# Reasoning in Games

## Lecture 3

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# Plan

✓ Day 1: Decision Theory

✓ Day 2: From Decisions to Games

- Day 3: Game Models
- Day 4: Modeling Deliberation (in Games)
- Day 5: Backward and Forward Induction, Concluding Remarks (Language-Based Games/ Variable Frame Theory, Behavioral Game Theory, . . . )

| | | B | |
|---|---|---|---|
| | | *L* | *R* |
| A | *U* | 2,1 | 0,0 |
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Strategic Game

| | | B: $L$ | B: $R$ |
|---|---|---|---|
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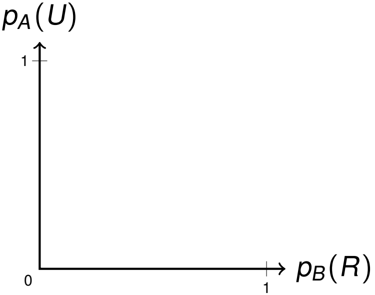
Strategic Game




Solution Space

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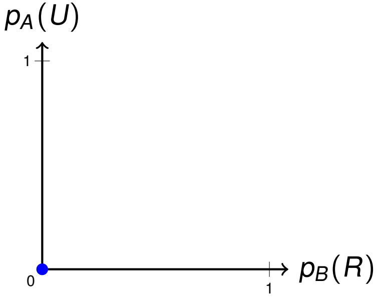


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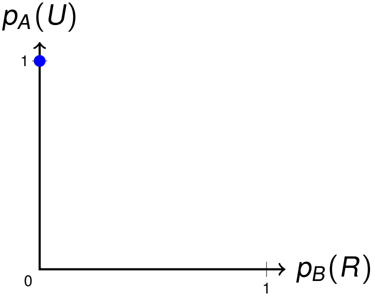


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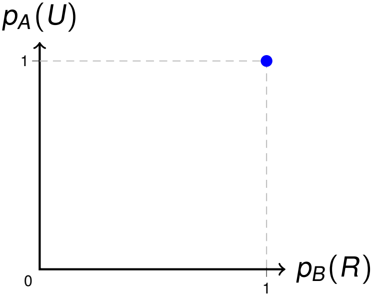


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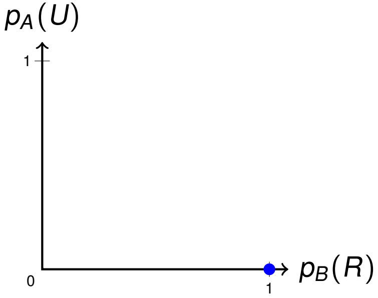


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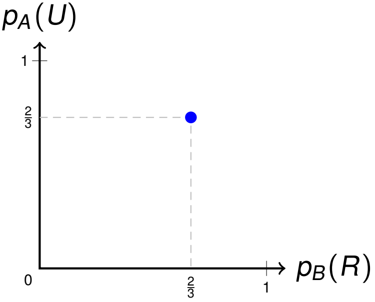


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- A **model of a game** is a completion of the partial specification of the Bayesian decision problems *and* a representation of a particular play of the game.

- There are no special rules of rationality telling one what to do in the absence of degrees of belief except: decide what you believe, and then **maximize (subjective) expected utility**.

# Models of Games

Suppose that $G$ is a game.

- Outcomes of the game: $S = \Pi_{i \in N} S_i$
- A profile is a vector $\vec{s} \in S$, specifying an action for each player
- Player $i$'s partial beliefs (or conjecture): $P_i \in \Delta(S_{-i})$

  $\Delta(X)$ *is the set of probabilities measures over* $X$

# Models of Games, continued

$G = \langle N, (S_i, u_i)_{i \in N} \rangle$ is a strategic (form of a) game.

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- $W$ is a set of *possible worlds* (possible outcomes of the game)
- $\mathbf{s}$ is a function $\mathbf{s} : W \to \Pi_{i \in N} S_i$, write $\mathbf{s}_i(w)$ for the *i*th component of $\mathbf{s}(w)$

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- If $\vec{s} \in \Pi_{i \in N} S_i$, then $[\vec{s}] = \{w \mid \mathbf{s}(w) = \vec{s}\}$; if $s_i \in S_i$, then $[s_i] = \{w \mid \mathbf{s}_i(w) = s_i\}$; and if $X \subseteq S$, $[X] = \bigcup_{s \in X} [s]$.

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- ***ex ante* beliefs**: For each $i \in N$, let $P_i \in \Delta(W)$ (the set of probability measures on $W$). Two assumptions:
  - $[s]$ is measurable for all strategy profiles $s \in S$
  - $P_i([s_i]) > 0$ for all $s_i \in S_i$

***ex interim*** **beliefs**: $P_{i,w} \in \Delta(S_{-i})$

- ...given player $i$'s choice: $P_{i,w}(\cdot) = P_i(\cdot \mid [\mathbf{s}_i(w)])$
- ...given all player $i$ *knows*: $P_{i,w}(\cdot) = P_i(\cdot \mid K_i)$, $\quad K_i \subseteq [\mathbf{s}_i(w)]$
- ...given all player $i$ *fully believes*: $P_{i,w}(\cdot) = P_i(\cdot \mid B_i)$, $\quad B_i \subseteq [\mathbf{s}_i(w)]$

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**Expected utility of strategy** $s_i \in S_i$: Given $P \in \Delta(S_{-i})$,

$$EU_{i,P}(s_i) = \sum_{s_{-i} \in S_{-i}} P(s_{-i}) u_i(s_i, s_{-i})$$

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# An Example

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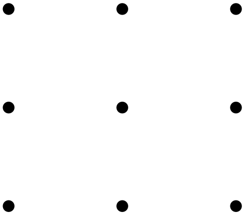

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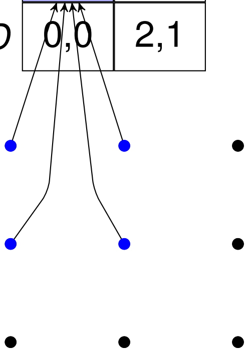

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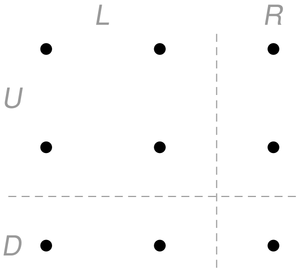

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| | | |
| --- | --- | --- |
| $\frac{1}{6}$ • | $\frac{1}{6}$ • | 0 • |
| $\frac{1}{6}$ • | 0 • | $\frac{1}{6}$ • |
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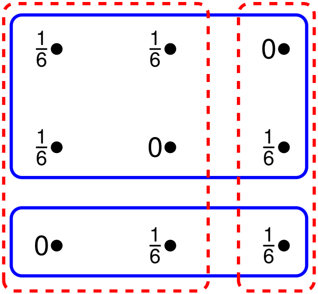

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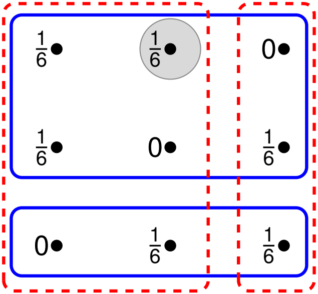

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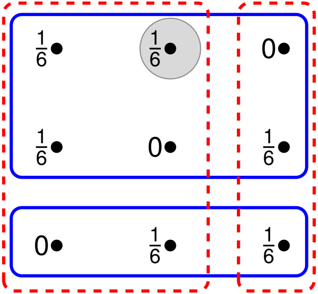


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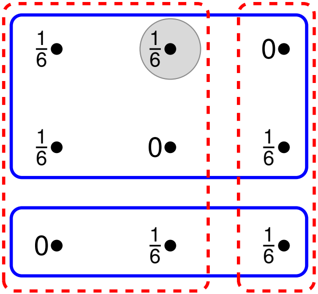


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$$1 \cdot P_A(L) + 0 \cdot P_A(R) \geq 0 \cdot P_A(L) + 2 \cdot P_A(R)$$

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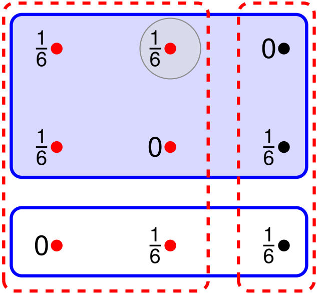


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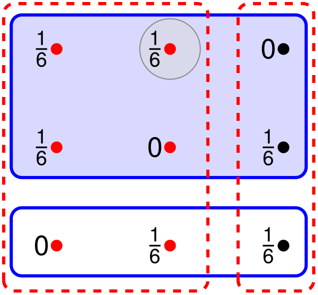


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$$1 \cdot \frac{3}{4} + 0 \cdot P_A(R) \geq 0 \cdot \frac{3}{4} + 2 \cdot P_A(R)$$

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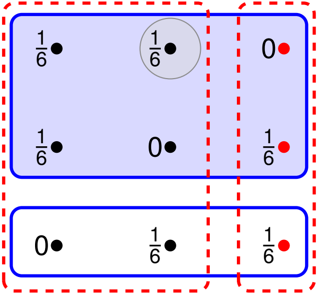


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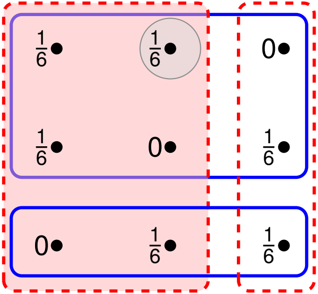


$$\color{blue}{2} \cdot \frac{3}{4} + \color{blue}{0} \cdot \frac{1}{4} \geq \color{red}{0} \cdot \frac{3}{4} + \color{red}{1} \cdot \color{blue}{\frac{1}{4}}$$

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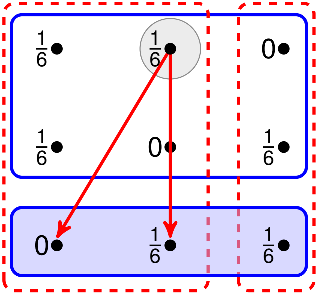


- Ann's choice is *optimal* (given her information)
- Bob's choice is *optimal* (given her information)
- Bob *considers it possible* Ann is *irrational*

$$1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \not\geq 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}$$

For any $P \in \Delta(S_{-i})$ and $s_i \in S_i$, $EU_{i,P}(s_i) = \sum_{s_{-i} \in S_{-i}} P(s_{-i}) u_i(s_i, s_{-i})$

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Each $P \in \Delta(W)$ is associated with $P^S \in \Delta(S)$ as follows: for all $s \in S$, $P^S(s) = P([s])$

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A mixed strategy $\sigma \in \Pi_{i \in N} \Delta(S_i)$, $P_\sigma \in \Delta(S)$, $P_\sigma(s) = \sigma_1(s_1) \cdots \sigma_n(s_n)$

# Characterizing Nash Equilibria

**Theorem** (Aumann). $\sigma$ is a Nash equilibrium of $G$ iff there exists a model $\mathcal{M}^G = \langle W, (P_i)_{i\in N}, \mathbf{s}\rangle$ such that:

- for all $i \in N$, $\mathrm{Rat}_i = W$;
- for all $i, j \in N$, $P_i = P_j$; and
- for all $i \in N$, $P_i^S = P_\sigma$.

# Correlation

**Correlation**: Players can improve their expected value by correlating their choices on an "outside signal"

# Correlated Strategies

| | $L$ | $R$ |
|---|---|---|
| $U$ | 2, 1 | 0, 0 |
| $D$ | 0, 0 | 1, 2 |

- Three Nash equilibria:
  - $(U, L)$: the payoff is $(2, 1)$
  - $(D, R)$: the payoff is $(1, 2)$
  - $([\frac{2}{3}(U), \frac{1}{3}D], [\frac{1}{3}(L), \frac{2}{3}(R)])$: the payoff is $(\frac{2}{3}, \frac{2}{3})$

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| | $L$ | $R$ |
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- Mixed Strategies: Each player conducts a private, independent lottery to choose their strategy.
- Conduct a *public* lottery: flip a fair coin and follow the strategy $(H \Rightarrow (U, L), T \Rightarrow (D, R))$. The payoff is $(1.5, 1.5)$.

Two extremes:

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2. A single, completely public lottery

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What about: a public lottery, but reveal only partial information about the outcome to each of the players?

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|---|---|---|
| $U$ | 6, 6 | 2, 7 |
| $D$ | 7, 2 | 0, 0 |

- Three Nash equilibria:
  - $(U, R)$: the payoff is $(2, 7)$
  - $(D, L)$: the payoff is $(7, 2)$
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|---|---|---|
| $U$ | 1/3 | 1/3 |
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- After conducting the lottery, an outside observer provides Ann with a recommendation to play the first component of the profile that was chosen, and Bob the second component.
- The expected payoff is $\frac{1}{3}(6, 6) + \frac{1}{3}(2, 7) + \frac{1}{3}(7, 2) = (5, 5)$ (which is outside the convex hull of the Nash equilibria)

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# Correlation

**Correlation**: Players can improve their expected value by correlating their choices on an "outside signal"

With more than 2 players...

- A player may believe that (some of) the other players strategy choices are **independent** or **correlated**.
- Two players can **agree** or **disagree** on the probabilities that the assign to a third player's choice of strategy.

# Characterizing Correlated Equilibrium

**Theorem** (Aumann). $\sigma$ is a correlated equilibrium of $G$ iff there exists a model $\mathcal{M}^G = \langle W, (P_i)_{i \in N}, \mathbf{s} \rangle$ such that:

- for all $i \in N$, $\text{Rat}_i = W$; and
- for all $i \in N$, $P_i^S = \sigma$.

# Rationalizability

A **best reply set** (BRS) is a sequence $(B_1, B_2, \ldots, B_n) \subseteq S = \Pi_{i \in N} S_i$ such that for all $i \in N$, and all $s_i \in B_i$, there exists $\mu_{-i} \in \Delta(B_{-i})$ such that $s_i$ is a best response to $\mu_{-i}$: I.e.,

$$b_i = \arg\max_{s_i \in S_i} EU_{i,\mu_{-i}}(s_i)$$

.

| | | 2 | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
| | $a_1$ | 0, 7 | 2, 5 | 7, 0 | 0, 1 |
| 1 | $a_2$ | 5, 2 | 3, 3 | 5, 2 | 0, 1 |
| | $a_3$ | 7, 0 | 2, 5 | 0, 7 | 0, 1 |
| | $a_4$ | 0, 0 | 0, -2 | 0, 0 | 10, -1 |

| | | 2 | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
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- $(a_2, b_2)$ is the unique Nash equilibria, hence $(\{a_2\}, \{b_2\})$ is a BRS

| | | 2 | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
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- $(a_2, b_2)$ is the unique Nash equilibria, hence $(\{a_2\}, \{b_2\})$ is a BRS
- $(\{a_1, a_3\}, \{b_1, b_3\})$ is a BRS

| | | 2 | | | |
|---|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
| | $a_1$ | 0, 7 | 2, 5 | 7, 0 | 0, 1 |
| 1 | $a_2$ | 5, 2 | 3, 3 | 5, 2 | 0, 1 |
| | $a_3$ | 7, 0 | 2, 5 | 0, 7 | 0, 1 |
| | $a_4$ | 0, 0 | 0, -2 | 0, 0 | 10, -1 |

- $(a_2, b_2)$ is the unique Nash equilibria, hence $(\{a_2\}, \{b_2\})$ is a BRS
- $(\{a_1, a_3\}, \{b_1, b_3\})$ is a BRS
- $(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ is a full BRS

**Theorem** (Bernheim; Pearce; Brandenburger and Dekel; ...).
$(B_1, B_2, \ldots, B_n)$ is a BRS for $G$ iff there exists a model $\mathcal{M}^G = \langle W, (P_i)_{i\in N}, \mathbf{s}\rangle$ such that for all $i \in N$, $\mathrm{Rat}_i = W$ and $[B_1 \times \cdots \times B_n] = W$.

| | | Bob | |
|---|---|---|---|
| | | $L$ | $R$ |
| Ann | $U$ | 2,2 | 4,1 |
| | $D$ | 1,4 | 3,3 |

Game 1

| | | Bob | |
|---|---|---|---|
| | | $L$ | $R$ |
| Ann | $U$ | 2,1 | 1,0 |
| | $D$ | 1,0 | 0,1 |

Game 2

Bob

| Ann | *L* | *R* |
|---|---|---|
| *U* | 2,2 | 4,1 |
| *D* | 1,4 | 3,3 |

Game 1

Bob

| Ann | *L* | *R* |
|---|---|---|
| *U* | 2,1 | 1,0 |
| *D* | 1,0 | 0,1 |

Game 2

**Game 1**: *U* strictly dominates *D* and *L* strictly dominates *R*.

| | Bob: $L$ | Bob: $R$ |
|---|---|---|
| Ann: $U$ | 2,2 | 4,1 |
| Ann: $D$ | 1,4 | 3,3 |

Game 1

| | Bob: $L$ | Bob: $R$ |
|---|---|---|
| Ann: $U$ | 2,1 | 1,0 |
| Ann: $D$ | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ strictly dominates $D$ and $L$ strictly dominates $R$.

**Game 2**: $U$ strictly dominates $D$

| Ann \ Bob | *L* | *R* |
|---|---|---|
| *U* | 2,2 | 4,1 |
| *D* | 1,4 | 3,3 |

Game 1

| Ann \ Bob | *L* | *R* |
|---|---|---|
| *U* | 2,1 | 1,0 |
| *D* | 1,0 | 0,1 |

Game 2

**Game 1**: *U* strictly dominates *D* and *L* strictly dominates *R*.

**Game 2**: *U* strictly dominates *D*, and *after removing D*, *L* strictly dominates *R*.

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 2,2 | 4,1 |
| $D$ | 1,4 | 3,3 |

Game 1

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 2,1 | 1,0 |
| $D$ | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ strictly dominates $D$ and $L$ strictly dominates $R$.

**Game 2**: $U$ strictly dominates $D$, and *after removing* $D$, $L$ strictly dominates $R$.

**Theorem**. In all models where the players are *rational* and there is *common belief of rationality*, the players choose strategies that survive iterative removal of strictly dominated strategies (and, conversely...).

# Comparing Dominance Reasoning and MEU

$G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
$X \subseteq S_{-i}$ (a set of strategy profiles for all players except $i$)

# Comparing Dominance Reasoning and MEU

$G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$
$X \subseteq S_{-i}$ (a set of strategy profiles for all players except $i$)

$s, s' \in S_i$, $s$ **strictly dominates** $s'$ with respect to $X$ provided

$$\forall s_{-i} \in X, \quad u_i(s, s_{-i}) > u_i(s', s_{-i})$$

# Comparing Dominance Reasoning and MEU

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$s, s' \in S_i$, $s$ **strictly dominates** $s'$ with respect to $X$ provided

$$\forall s_{-i} \in X, \quad u_i(s, s_{-i}) > u_i(s', s_{-i})$$

$p \in \Delta(X)$, $s$ is a **best response** to $p$ with respect to $X$ provided

$$\forall s' \in S_i, \quad EU(s, p) \geq EU(s', p)$$

| Ann | | Bob: $L$ | $R$ |
|---|---|---|---|
| | $U$ | 5, * | 1, * |
| | $M$ | 1, * | 5, * |
| | $D$ | 2, * | 2, * |

$D$ is strictly dominated by $(0.5U, 0.5M)$.

# Strict Dominance and MEU

**Proposition**. Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategic game and $X \subseteq S_{-i}$. A strategy $s_i \in S_i$ is strictly dominated (possibly by a mixed strategy) with respect to $X$ iff there is no probability measure $p \in \Delta(X)$ such that $s_i$ is a best response to $p$.

# Important Issue: Correlated Beliefs

| $x$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,3 | 1,0,3 |
| $d$ | 0,1,0 | 0,0,0 |

| $y$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,2 | 1,0,0 |
| $d$ | 0,1,0 | 1,1,2 |

| $z$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,0 | 1,0,0 |
| $d$ | 0,1,3 | 0,0,3 |

# Important Issue: Correlated Beliefs

| $x$ | $l$ | $r$ |
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| $u$ | 1,1,2 | 1,0,0 |
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| $z$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,0 | 1,0,0 |
| $d$ | 0,1,3 | 0,0,3 |

- Note that $y$ is not strictly dominated for Charles.

# Important Issue: Correlated Beliefs

| $x$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,3 | 1,0,3 |
| $d$ | 0,1,0 | 0,0,0 |

| $y$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,2 | 1,0,0 |
| $d$ | 0,1,0 | 1,1,2 |

| $z$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,0 | 1,0,0 |
| $d$ | 0,1,3 | 0,0,3 |

- Note that $y$ is not strictly dominated for Charles.
- It is easy to find a probability measure $p \in \Delta(S_A \times S_B)$ such that $y$ is a best response to $p$. Suppose that $p(u,l) = p(d,r) = \frac{1}{2}$. Then, $EU(x,p) = EU(z,p) = 1.5$ while $EU(y,p) = 2$.

# Important Issue: Correlated Beliefs

| $x$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,3 | 1,0,3 |
| $d$ | 0,1,0 | 0,0,0 |

| $y$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,2 | 1,0,0 |
| $d$ | 0,1,0 | 1,1,2 |

| $z$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,0 | 1,0,0 |
| $d$ | 0,1,3 | 0,0,3 |

- Note that $y$ is not strictly dominated for Charles.
- It is easy to find a probability measure $p \in \Delta(S_A \times S_B)$ such that $y$ is a best response to $p$. Suppose that $p(u, l) = p(d, r) = \frac{1}{2}$. Then, $EU(x, p) = EU(z, p) = 1.5$ while $EU(y, p) = 2$.
- However, there is no probability measure $p \in \Delta(S_A \times S_B)$ such that $y$ is a best response to $p$ and $p(u, l) = p(u) \cdot p(l)$.

| $x$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,3 | 1,0,3 |
| $d$ | 0,1,0 | 0,0,0 |

| $y$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,2 | 1,0,0 |
| $d$ | 0,1,0 | 1,1,2 |

| $z$ | $l$ | $r$ |
|---|---|---|
| $u$ | 1,1,0 | 1,0,0 |
| $d$ | 0,1,3 | 0,0,3 |

- To see this, suppose that $a$ is the probability assigned to $u$ and $b$ is the probability assigned to $l$. Then, we have:
  - The expected utility of $y$ is $2ab + 2(1 - a)(1 - b)$;
  - The expected utility of $x$ is $3ab + 3a(1 - b) = 3a(b + (1 - b)) = 3a$; and
  - The expected utility of $z$ is $3(1 - a)b + 3(1 - a)(1 - b) = 3(1 - a)(b + (1 - b)) = 3(1 - a)$.

Let $P \in \Delta(X)$ be a probability measure, the **support** of $P$ is $\text{supp}(P) = \{x \in X \mid P(x) > 0\}$.

A probability measure $P \in \Delta(X)$ is said to be a **full support** probability measure on $X$ provided $\text{supp}(P) = X$.

|  |  | Bob |  |
| --- | --- | --- | --- |
|  |  | $L$ | $R$ |
| Ann | $U$ | 3,3 | 1,1 |
|  | $D$ | 2,2 | 2,2 |

Is $R$ rationalizable?

| | | Bob | |
|---|---|---|---|
| | | $L$ | $R$ |
| Ann | $U$ | 3,3 | 1,1 |
| | $D$ | 2,2 | 2,2 |

Is $R$ rationalizable?

There is no *full support* probability such that $R$ is a best response

| | | Bob | |
|---|---|---|---|
| | | $L$ | $R$ |
| Ann | $U$ | 3,3 | 1,1 |
| | $D$ | 2,2 | 2,2 |

Is $R$ rationalizable?

There is no *full support* probability such that $R$ is a best response

Should Ann assign probability 0 to $R$ or probability $>0$ to $R$?

# Strategic Reasoning and Admissibility

"The argument for deletion of a weakly dominated strategy for player *i* is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur."

Mas-Colell, Whinston and Green. *Introduction to Microeconomics*. 1995.

# A Puzzle

R. Cubitt and R. Sugden. *Rationally Justifiable Play and the Theory of Non-cooperative games*. Economic Journal, 104, pgs. 798 - 803, 1994.

R. Cubitt and R. Sugden. *Common reasoning in games: A Lewisian analysis of common knowledge of rationality*. Manuscript, 2011.

Game 1

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 1,1 | 0,0 |
| $D$ | 0,0 | 0,0 |

Game 2

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 1,1 | 1,0 |
| $D$ | 1,0 | 0,1 |

**Game 1**

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 1,1 | 0,0 |
| $D$ | 0,0 | 0,0 |

Game 1

**Game 2**

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 1,1 | 1,0 |
| $D$ | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

| Ann \ Bob | *L* | *R* |
|---|---|---|
| *U* | 1,1 | 0,0 |
| *D* | 0,0 | 0,0 |

Game 1

| Ann \ Bob | *L* | *R* |
|---|---|---|
| *U* | 1,1 | 1,0 |
| *D* | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

**Game 2**: $U$ weakly dominates $D$

| Ann \ Bob | $L$ | $R$ |
| --- | --- | --- |
| $U$ | 1,1 | 0,0 |
| $D$ | 0,0 | 0,0 |

Game 1

| Ann \ Bob | $L$ | $R$ |
| --- | --- | --- |
| $U$ | 1,1 | 1,0 |
| $D$ | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

**Game 2**: $U$ weakly dominates $D$, and *after removing* $D$, $L$ strictly dominates $R$.

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 1,1 | 0,0 |
| $D$ | 0,0 | 0,0 |

Game 1

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $U$ | 1,1 | 1,0 |
| $D$ | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

**Game 2**: But, now what is the reason for not playing $D$?

| | | Bob | |
| --- | --- | --- | --- |
| | | $L$ | $R$ |
| Ann | $U$ | 1,1 | 0,0 |
| | $D$ | 0,0 | 0,0 |

Game 1

| | | Bob | |
| --- | --- | --- | --- |
| | | $L$ | $R$ |
| Ann | $U$ | 1,1 | 1,0 |
| | $D$ | 1,0 | 0,1 |

Game 2

**Game 1**: $U$ weakly dominates $D$ and $L$ weakly dominates $R$.

**Game 2**: But, now what is the reason for not playing $D$?

**Theorem** (Samuelson). There is no model of Game 2 satisfying common knowledge of rationality (where rationality incorporates weak dominance).

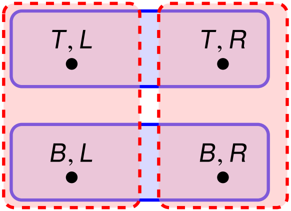
# Common Knowledge of Admissibility

| | | Bob | |
|---|---|---|---|
| | | $L$ | $R$ |
| Ann | $T$ | 1,1 | 1,0 |
| | $B$ | 1,0 | 0,1 |

There is no model of this game with *common knowledge* of admissibility.

# Common Knowledge of Admissibility

| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $T$ | 1,1 | 1,0 |
| $B$ | 1,0 | 0,1 |

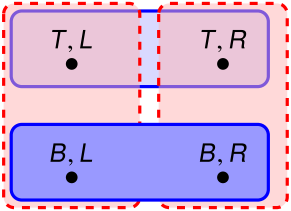



The "full" model of the game

# Common Knowledge of Admissibility

Bob

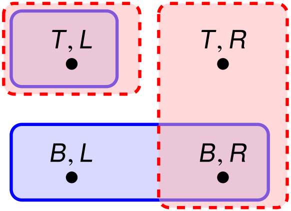
| Ann | | $L$ | $R$ |
|---|---|---|---|
| | $T$ | 1,1 | 1,0 |
| | $B$ | 1,0 | 0,1 |




The "full" model of the game: *$B$ is not admissible given Ann's information*

# Common Knowledge of Admissibility

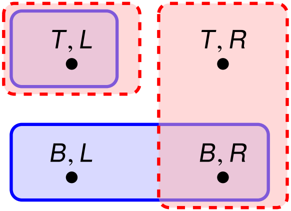
| Ann \ Bob | $L$ | $R$ |
|---|---|---|
| $T$ | 1,1 | 1,0 |
| $B$ | 1,0 | 0,1 |




What is wrong with this model?

# Common Knowledge of Admissibility

| Ann \ Bob | $L$ | $R$ |
| --- | --- | --- |
| $T$ | 1,1 | 1,0 |
| $B$ | 1,0 | 0,1 |




**Privacy of Tie-Breaking/No Extraneous Beliefs**: If a strategy is *rational* for an opponent, then it cannot be "ruled out".

# Summary

- Game models describe the *informational context* of a game.

- Game models can be used to *characterize* different solution concepts (e.g., iterated strict dominance, iterated weak dominance, Nash equilibrium, correlated equilibrium,...)

Tomorrow: Deliberation in Games