow's Independence of Irrelevant Alternatives (IIA) says 'yes'.

We say 'no': if P′ is sufficiently *incoherent*, we may need to suspend judgment on many defeat relations that could be coherently accepted in P.

W. Holliday and EP (2021). *Axioms for Defeat in Democratic Elections*. Journal of Theoretical Politics.

In the context of the following perfectly coherent profile P, the margin of n for a over b should be sufficient for a to defeat b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
c	c	$\boldsymbol{b}$



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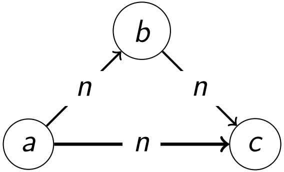

Yet in the following P′ with $\mathrm{P}'_{|\{a,b\}} = \mathrm{P}_{|\{a,b\}}$, no VCCR satisfying Anonymity, Neutrality, and Availability can say that a defeats b:

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$\boldsymbol{a}$	$\boldsymbol{b}$	c
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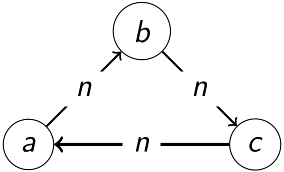

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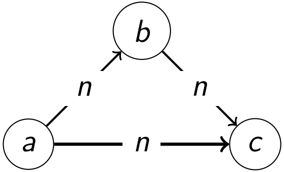

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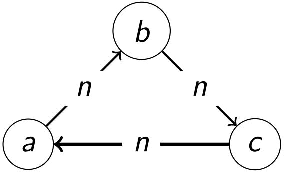

This is a counterexample to IIA as a plausible axiom.

Coherent IIA

Key idea: Move away from the local evaluation of x vs. y when increasing incoherence demands we be more conservative in locking in relations of defeat.

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Although there is a perfectly reasonable notion of the *advantage* of x over y that only depends on how voters rank x vs. y, whether that intrinsic advantage is sufficient for x to *defeat* y may depend on a *standard* that takes into whether the electorate is incoherent with respect to a set of candidates including x, y.

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The advantage-standard idea is formalized in:

W. Holliday and M. Kelley (2022). *Escaping Arrow's Theorem*. arXiv:2108.01134.

Theorem (Patty and Penn, 2014)

Arrow's IIA condition is equivalent to the condition of unilateral flip independence: if two profiles are alike except that one voter flips one pair of adjacent candidates on her ballot, then the defeat relations for the two profiles can differ at most on the flipped candidates.

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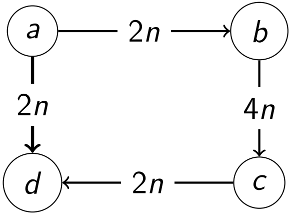
This theorem "demonstrates a fundamental basis of the normative appeal of IIA" (p. 52, Penn and Patty, 2014).

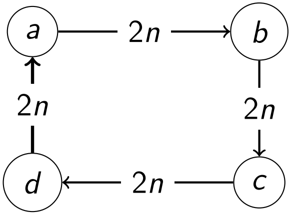
However, unilateral flip independence makes the same mistake as IIA in ignoring how context can affect the standard for defeat (let $n = 1$ and consider the middle voter in the previous example).

Maskin (2020) proposes a weakening of IIA called Modified IIA, which states for all profiles P and P$'$, if $\mathsf{P}_{\{x,y\}} = \mathsf{P}'_{\{x,y\}}$, and for each voter i and candidate z, i ranks z in between x and y in P if and only if i ranks z in between x and y in P$'$, then x defeats y in P if and only if x defeats y in P$'$. (cf. Saari, 1994, 1995, 1998)

n	n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
$\boldsymbol{b}$	c	$\boldsymbol{b}$	$\boldsymbol{b}$
c	d	c	d
d	$\boldsymbol{a}$	d	c



n	n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c	d
$\boldsymbol{b}$	c	d	$\boldsymbol{a}$
c	d	$\boldsymbol{a}$	$\boldsymbol{b}$
d	$\boldsymbol{a}$	$\boldsymbol{b}$	c



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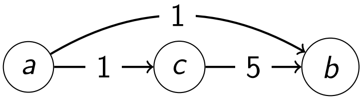
Key idea: the operations described in 2 cannot increase cyclic incoherence.

Note: this is a variable-candidate axiom, so it is best compared to what we call VIIA (see our "Axioms for Defeat in Democratic Elections").

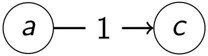
Violations of Coherent IIA

2	3
c	a
b	c
a	b



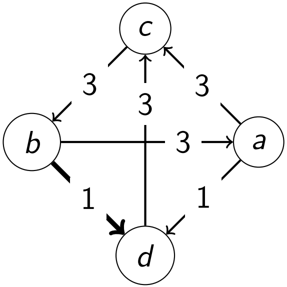
Borda: c defeats a

2	3
c	a
a	c

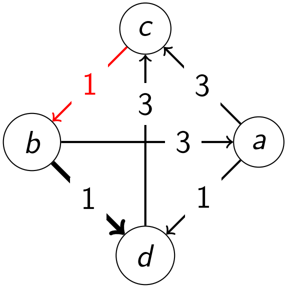


Borda winner: a defeats c

Violations of Coherent IIA



Beat Path: d defeats b



Beat Path: d doesn't defeat b

Coherent IIA and acyclicity

Proposition

Coherent IIA implies Weak IIA.

Coherent IIA and acyclicity

Proposition

Coherent IIA implies Weak IIA.

There is an acyclic VCCR satisfying Coherent IIA: The Split Cycle defeat relation

W. Holliday and EP (2021). *Axioms for Defeat in Democratic Elections*. Forthcoming in Public Choice.

W. Holliday and EP (2022). *Axioms for Defeat in Democratic Elections*. Journal of Theoretical Politics, `https://arxiv.org/pdf/2008.08451.pdf`.

Y. Ding, W. Holliday and EP (2022). *A Full Characterization of Split Cycle*. manuscript.

Social choice correspondence

A **voting method** is a function F on the domain of all profiles such that for any profile P, $\varnothing \neq F(\mathrm{P}) \subseteq X(\mathrm{P})$ (also called a **variable social choice correspondence** VSCC).

- A (V, X)-SCC is a social choice correspondence defined on (V, X)-profiles.
- A voting method F is **resolute** if for all P, $|F(\mathrm{P})| = 1$. Resolute SCCs are called **social choice functions**.

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There are many examples of voting methods.

See `https://pref_voting.readthedocs.io` for a Python package that provides computational tools to study different voting methods.

Positional scoring rules

A *scoring vector* is a vector $\langle s_1, \ldots, s_n \rangle$ of numbers such that for each $m \in \{1, \ldots, n-1\}$, $s_m \geq s_{m+1}$.

Given a profile $\mathbf{P}$ with $|X(\mathbf{P})| = n$, $x \in X(\mathbf{P})$, a scoring vector $\vec{s}$ of length n, and $i \in V(\mathbf{P})$, define $score_{\vec{s}}(x, \mathbf{P}_i) = s_r$ where $r = Rank(x, \mathbf{P}_i)$.

Let $score_{\vec{s}}(x, \mathbf{P}) = \sum_{i \in V(\mathbf{P})} score_{\vec{s}}(x, \mathbf{P}_i)$. A voting method F is a **positional scoring rule** if there is a map $\mathcal{S}$ assigning to each natural number n a scoring vector of length n such that for any profile $\mathbf{P}$ with $|X(\mathbf{P})| = n$,

$$F(\mathbf{P}) = \mathrm{argmax}_{x \in X(\mathbf{P})} score_{\mathcal{S}(n)}(x, \mathbf{P}).$$

Examples

Borda:	$\mathcal{S}(n) = \langle n-1, n-2, \ldots, 1, 0\rangle$
Plurality:	$\mathcal{S}(n) = \langle 1, 0, \ldots, 0\rangle$
Anti-Plurality:	$\mathcal{S}(n) = \langle 1, 1, \ldots, 1, 0\rangle$

1	3	2	4
a	b	b	c
c	a	c	a
b	c	a	b

Borda winner	c
Plurality winner	b
Anti-Plurality winner	a

Iterative procedures: Instant Runoff

- If some alternative is ranked first by an absolute majority of voters, then it is declared the winner.
- Otherwise, the alternative ranked first be the fewest voters (the plurality loser) is eliminated.
- Votes for eliminated alternatives get transferred: delete the removed alternatives from the ballots and “shift” the rankings (e.g., if 1st place alternative is removed, then your 2nd place alternative becomes 1st).

Also known as Ranked-Choice, STV, Hare

How should you deal with ties? (e.g., multiple alternatives are plurality losers)

Iterative procedures

Variants:

- Plurality with runoff: remove all candidates except top two plurality score;
- Coombs: remove candidates with most last place votes;
- Baldwin: remove candidate with smallest Borda score;
- Nanson: remove candidates with below average Borda score

Example

1	1	1	1	1
c	b	a	b	d
a	d	b	c	a
d	a	c	d	b
b	c	d	a	c

Instant Runoff	$\{b\}$
Plurality with Runoff	$\{a, b\}$
Coombs	$\{d\}$
Baldwin	$\{a, b, d\}$
Strict Nanson	$\{a\}$

Condorcet criteria

The **Condorcet winner** in a profile P is a candidate $x \in X(\mathrm{P})$ that is the maximum of the majority ordering, i.e., for all $y \in X(\mathrm{P})$, if $x \neq y$, then $\mathit{Margin}_{\mathrm{P}}(x, y) > 0$.

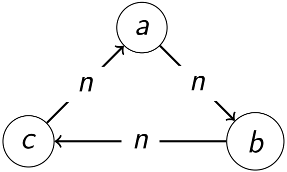
The **Condorcet loser** in a profile P is a candidate $x \in X(\mathrm{P})$ that is the minimum of the majority ordering, i.e., for all $y \in X(\mathrm{P})$, if $x \neq y$, then $\mathit{Margin}_{\mathrm{P}}(y, x) < 0$.

A voting method F is **Condorcet consistent**, if for all P, if x is a Condorcet winner in P, then $F(\mathrm{P}) = \{x\}$.

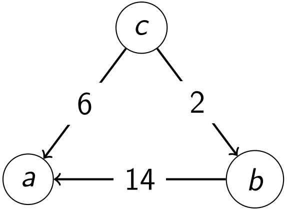
A voting method F is susceptible to the **Condorcet loser paradox** (also known as *Borda's paradox*) if there is some P such that x is a Condorcet loser in P and $x \in F(\mathrm{P})$.

Condorcet paradox

n	n	n
a	b	c
b	c	a
c	a	b



20	13	21	14	22	10
a	a	c	b	b	c
c	b	b	a	c	a
b	c	a	c	a	b



Condorcet winner: c

Instant Runoff winner: b

Plurality winner: b

Borda winner: b

Theorem (Smith 1973, Young 1974)

A voting method satisfies Anonymity, Neutrality and **Reinforcement** *if and only if F is a scoring rule.*

Saari's argument, Balinski and Laraki (2010, pg. 77); Zwicker (2016, Proposition 2.5): Multiple districts paradox, *f cancels properly*.

2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c

- no Condorcet winner in the left profile
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

Condorcet consistent voting methods

- Minimax
- Copeland
- Beat Path
- Ranked Pairs
- Split Cycle

Minimax: For a profile P, The Minimax winners in P are:

$$\text{argmin}_{x \in X(\mathrm{P})} \max\{\mathit{Margin}_{\mathrm{P}}(y, x) \mid y \in X(\mathrm{P})\}$$

Copeland/**Llull**: For $\alpha \in [0, 1]$, the Copeland$_\alpha$ score of a in P is the number of $b \in X(\mathrm{P})$ such that $\mathit{Margin}_{\mathrm{P}}(a, b) > 0$ plus α times the number of $b \in X(\mathrm{P})$ such that $\mathit{Margin}_{\mathrm{P}}(a, b) = 0$. *Copeland*(P) (resp. *Llull*(P)) is the set of candidates with maximal Copeland$_{1/2}$ (resp. $\mathit{Copeland}_1$) score in P.

Schulze Beat Path

For $a, b \in X(\mathrm{P})$, a *path from a to b in* P is a sequence $\rho = x_1, \ldots, x_n$ of distinct candidates in $X(\mathrm{P})$ with $x_1 = a$ and $x_n = b$ such that for $1 \leq k \leq n-1$, $Margin_{\mathrm{P}}(x_k, x_{k+1}) > 0$.

The *strength of* ρ is $\min\{Margin_{\mathrm{P}}(x_k, x_{k+1}) \mid 1 \leq k \leq n-1\}$.

Then a defeats b in P according to Beat Path if the strength of the strongest path from a to b is greater than the strength of the strongest path from b to a.

$BP(\mathrm{P})$ is the set of undefeated candidates.

Tideman Ranked Pairs, I

For a profile P and $T \in \mathcal{L}\big(\{(x, y) \mid x \neq y \text{ and } Margin_{\mathrm{P}}(x, y) \geq 0\}\big)$, called the *tie-breaking ordering*

A pair (x, y) of candidates has a *higher priority* than a pair (x', y') of candidates according to T when either $Margin_{\mathrm{P}}(x, y) > Margin_{\mathrm{P}}(x', y')$ or $Margin_{\mathrm{P}}(x, y) = Margin_{\mathrm{P}}(x', y')$ and $(x, y)\ T\ (x', y')$.

Tideman Ranked Pairs, II

We construct a *Ranked Pairs ranking* $\succ_{\mathrm{P},T} \in \mathcal{L}(X)$ as follows:

1. Initialize $\succ_{\mathrm{P},T}$ to $\varnothing$.
2. If all pairs (x, y) with $x \neq y$ and $Margin_{\mathrm{P}}(x, y) \geq 0$ have been considered, then return $\succ_{\mathrm{P},T}$. Otherwise let (a, b) be the pair with the highest priority among those with $a \neq b$ and $Margin_{\mathrm{P}}(a, b) \geq 0$ that have not been considered so far.
3. If $\succ_{\mathrm{P},T} \cup \{(a, b)\}$ is acyclic, then add (a, b) to $\succ_{\mathrm{P},T}$; otherwise, add (b, a) to $\succ_{\mathrm{P},T}$. Go to step 2.

When the procedure terminates, $\succ_{\mathrm{P},T}$ is a linear order.

The set $RP(\mathrm{P})$ of Ranked Pairs winners is the set of all $x \in X(\mathrm{P})$ such that x is the maximum of $\succ_{\mathrm{P},T}$ for some tie-breaking ordering T.

Split Cycle

Split Cycle defeat: a candidate a defeats a candidate b just in case

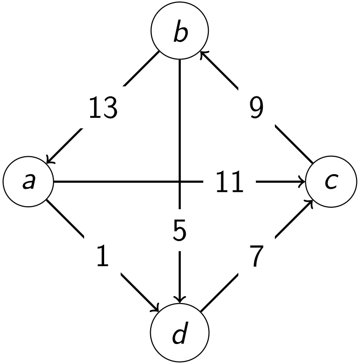
- the majority margin of a over b is greater than 0, and
- for every majority cycle containing a and b, the margin of a over b is greater than the smallest margin between consecutive candidates in the cycle.

The Split Cycle winners are the undefeated candidates.

An intuitive way defeat relation is as follows:

1. In each majority cycle, identify the wins with the smallest margin in that cycle.
2. After completing step 1 for all cycles, discard the identified wins. All remaining wins count as defeats.

Example



Minimax: $\{d\}$
Copeland: $\{a, b\}$
Beat Path: $\{d\}$
Ranked Pairs: $\{b\}$
Split Cycle: $\{b, d\}$

We are interested in voting methods that:

1. respond in a reasonable way to **new candidates** joining the election;
2. respond in a reasonable way to **new voters** joining the election.

Key idea: Unequivocal increase in support for a candidate should not result in that candidate going from being a winner to being a loser.

1. *monotonicity*: if a candidate x is a winner given a preference profile P, and P′ is obtained from P by one voter moving x up in their ranking, then x should still be a winner given P′.
   (fixed-electorate axiom)

2. *positive involvement*: if a candidate x is a winner given P, and P^* is obtained from P by adding a new voter who ranks x in first place, then x should still be a winner given P^*.
   (variable-electorate axiom)

More-is-Less Paradox: Instant Runoff

6	5	4	2
a	c	b	b
b	a	c	a
c	b	a	c

6	5	4	2
a	c	b	a
b	a	c	b
c	b	a	c

More-is-Less Paradox: Instant Runoff

6	5	4	2
a	c	b	b
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c	b	a	c

More-is-Less Paradox: Instant Runoff

6	5	4	2
a	c	b	b
b	a	c	a
c	b	a	c

Instant Runoff Winner: a

6	5	4	2
a	c	b	a
b	a	c	b
c	b	a	c

More-is-Less Paradox: Instant Runoff

6	5	4	2
a	c	b	b
b	a	c	a
c	b	a	c

Instant Runoff Winner: a

6	5	4	2
a	c	b	a
b	a	c	b
c	b	a	c

Instant Runoff Winner: c

More-is-Less Paradox: Instant Runoff

6	5	4	2
a	c	b	b
b	a	c	a
c	b	a	c

Instant Runoff Winner: a

6	5	4	2
a	c	b	a
b	a	c	b
c	b	a	c

Instant Runoff Winner: c

Violating Positive Involvement: Coombs

2	2	1	1	2	1	1
c	b	d	d	c	a	b
a	a	c	a	b	d	d
b	c	b	c	d	b	a
d	d	a	b	a	c	c

Coombs winner: $\{b\}$

(the order of elimination is d, c)

2	2	1	1	2	1	1	1
c	b	d	d	c	a	b	b
a	a	c	a	b	d	d	d
b	c	b	c	d	b	a	c
d	d	a	b	a	c	c	a

Coombs winner: $\{c\}$

(a and d are tied for the most last place votes)

Breaking Ties

There are many tiebreaking rules: non-anonymous, non-neutral, random

Parallel universe tiebreaking: x is a winner if x wins according to some tiebreaking rule.

S. Obraztsova, E. Elkind and N. Hazon. *Ties Matter: Complexity of Voting Manipulation Revisited*. Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence.

J. Wang, S. Sikdar, T. Shepherd, Z. Zhao, C. Jiang and L. Xia. *Practical Algorithms for Multi-Stage Voting Rules with Parallel Universes Tiebreaking*. Proceedings of AAAI, 2019.

Violating Positive Involvement: Coombs PUT

1	1	1	1	1
a	c	b	c	d
c	d	d	a	b
b	b	a	b	a
d	a	c	d	c

Coombs winner: $\{a, b\}$

1	1	1	1	1	1
a	c	b	c	d	a
c	d	d	a	b	d
b	b	a	b	a	b
d	a	c	d	c	c

Coombs winner: $\{b, d\}$

No Show Paradox

The term "No Show Paradox" was introduced by Fishburn and Brams for violations of what is now called *negative involvement*: Adding a new voter who ranks a candidate last should not result in the candidate going from being a loser to a winner.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine, 56(4), pp. 207 - 214, 1983.

D. Saari. *Basic Geometry of Voting*. Springer, 1995.

No Show Paradox

Moulin changed the meaning of "No Show Paradox" to refer to violations of participation: A resolute voting method satisfies participation if adding a new voter who ranks x above y cannot result in a change from x being the unique winner to y being the unique winner.

H. Moulin. *Condorcet's Principle Implies the No Show Paradox*. Journal of Economic Theory 45(1), pp. 53 - 64, 1988.

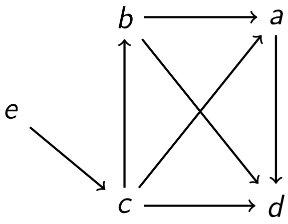
No Show Paradox

Peréz concludes that the Strong No Show Paradox is a common flaw of many *Condorcet consistent* voting methods, which are methods that always pick a Condorcet winner—a candidate who is majority preferred to every other candidate—if one exists.

J. Pérez. *The Strong No Show Paradoxes are a common flaw in Condorcet voting correspondences*. Social Choice and Welfare 18(3), pp. 601 - 616, 2001.

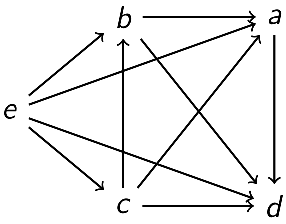
Violating Positive Involvement: Copeland

2	1	1
e	c	a
c	b	d
b	a	b
a	d	e
d	e	c



Copeland winners: $\{c\}$

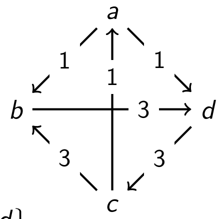
2	1	1	
e	c	a	c
c	b	d	e
b	a	b	d
a	d	e	c
d	e	c	a



Copeland winners: $\{e\}$

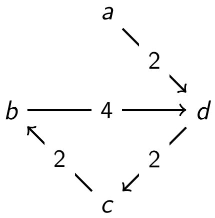
Violating Positive Involvement: Beat Path

1	1	1	1	2	1	1	1	1	1
a	d	c	c	a	b	a	d	d	b
d	c	a	b	c	d	b	c	b	a
c	b	b	d	b	c	d	a	c	d
b	a	d	a	d	a	c	b	a	c



Beat Path winners: $\{a, b, c, d\}$

1	1	1	1	2	1	1	1	1	1	1
a	d	c	c	a	b	a	d	d	b	b
d	c	a	b	c	d	b	c	b	a	a
c	b	b	d	b	c	d	a	c	d	c
b	a	d	a	d	a	c	b	a	c	d



Beat Path winners: $\{a\}$

A logic for resolute social choice correspondences

G. Ciná and U. Endriss. *Proving classical theorems of social choice theory in modal logic*. Autonomous Agents and Multi-Agent Systems, 30, pp. 963 - 989, 2016.

N. Troquard, W. van der Hoek, and M. Wooldridge. *Reasoning about social choice functions*. Journal of Philosophical Logic 40(4), 473 - 498 (2011).

T. Agotnes, W. van der Hoek, and M. Wooldridge. *On the logic of preference and judgment aggregation*. Journal of Autonomous Agents and Multiagent Systems 22(1), 4 - 30 (2011).

Language

Atomic Propositions:

- $Pref[V, X] := \{p^i_{x \succeq y} \mid i \in V, x, y \in X\}$ is the set of preference atomic propositions, where $p^i_{x \preceq y}$ means i prefers y to x.
- Each $x \in X$ is an atomic proposition.

Modality:

- $\Diamond_C \varphi$: C can *ensure* the truth of φ.

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Modality:

- $\Diamond_C \varphi$: C can *ensure* the truth of φ.

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond_C \varphi$$

Model

A **model** is a triple $M = \langle N, X, F \rangle$, consisting of a finite set of agents N (with $n = |N|$), a finite set of alternatives X, and a resolute SCC $F : \mathcal{L}(X)^V \to X$.

A **world** is a profile $(\mathrm{P}_1, \ldots, \mathrm{P}_n)$

Truth

Let $w = (\mathrm{P}_1, \ldots, \mathrm{P}_n)$

- $M, w \models p^i_{x \succeq y}$ iff $x\mathrm{P}_i y$
- $M, w \models x$ if and only if $F(\mathrm{P}_1, \ldots, \mathrm{P}_n) = x$
- $M, w \models \neg\varphi$ if and only if $M, w \not\models \varphi$
- $M, w \models \varphi \wedge \psi$ if and only if $M, w \models \varphi$ and $M, w \models \psi$
- $M, w \models \Diamond_C \varphi$ if and only if $M, w' \models \varphi$ for some $w' = (\mathrm{P}'_1, \ldots, \mathrm{P}'_n)$ with $\mathrm{P}_j = \mathrm{P}'_j$ for all $j \in N - C$.

(1) $p^i_{x \succeq x}$

(2) $p^i_{x \succeq y} \leftrightarrow \neg p^i_{y \succeq x}$ for $x \neq y$

(3) $p^i_{x \succeq y} \wedge p^i_{y \succeq y} \rightarrow p^i_{x \succeq z}$

(1) $p^i_{x \succeq x}$

(2) $p^i_{x \succeq y} \leftrightarrow \neg p^i_{y \succeq x}$ for $x \neq y$

(3) $p^i_{x \succeq y} \wedge p^i_{y \succeq y} \rightarrow p^i_{x \succeq z}$

$$ballot_i(w) \;=\; p^i_{x_1 \succeq x_2} \wedge \cdots \wedge p^i_{x_{m-1} \succeq x_m}$$

$$profile(w) \;=\; ballot_1(w) \wedge \cdots \wedge ballot_n(w)$$

(4) all propositional tautologies

(5) $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$ (K(i))

(6) $\Box_i\varphi \to \varphi$ (T(i))

(7) $\varphi \to \Box_i\Diamond_i\varphi$ (B(i))

(8) $\Diamond_i\Box_j\varphi \leftrightarrow \Box_j\Diamond_i\varphi$ (confluence)

(9) $\Box_{C_1}\Box_{C_2}\varphi \leftrightarrow \Box_{C_1\cup C_2}\varphi$ (union)

(10) $\Box_\emptyset\varphi \leftrightarrow \varphi$ (empty coalition)

(11) $(\Diamond_i p \wedge \Diamond_i\neg p) \to (\Box_j p \vee \Box_j\neg p)$, where $i \neq j$ (exclusiveness)

(12) $\Diamond_i ballot_i(w)$ (ballot)

(13) $\Diamond_{C_1}\delta_1 \wedge \Diamond_{C_2}\delta_2 \to \Diamond_{C_1\cup C_2}(\delta_1 \wedge \delta_2)$ (cooperation)

(14) $\bigvee_{x\in X}(x \wedge \bigwedge_{y\in X\setminus\{x\}}\neg y)$ (resoluteness)

(15) $(profile(w) \wedge \varphi) \to \Box_N(profile(w) \to \varphi)$ (functionality)

Theorem (Cinà and Endriss) The logic $L[V, X]$ is sound and complete w.r.t. the class of models of resolute social choice correpsondences.

Pareto

$$Par := \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} \left[\left(\bigwedge_{i \in N} p^i_{x \succeq y} \right) \to \neg y \right]$$

IIA

$$IIA \;:= \bigwedge_{w \in \mathcal{L}(X)^n} \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} [\Diamond_V(\mathit{profile}(w) \wedge x) \rightarrow (\mathit{profile}(w)(x,y) \rightarrow \neg y)]$$

- $N^w_{x \succeq y} = \bigwedge \{p^i_{x \succeq y} \mid x \mathrm{P}_i y \text{ in } w\}$
- $\mathit{profile}(w)(x,y) := N^w_{x \succeq y} \wedge N^w_{y \succeq x}$

Dictatorship

$$Dic \quad := \bigvee_{i \in N} \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} (p^i_{x \succeq y} \rightarrow \neg y)$$

Arrow's Theorem

Theorem (Ciná and Endriss) Consider a logic $L[V, X]$ with a language parameterised by X such that $|X| > 3$. Then we have:

$$\vdash Par \land IIA \rightarrow Dic$$

Strong Monotonicity

$$SM := \bigwedge_{w \in \mathcal{L}(X)^n} \bigwedge_{x \in X} \left[\Diamond_V (profile(w) \wedge x) \wedge \left(\bigwedge_{y \in X \setminus \{x\}} N^w_{x \succeq y} \right) \to x \right]$$

Surjectivity

$$Sur \; := \bigwedge_{x \in X} \; \bigwedge_{w \in \mathcal{L}(X)^V} \Diamond_V(\mathit{profile}(w) \wedge x)$$

Theorem (Cinà and Endriss) Consider a logic $L[V, X]$ with a language parameterised by X such that $|X| \geq 3$. Then we have:

$$\vdash SM \wedge Sur \rightarrow Dic$$