# Logics for Social Choice Theory

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Lecture 1

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# Plan

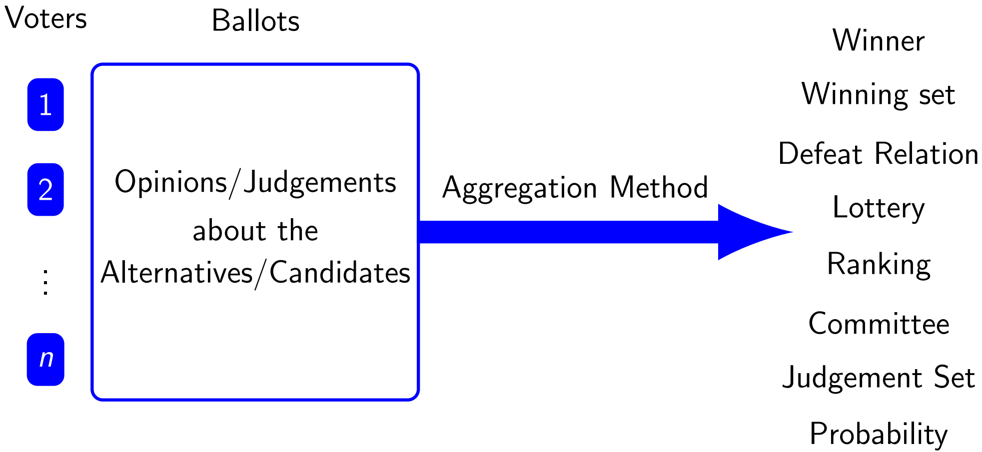
1.

# Social Choice Theory

"Social choice theory is the study of collective decision processes and procedures. It is not a single theory, but a cluster of models and results concerning the aggregation of individual inputs (e.g., votes, preferences, judgments, welfare) into collective outputs (e.g., collective decisions, preferences, judgments, welfare)."

C. List. *Social Choice Theory*. Stanford Encyclopedia of Philosophy, 2013.

# Social Choice Theory



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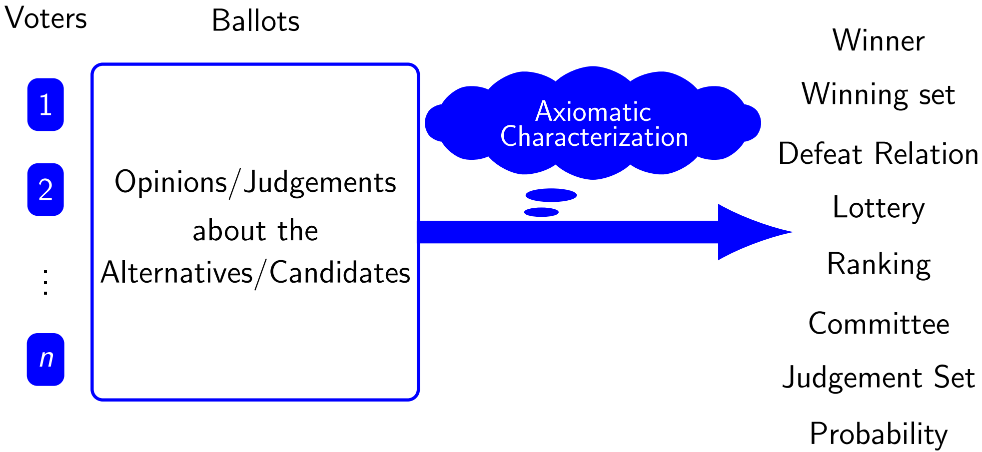

# Profiles

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## Definition

Given nonempty finite $V \subseteq \mathcal{V}$ and $X \subseteq \mathcal{X}$, a $(V, X)$-**profile** is a function $\mathsf{P}$ assigning to each $i \in V$ a binary relation $\mathsf{P}_i$ on $X$.

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We write $V(\mathbf{P})$ for P's set of voters and $X(\mathbf{P})$ for P's set of candidates.

# Collective choice rules

## Definition

A $(V, X)$-**collective choice rule** (or $(V, X)$-CCR) is a function $f$ such that for any $(V, X)$-profile $\mathsf{P} \in \mathrm{dom}(f)$, $f(\mathsf{P})$ is a binary relation on $X$.

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We assume that $f(\mathrm{P})$ is at least asymmetric, reflecting our interpretation of $(x, y) \in f(\mathrm{P})$ as meaning that $x$ is *strictly socially preferred* to $y$, or $x$ *defeats* $y$.

# Preferences (Rankings)

Suppose that $B \subseteq X \times X$ is a binary relation.

asymmetry: if $xBy$, then *not* $yBx$;

negative transitivity: if $xBy$, then $xBz$ or $zBy$.

# Negative transitivity

if $xBy$, then $xBz$ or $zBy$

Negative transitivity is equivalent to the condition that if *not* $xBz$ and *not* $zBy$, then *not* $xBy$, which explains the name.

# Negative transitivity

$$\text{if } xBy\text{, then } xBz \text{ or } zBy$$

Negative transitivity is equivalent to the condition that if *not* $xBz$ and *not* $zBy$, then *not* $xBy$, which explains the name.

Together negative transitivity and asymmetry imply that $B$ is transitive:

transitivity: if $xBy$ and $yBz$, then $xBz$.

$B$ is a *strict weak order* if and only if $B$ satisfies asymmetry and negative transitivity

$B$ is a *strict linear order* if and only if it satisfies asymmetry, transitivity, and weak completeness: for all $x, y \in X$, if $x \neq y$, then $xBy$ or $yBx$.

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$\mathcal{B}(X)$ is the set of all asymmetric binary relations on $X$;
$\mathcal{O}(X)$ is the set of all strict weak orders on $X$;
$\mathcal{L}(X)$ is the set of all strict linear orders on $X$.

# Non-compariability

Let $xNy$ if and only if neither $xBy$ nor $yBx$. We call $N$ the relation of *non-comparability*.

If $B$ is a strict weak order, then $N$ satisfies the following for all $x, y, z \in X$:

transitivity of non-comparability: if $xNy$ and $yNz$, then $xNz$.

# Profiles

A $(V, X)$-**profile** P of strict weak orders is an element of $\mathcal{O}(X)^V$, i.e., a function assigning to each $i \in V$ a relation $P_i \in O(X)$.

For $x, y \in X$, let:

$$\begin{aligned} \mathrm{P}(x,y) &= \{i \in V \mid xP_iy\}; \\ \mathrm{P}_{|\{x,y\}} &= \text{the function assigning to each } i \in V \\ &\quad\ \text{the relation } P_i \cap \{x,y\}^2. \end{aligned}$$

# Postulates: Domain conditions

universal domain (UD): $\mathrm{dom}(f) = O(X)^V$.

linear domain (LD): $\mathrm{dom}(f) = L(X)^V$.

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$f : \mathcal{L}(X)^V \to \mathcal{B}(X)$

# Postulates: Codomain conditions (rationality postulates)

transitive rationality (TR): for all $\mathsf{P} \in \mathrm{dom}(f)$, $f(\mathsf{P})$ is transitive.

full rationality (FR): for all $\mathsf{P} \in \mathrm{dom}(f)$, $f(\mathsf{P})$ is a strict weak order.

# Interprofile conditions

independence of irrelevant alternatives (IIA): for all $\mathsf{P}, \mathsf{P}' \in \mathrm{dom}(f)$ and $x, y \in X$, if $\mathsf{P}_{|\{x,y\}} = \mathsf{P}'_{|\{x,y\}}$, then $xf(\mathsf{P})y$ if and only if $xf(\mathsf{P}')y$.

# Postulates: Decisiveness conditions

Pareto (P): for all $\mathrm{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if $\mathrm{P}(x, y) = V$, then $xf(\mathrm{P})y$.

dictatorship: there is an $i \in V$ such that for all $\mathrm{P} \in \mathrm{dom}(f)$ and $x, y \in X$, if $xP_iy$, then $xf(\mathrm{P})y$.

# Arrow's Theorem

**Theorem (Arrow, 1951)** Assume that $|X| \geq 3$ and $V$ is finite. Then any $(V, X)$-CCR satisfying UD, IIA, FR, and P is a dictatorship.

K. Arrow. *Social Choice and Individual Values*. Yale University Press (1951, 2nd ed., 1963, 3rd ed., 2012).

M. Morreau (2019). *Arrow's Theorem*. Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/entries/arrows-theorem/.

J. S. Kelly (1978). *Arrow Impossibility Theorems*. New York: Academic Press.

Eric Maskin and Amartya Sen (2014). *The Arrow Impossibility Theorem*. (Kenneth J. Arrow Lecture Series), Columbia University Press.

J. Geanakoplos (2005). *Three brief proofs of Arrow's Impossibility Theorem*. Economic Theory, 26, pp. 211 - 215.

| 1 | 2 | Society |
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$$\left.\begin{array}{c} c \succ b \\ a \sim c \\ b \succ a \end{array}\right\} \text{(FR)} \Longrightarrow \quad b \succ b, \text{ contradiction!}$$

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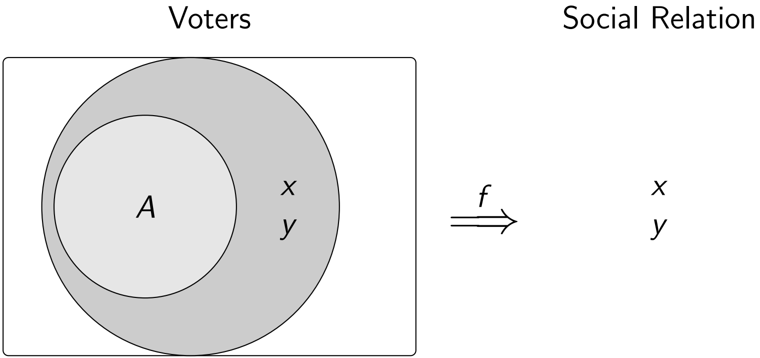
W. Holliday and EP (2020). *Arrow's Decisive Coalitions*. Social Choice and Welfare Social, 54, pp. 463 - 505.

The goal of this paper is a fine-grained analysis of reasoning about decisive coalitions, formalizing how the concept of a decisive coalition gives rise to a social choice theoretic language and logic all of its own.

# Decisive coalitions

A coalition $A \subseteq V$ is **decisive for** $x$ **over** $y$ **according to** $f$ if for all $(V, X)$-profiles $\mathbf{P}$, if $x \mathbf{P}_i y$ for all $i \in A$, then $x f(\mathbf{P}) y$.



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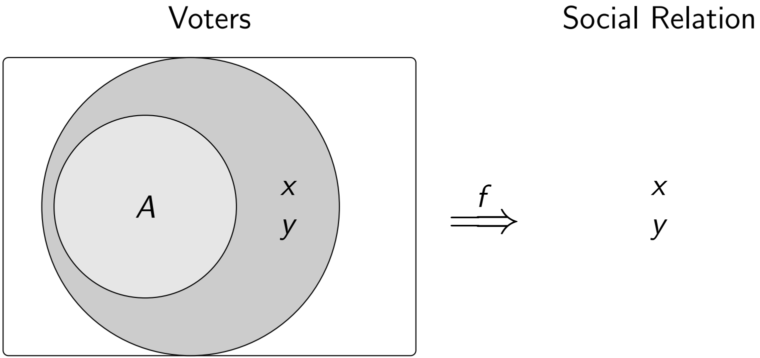


$A$ is **decisive according to** $f$ if for all distinct $x, y$, $A$ is decisive for $x$ over $y$.

# Decisive representation

## Definition

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Let $f$ be a $(V, X)$-CCR. Define $D_f : X^2 \to \wp(\wp(V))$ as follows:

$$D_f(x, y) = \{A \subseteq V \mid A \text{ is decisive for } x \text{ over } y \text{ according to } f\}.$$

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## Definition

Let $K$ be a class of $(V, X)$-CCRs. A function $D : X^2 \to \wp(\wp(V))$ is **decisively representable in** $K$ if $D$ is decisively represented by some $f \in K$.

## Theorem (H. and Pacuit 2020)

Let $V$ and $X$ be nonempty sets with $|X| \geq 3$. A function $D : X^2 \to \wp(\wp(V))$ is decisively representable in the class of $(V, X)$-CCRs satisfying IIA and transitivity (resp. the SWF condition) if and only if

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1. $A \in D(x, x)$ if and only if $A \neq \varnothing$;
2. if $A \in D(x, y)$ and $A \cap B = \varnothing$, then $B \notin D(y, x)$;
3. for transitive CCRs: if $A \in D(x, y)$, $B \in D(y, z)$, and $A \cap B \subseteq C \subseteq A \cup B$, then $C \in D(x, z)$;
4. for SWFs: if $A \in D(x, y)$ and $B \cap C \subseteq A \subseteq B \cup C$, then $B \in D(x, z)$ or $C \in D(z, y)$;
5. if $A \in D(x, y)$ and $A \subseteq B$, then $B \in D(x, y)$.

# Complete logics and formal proofs

We turn the representation theorem on the previous slide into a completeness theorem for a formal logic for reasoning about decisive coalitions, using atomic formulas of the form $D_{x>y}(t)$ where $t$ is a Boolean algebraic term.

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In the semantics, we evaluate formulas at CCRs: $f \models \varphi$.

# Language, I

Let Coal be a nonempty set, called the set of **coalition labels**. The set Term of **coalition terms** is generated by the following grammar:

$$t := a \mid 0 \mid 1 \mid -t \mid (t \sqcap t) \mid (t \sqcup t)$$

where $a \in \mathrm{Coal}$.

# Language, II

Let Alt be a set with $|\mathrm{Alt}| = |X|$, called the set of *alternative labels*. The set Form of *formulas* is generated by the following grammar:

$$\varphi ::= t \equiv t \mid D_{x>y}(t) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi)$$

where $t \in$ Term and $x, y \in$ Alt.

We define the following abbreviation:

$$\begin{aligned} (s \sqsubseteq t) &:= s \sqcap t \equiv s \\ D(t) &:= \bigwedge_{x,y \in \mathrm{Alt},\, x \neq y} D_{x>y}(t), \end{aligned}$$

# Semantics, I

**Definition**. A **coalition labeling** is a function $\alpha$ mapping each coalition label to a subset of $V$, i.e., $\alpha : \text{Coal} \rightarrow \wp(V)$. We extend $\alpha$ to a function $\dot{\alpha} : \text{Term} \rightarrow \wp(V)$ as follows:

1. $\dot{\alpha}(a) = \alpha(a)$ for $a \in \text{Coal}$;
2. $\dot{\alpha}(0) = \varnothing$;
3. $\dot{\alpha}(1) = V$;
4. $\dot{\alpha}(-t) = \dot{\alpha}(t)^c$;
5. $\dot{\alpha}(s \sqcap t) = \dot{\alpha}(s) \cap \dot{\alpha}(t)$;
6. $\dot{\alpha}(s \sqcup t) = \dot{\alpha}(s) \cup \dot{\alpha}(t)$.

# Semantics, II

**Definition.** An **alternative labeling** is a function $\beta$ mapping each alternative label to an element of $X$, i.e., $\beta : \mathsf{Alt} \to X$.

# Semantics, III

**Definition.** Let $f$ be a CCR, $\alpha$ a coalition labeling, and $\beta$ an alternative labeling. We inductively define the notion of a formula $\varphi$ being *true of* $f$ *relative to* $\alpha, \beta$ (notation: $f \models_{\alpha,\beta} \varphi$) as follows:

1. $f \models_{\alpha,\beta} s \equiv t$ if and only if $\dot{\alpha}(s) = \dot{\alpha}(t)$;
2. $f \models_{\alpha,\beta} D_{x>y}(t)$ if and only if $\dot{\alpha}(t) \in D_f(\beta(x), \beta(y))$;
3. $f \models_{\alpha,\beta} \neg\varphi$ if and only if $f \not\models_{\alpha} \varphi$;
4. $f \models_{\alpha,\beta} \varphi \wedge \psi$ if and only if $f \models_{\alpha,\beta} \varphi$ and $f \models_{\alpha,\beta} \psi$;
5. $f \models_{\alpha,\beta} \varphi \vee \psi$ if and only if $f \models_{\alpha,\beta} \varphi$ or $f \models_{\alpha,\beta} \psi$;
6. $f \models_{\alpha,\beta} \varphi \to \psi$ if and only if $f \not\models_{\alpha,\beta} \varphi$ or $f \models_{\alpha,\beta} \psi$.

# Semantics, III

We say that $\varphi$ is simply **true of** $f$ if and only if $\varphi$ is true of $f$ relative to every coalition labeling and alternative labeling.

Since the semantics supplies a notion of a formula $\varphi$ being true of a CCR $f$, for any class $K$ of CCRs we can ask the following key logical question:

> Is there a finite formal calculus for deriving all and only the formulas that are true of all CCRs in $K$?

# Logic, I

A *decisiveness logic* is any set L of formulas—called the *theorems* of L—that contains all instances of the following axioms 1–3 and is closed under rules 4 and 5:

1. all valid equations $s \equiv t$;
2. Leibniz's law $s \equiv t \to (\varphi[s/u] \leftrightarrow \varphi[t/u])$, where $\psi[u'/u]$ is the result of replacing all occurrences in $\psi$ of the term $u$ by the term $u'$;
3. all tautologies of propositional logic;
4. if $\varphi$ and $\varphi \to \psi$ are theorems of L, then $\psi$ is a theorem of L;
5. if $\varphi$ is a theorem of L, then so is any formula obtained from $\varphi$ by uniformly substituting coalition terms for coalition labels in $\varphi$, or by uniformly substituting alternative labels that do not occur in $\varphi$ for alternative labels in $\varphi$.

# Logic, III

Let $\overline{\mathsf{T}}$ be the smallest decisiveness logic that contains the following axioms for $a, b, c \in \mathsf{Coal}$ and $x, y, z \in \mathsf{Alt}$ such that $x \neq y$, $x \neq z$, and $y \neq z$:

1. $D_{x>x}(a) \leftrightarrow \neg(a \equiv 0)$;
2. $(D_{x>y}(a) \wedge ((a \sqcap b) \equiv 0)) \rightarrow \neg D_{y>x}(b)$;
3. $(D_{x>y}(a) \wedge (a \sqsubseteq b)) \rightarrow D_{x>y}(b)$;
4. transitivity axiom:
   $\big(D_{x>y}(a) \wedge D_{y>z}(b) \wedge (a \sqcap b \sqsubseteq c) \wedge (c \sqsubseteq a \sqcup b)\big) \rightarrow D_{x>z}(c)$.

Let $\overline{\mathsf{W}}$ be the smallest decisiveness logic that contains the axioms of $\overline{\mathsf{T}}$ as well as the following for $a, b, c \in \mathsf{Coal}$ and $x, y, z \in \mathsf{Alt}$ such that $x \neq y$, $x \neq z$, and $y \neq z$:

5. negative transitivity axiom:
   $\big(D_{x>y}(a) \wedge (b \sqcap c \sqsubseteq a) \wedge (a \sqsubseteq b \sqcup c)\big) \rightarrow \big(D_{x>z}(b) \vee D_{z>y}(c)\big)$.

# Soundness and Completeness Theorems

1. Soundness: if $\varphi$ is a theorem of $\overline{\mathsf{T}}$ (resp. $\overline{\mathsf{W}}$), then for any nonempty set $V$, $\varphi$ is true of all CCRs satisfying UD, IIA, and TR (resp. FR), according to the decisiveness semantics.
2. Completeness: if for any finite nonempty sets $V$, $\varphi$ is true of all CCRs satisfying UD, IIA, and TR (resp. FR), then $\varphi$ is a theorem of $\overline{\mathsf{T}}$ (resp. $\overline{\mathsf{W}}$), according to the decisiveness semantics.
3. The set of theorems of $\overline{\mathsf{T}}$ (resp. $\overline{\mathsf{W}}$) is decidable.

# Existential assumptions, I

$$Pareto := D(1).$$

Yet we are also interested in weaker assumptions (implied by Pareto, assuming UD).

Let the *existential assumption* (EA) be

$$EA := \bigwedge_{x,y \in \mathrm{Alt}} D_{x>y}(c_{x,y}).$$

# Existential assumptions, II

Let the *weak existential assumption* (WEA) be

$$WEA := \bigwedge_{x,y \in \mathsf{Alt},\, x \neq y} \neg D_{x>y}(c_{x,y}).$$

If we interpret $D$ as decisiveness, then WEA is equivalent to the well-known condition of

> non-imposition (NI): a CCR $f$ satisfies non-imposition if and only if for every $x, y \in X$, there exists a profile $\mathrm{P} \in \mathrm{dom}(f)$ such that *not* $xf(\mathrm{P})y$.

# Existential assumptions, III

Finally, let *non-emptiness* (NE) be

$$NE := D_{s>t}(e).$$

Note that EA implies WEA and NE, but not vice versa.

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By a "formal proof of Arrow's Theorem" here we mean formal proofs of the formulas expressing that *the family of decisive coalitions is an ultrafilter*.

That is the social-choice theoretic content of Arrow's proof, and then some basic set theory—any ultrafilter on a finite set is principal—delivers the dictatorship.

# Transitivity Axiom

$$\left(D_{x>y}(a) \wedge D_{y>z}(b) \wedge (a \sqcap b \sqsubseteq \mathsf{c}) \wedge (\mathsf{c} \sqsubseteq a \sqcup b)\right) \rightarrow D_{x>z}(\mathsf{c})$$

$$\left(D_{x>y}(a) \wedge D_{y>z}(b) \wedge (a \sqcap b \sqsubseteq c) \wedge (c \sqsubseteq a \sqcup b)\right) \rightarrow D_{x>z}(c)$$

# Intersection

$$\left(D_{x>y}(a) \wedge D_{y>z}(b) \wedge (a \sqcap b \sqsubseteq a \sqcap b) \wedge (a \sqcap b \sqsubseteq a \sqcup b)\right)$$
$$\rightarrow D_{x>z}(a \sqcap b)$$

**Lemma**. Assume $x \neq y$, $y \neq z$, and $x \neq z$.

$\vdash_{\mathsf{T}} \left(D_{x>z}(a) \wedge D_{z>y}(b)\right) \rightarrow D_{x>y}(a \sqcap b)$.

**Proof**.

1. $\left(D_{x>y}(a) \wedge D_{y>z}(b) \wedge (a \sqcap b \sqsubseteq a \sqcap b) \wedge (a \sqcap b \sqsubseteq a \sqcup b)\right) \rightarrow D_{x>z}(a \sqcap b)$
2. $(a \sqcap b \sqsubseteq a \sqcap b) \wedge (a \sqcap b \sqsubseteq a \sqcup b)$, true Boolean inequalities
3. $\left(D_{x>z}(a) \wedge D_{z>y}(b)\right) \rightarrow D_{x>y}(a \sqcap b)$ from (1) and (2).

1. $\vdash_{\mathsf{T}} EA \to ((D(a) \wedge D(b)) \to D(a \sqcap b))$.
2. $\vdash_{\mathsf{T}} D(1) \to ((D(a) \wedge D(b)) \to D(a \sqcap b))$.

# Contagion Lemma from EA

$$\big(D_{x>y}(a) \wedge D_{y>v}(c) \wedge (a \sqcap c \sqsubseteq \text{a}) \wedge (\text{a} \sqsubseteq a \sqcup c)\big) \rightarrow D_{x>v}(\text{a})$$

**Lemma** Assume $x \neq y$, $x \neq v$, $y \neq v$, $x \neq w$, and $y \neq w$.

1. $\vdash_{\mathsf{T}} D_{y>v}(c) \rightarrow \big(D_{x>y}(a) \rightarrow D_{x>v}(a)\big)$.
2. $\vdash_{\mathsf{T}} D_{w>x}(c) \rightarrow \big(D_{x>y}(a) \rightarrow D_{w>y}(a)\big)$.

For part 1, we have:

1. $\big(D_{x>y}(a) \wedge D_{y>v}(c) \wedge (a \sqcap c \sqsubseteq a) \wedge (a \sqsubseteq a \sqcup c)\big) \rightarrow D_{x>v}(a)$, instance of transitivity axiom
2. $(a \sqcap c \sqsubseteq a) \wedge (a \sqsubseteq a \sqcup c)$, valid Boolean inequalities
3. $\big(D_{x>y}(a) \wedge D_{y>v}(c)\big) \rightarrow D_{x>v}(a)$ from 1 and 2 by propositional logic
4. $D_{y>v}(c) \rightarrow \big(D_{x>y}(a) \rightarrow D_{x>v}(a)\big)$ from 3 by propositional logic.

The proof for part 2 is analogous, starting with:

(1′) $\big(D_{w>x}(c) \wedge D_{x>y}(a) \wedge (c \sqcap a \sqsubseteq a) \wedge (a \sqsubseteq c \sqcup a)\big) \rightarrow D_{w>y}(a)$, instance of transitivity axiom.

# Contagion Lemma from Pareto

**Lemma.** Assume $x \neq y$.

1. $\vdash_{\mathsf{T}} EA \rightarrow (D_{x>y}(a) \rightarrow D(a))$.
2. $\vdash_{\mathsf{T}} D(1) \rightarrow (D_{x>y}(a) \rightarrow D(a))$.

# Filter Lemmas

- $\vdash_{\mathrm{T}} D(1) \to \big((D(a) \wedge a \sqsubseteq b) \to D(b)\big)$.
- $\vdash_{\mathrm{T}} D(1) \to \big(D(a \sqcap b) \to (D(a) \wedge D(b))\big)$.
- $\vdash_{\mathrm{T}} D(1) \to \neg D(0)$.

- **Ultrafilter Lemma** $\vdash_{\mathrm{W}} (WEA \wedge NE) \to (D(a) \vee D(-a))$.

# Arrow's Theorem

In the logic W, we have:

1. $\vdash_W D(1) \rightarrow \neg D(0)$;
2. $\vdash_W D(1) \rightarrow (D(a) \vee D(-a))$;
3. $\vdash_W D(1) \rightarrow \big(D(a \sqcap b) \leftrightarrow (D(a) \wedge D(b))\big)$.

# Stronger Arrow's Theorem

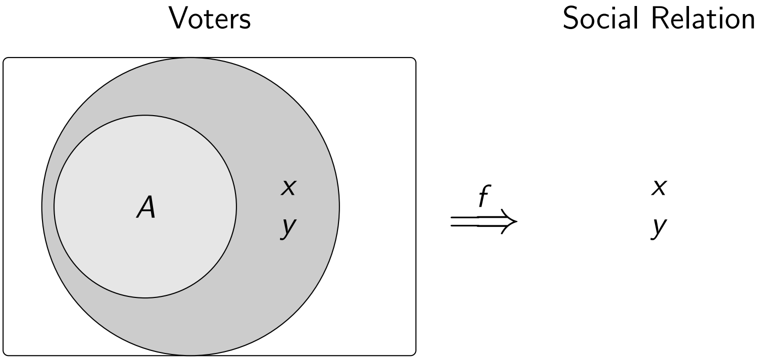
In the logics T and W, we have:

1. $\vdash_{\mathrm{T}} D(1) \to \neg D(0)$;
2. $\vdash_{\mathrm{W}} (WEA \wedge NE) \to (D(a) \vee D(-a))$;
3. $\vdash_{\mathrm{T}} EA \to \big((D(a) \wedge D(b)) \to D(a \sqcap b)\big)$ and $\vdash_{\mathrm{W}} WEA \to \big((D(a) \wedge D(b)) \to D(a \sqcap b)\big)$;
4. $\vdash_{\mathrm{T}} D(1) \to \big(D(a \sqcap b) \to (D(a) \wedge D(b))\big)$.

- Other notions of decisiveness and related impossibility theorems
- Escaping impossibility

# Decisive coalitions

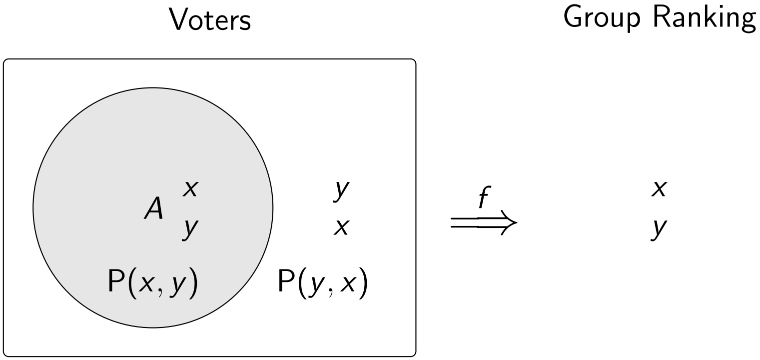
A coalition $A \subseteq V$ is **decisive for** $x$ **over** $y$ **according to** $f$ if for all $(V, X)$-profiles $\mathbf{P}$, if $x \mathbf{P}_i y$ for all $i \in A$, then $x f(\mathbf{P}) y$.




$A$ is **decisive according to** $f$ if for all distinct $x, y$, $A$ is decisive for $x$ over $y$.

# Almost decisive coalitions

A coalition $A$ is **almost decisive for** $x$ **over** $y$ **according to** $f$ if and only if for all $\mathrm{P} \in \mathrm{dom}(f)$, if $A = \mathrm{P}(x, y)$ and $A^c = \mathrm{P}(y, x)$, then $xf(\mathrm{P})y$




$A$ is **almost decisive according to** $f$ if for all distinct $x, y$, $A$ is almost decisive for $x$ over $y$.

# Another Representation Theorem

**Theorem** Let $X$ and $V$ be nonempty sets with $|X| \geq 3$. A function $D : X^2 \to \wp(\wp(V))$ is *almost-decisively representable in the class of CCRs for $\langle X, V \rangle$ satisfying LD, IIA, and TR (resp. FR)* if and only if for all $A, B, C \subseteq V$ and $x, y, z \in X$ with $x \neq y$, $y \neq z$, and $x \neq z$:

1. $A \in D(x, x)$;
2. if $A \in D(x, y)$, then $A^c \notin D(y, x)$;
3. for TR: if $A \in D(x, y)$, $B \in D(y, z)$, and $A \cap B \subseteq C \subseteq A \cup B$, then $C \in D(x, z)$;
4. for FR: if $A \in D(x, y)$ and $B \cap C \subseteq A \subseteq B \cup C$, then $B \in D(x, z)$ or $C \in D(z, y)$.

# Almost decisive logics

Sound and complete logic with respect to the *almost decisiveness semantics*, defined as above except with a modified clause for $D$:

2.′ $f \models_{\alpha,\beta} D_{x>y}(t)$ if and only if $\dot{\alpha}(t) \in \widehat{D}_f(\beta(x), \beta(y))$.

# Oligarchies

Let $f$ be a CCR for $\langle X, V \rangle$. For any $x, y \in X$ and $A \subseteq V$:

> $A$ is **almost semi-decisive for** $x$ **over** $y$ **according to** $f$ if and only if for all $\mathrm{P} \in \mathrm{dom}(f)$, if $A = \mathrm{P}(x, y)$ and $A^c = \mathrm{P}(y, x)$, then *not* $yf(\mathrm{P})x$;

$$
\begin{aligned}
S_{x>y}(a) &:= \neg D_{y>x}(-a) \\
S(a) &:= \bigwedge_{x,y \in \mathrm{Alt}} S_{x>y}(a).
\end{aligned}
$$

# Oligarchies

Let $f$ be a CCR and $A \subseteq V$.

1. $A$ is an *almost oligarchy* according to $f$ if and only if $A$ is almost decisive according to $f$ and for each $i \in A$, $\{i\}$ is almost semi-decisive according to $f$.

**Theorem** Assume that $|X| \geq 3$ and $V$ is finite. If a CCR $f$ satisfies UD, IIA, EA, and TR, then there exists a strong almost oligarchy according to $f$.

A. Gibbard (2014). *Intransitive social indifference and the Arrow dilemma*. Review of Economic Design, 18, pp. 3 - 10.

# Inverse Decisiveness

Let $f$ be a CCR for $\langle X, V \rangle$. For any $x, y \in X$ and $A \subseteq V$:

1. $A$ is *almost inversely decisive for* $x$ *over* $y$ *according to* $f$ if and only if for all $\mathsf{P} \in \mathrm{dom}(f)$, if $A = \mathsf{P}(x, y)$ and $A^c = \mathsf{P}(y, x)$, then $yf(\mathsf{P})x$;
2. $A$ is *inversely decisive for* $x$ *over* $y$ *according to* $f$ if and only if for all $\mathsf{P} \in \mathrm{dom}(f)$, if $A \subseteq \mathsf{P}(x, y)$, then $yf(\mathsf{P})x$;

$$
\begin{aligned}
I_{x>y}(a) &:= D_{y>x}(-a); \\
I(a) &:= D(-a).
\end{aligned}
$$

# Almost Wilson's Theorem

In the logics T and W, we have:

1. $\vdash_{\mathrm{W}} (WEA \wedge NE) \rightarrow (D(1) \vee I(1))$;
2. $\vdash_{\mathrm{W}} D(1) \rightarrow (D(a) \vee D(-a))$;
3. $\vdash_{\mathrm{T}} D(1) \rightarrow \big(D(a \sqcap b) \leftrightarrow (D(a) \wedge D(b))\big)$;
4. $\vdash_{\mathrm{W}} I(1) \rightarrow (I(a) \vee I(-a))$;
5. $\vdash_{\mathrm{T}} I(1) \rightarrow \big(I(a \sqcap b) \leftrightarrow (I(a) \wedge I(b))\big)$.

R. Wilson (1972). *Social choice theory without the Pareto principle.* Journal of Economic Theory, 5, pp. 478 - 486.

# Dropping IIA

So far, all of our results have assumed IIA. We show how our approach can be applied in a setting without IIA, namely the setting of Sen's Impossibility Theorem concerning the "Paretian liberal"

# Dropping IIA

For any $S \subseteq \mathbb{N}$:

- $\mathrm{AR}_S$: for any $\mathrm{P} \in \mathrm{dom}(f)$ and $n \in S$, there are no distinct $x_0, \ldots, x_n \in X$ such that for all $k < n$, $x_k f(\mathrm{P}) x_{k+1}$ and $x_n f(\mathrm{P}) x_0$.

**Theorem (Sen's Impossibility Theorem)**

$$\vdash_{\overline{\mathrm{A}}_{\{1,2,3\}}} EA \rightarrow \neg\big(D_{x>y}(a) \wedge D_{y>x}(a) \wedge D_{z>w}(b) \wedge D_{w>z}(b) \wedge (a \sqcap b \equiv 0)\big).$$

A. Sen (1970). *The impossibility of a Paretian liberal.* Journal of Political Economy, 78(1), pp. 15 - 157.

# Escaping impossibility

Key assumptions in Arrow's Theorem:

- The number of voters is finite

  P. Fishburn (1970). *Arrow's impossibility theorem: concise proof and infinitely many voters*. Journal of Economic Theory, 2, pp. 103 - 106.

- Universal domain

  W. Gaertner (2001). *Domain Conditions in Social Choice Theory*. Cambridge University Press.

  E. Elkind, M. Lackner, and D. Peters (2022). *Preference Restrictions in Computational Social Choice: A Survey*. `https://arxiv.org/abs/2205.09092`.

- There are at least 3 alternatives
- IIA