Tools for Formal Epistemology: Doxastic Logic, Probability and Default Logic

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Lecture 1

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Doxastic Logic: Models

Model: $\langle W, R, V \rangle$

States/possible worlds: $W \neq \emptyset$

Quasi-partitions: $R \subseteq W \times W$ is serial, transitive and Euclidean

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- *transitive*: for all $w, v, u \in W$, if $w \, R \, v$ and $v \, R \, u$, then $w \, R \, u$
- *Euclidean*: for all $w, v, u \in W$, if $w \, R \, v$ and $w \, R \, u$, then $v \, R \, u$

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Valuation function: $V : \text{At} \to \wp(W)$, where At is a set of atomic propositions.

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Belief operators: $\mathcal{M}, w \models B\varphi$ iff for all v, if $w\, R\, v$, then $\mathcal{M}, v \models \varphi$.

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$$\mathcal{M}, w \models B\varphi \text{ iff } R(w) \subseteq [\![\varphi]\!]^{\mathcal{M}}$$

$\{v \mid w\, R\, v\}$ $\qquad$ $\{v \mid \mathcal{M}, w \models \varphi\}$

Doxastic Logic: **KD45**

K	$B(\varphi \to \psi) \to (B\varphi \to B\psi)$
D	$B\varphi \to \neg B \neg \varphi$
4	$B\varphi \to BB\varphi$
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The logic **KD45** adds the above axiom schemes to an axiomatization of classical propositional logic with the rules Modus Ponens, Substitution of Equivalents, and Necessitation (from φ infer $B\varphi$).

KD45 is sound and strongly complete with respect to all quasi-partition frames.

Exercise: Show that the following axiom schemes and rules are valid on quasi-partition models and are theorems of **KD45**:

- agglomeration: $(B\varphi \wedge B\psi) \rightarrow B(\varphi \wedge \psi)$
- consistency: $\neg B\bot$
- monotonicity: From $\varphi \rightarrow \psi$ infer $B\varphi \rightarrow B\psi$

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- secondary-reflexivity: for all $w, v \in W$, if $w\,R\,v$ then $v\,R\,v$
  $B(B\varphi \to \varphi)$
- correctness of own beliefs:
  $B\neg B\varphi \to \neg B\varphi$
  for all w, there is a v such that $w\,R\,v$ and for all z if $v\,R\,z$ then $w\,R\,z$
  $BB\varphi \to B\varphi$
  density: for all w and v if $w\,R\,v$ then there is a z such that $w\,R\,z$ and $z\,R\,v$

- Defining beliefs from *evidence*.
- Defining beliefs from *knowledge*.
- Extending the basic logic of beliefs:
  - Conditional beliefs
  - Strong belief and robust belief
- Paradoxes of belief
  - The Buriden-Burge Paradox
  - Can an ideally rational agent be modest about her beliefs?
  - Prior's Theorem
  - Problems with Agglomeration: The Preface Paradox and The Lottery Paradox

Defining beliefs from evidence

J. van Benthem and EP. *Dynamic logics of evidence-based beliefs*. Studia Logica, 99(61), 2011.

J. van Benthem, D. Fernández-Duque and EP. *Evidence and plausibility in neighborhood structures*. Annals of Pure and Applied Logic, 165, pp. 106-133.

Evidence Models: Basic Assumptions

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3. Despite the fact that sources may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.

Evidential States

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In addition, much of the literature would suggest a 'monotonicity' assumption:

If the agent has evidence X *and* $X \subseteq Y$ *then the agent has evidence* Y*.*

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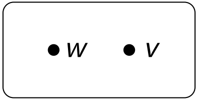
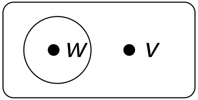
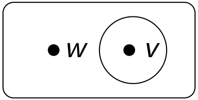


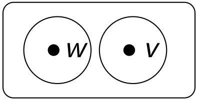
There is no evidence for or against p.



There is evidence that supports p.



There is evidence that rejects p.



There is evidence that supports p and also evidence that rejects p.

Evidence Model

Evidence model: $\mathcal{M} = \langle W, E, V \rangle$

- W is a non-empty set of worlds,
- $V : \mathsf{At} \to \wp(W)$ is a valuation function, and
- $E \subseteq W \times \wp(W)$ is an evidence relation

  $E(w) = \{X \mid w\, E\, X\}$ and $X \in E(w)$: "the agent accepts X as evidence at state w".

Uniform evidence model (E is a constant function): $\langle W, \mathcal{E}, V \rangle, w$ where $\mathcal{E}$ is the fixed family of subsets of W related to each state by E.

Assumptions

(Cons) For each state w, $\emptyset \notin E(w)$.

(Triv) For each state w, $W \in E(w)$.

The Basic Language Ł of Evidence and Belief

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid B\varphi \mid A\varphi$$

- $\Box\varphi$: "the agent has evidence that φ is true" (i.e., "the agent has evidence for φ")
- $B\varphi$ says that "the agents believes that φ is true" (based on her evidence)
- $A\varphi$: "φ is true in all states" (for technical convenience/knowledge)

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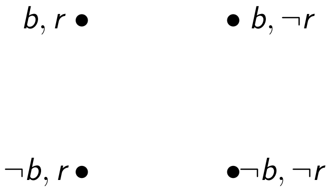

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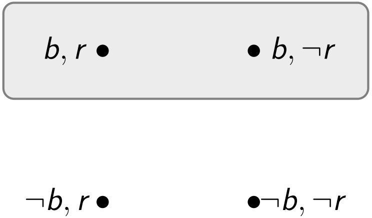

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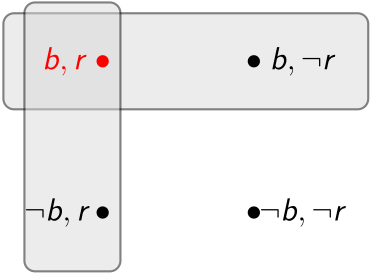

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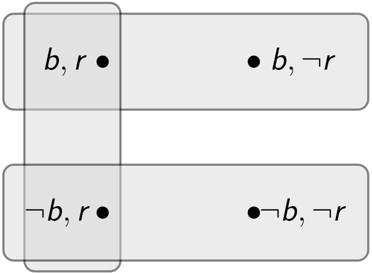

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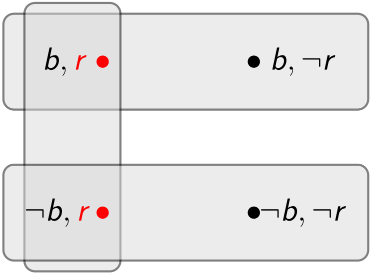

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Defining Beliefs

w-**scenario**: A maximal family of evidence sets $\mathcal{X} \subseteq E(w)$ that has the **finite intersection property** (f.i.p.: for each finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{X}$, $\bigcap_{1 \leq i \leq n} X_i \neq \emptyset$).

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An agent believes φ at w if each w-scenario implies that φ is true (i.e., φ is true at each point in the intersection of each w-scenario).

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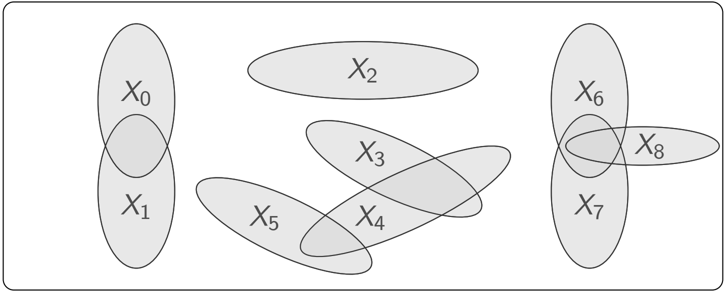

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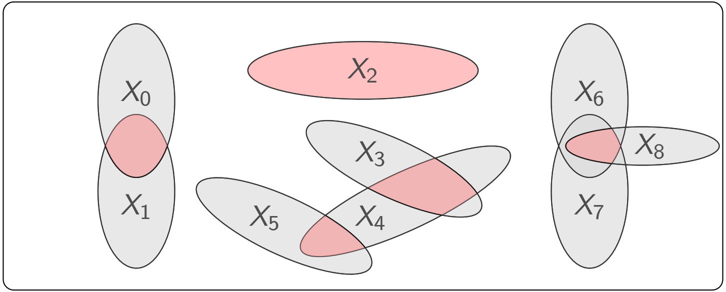

Our definition of belief is very conservative, many other definitions are possible (there exists a w-scenario, "most" of the w-scenarios,...)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \mathsf{At})$
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- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B\varphi$ for each maximal f.i.p. $\mathcal{X} \subseteq E(w)$ and for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$

Notation for the truth set: $[\![\varphi]\!]^{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$

Flat Evidence Models

An evidence model $\mathcal{M}$ is **flat** if every scenario on $\mathcal{M}$ has non-empty intersection.

Proposition. The formula $\Box\varphi \rightarrow \langle B \rangle \varphi$ is valid on the class of flat evidence models, but not on the class of all evidence models.

Exercises

1. Prove that $(\Box\varphi \wedge A\psi) \leftrightarrow \Box(\varphi \wedge A\psi)$ is valid on all evidence models.
2. Prove that $B\varphi \to AB\varphi$ is valid on all uniform evidence models.

Defining Beliefs from Knowledge

R. Stalnaker (2006). *On logics of knowledge and belief*. Philosophy Studies,128,169-199.

A. Baltag, N. Bezhanishvili, A. Özgün, and S. Smets (2019). *A Topological Approach to Full Belief*. Journal of Philosophical Logic, 48(2), pp. 205 - 244.

A. Bjorndahl and A. Özgün (2020). *Logic and Topology for Knowledge, Knowability, and Belief*. The Review of Symbolic Logic, 13(4), pp. 748-775.

Stalnaker's Axioms

Stalnaker bases his analysis on a conception of belief as 'subjective certainty': From the point of the agent in question, her belief is subjectively indistinguishable from her knowledge.

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Bi-modal language of knowledge and belief: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid B\psi$
Define $\langle K\rangle\varphi$ as $\neg K\neg\varphi$ and $\langle B\rangle\varphi$ as $\neg B\neg\varphi$

Stalnaker's Axioms

K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
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KB	$K\varphi \to B\varphi$
FB	$B\varphi \to BK\varphi$

Proposition (Stalnaker). The following equivalence is a theorem of the propositional modal logic that contains the previous axiom schemas (with Modus Ponens and Necessitation for both K and B):

$$B\varphi \leftrightarrow \langle K \rangle K\varphi$$

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Moreover, all of the axioms of **KD45** and the (.2)-axiom $\langle K \rangle K\varphi \to K\langle K \rangle\varphi$ are provable.

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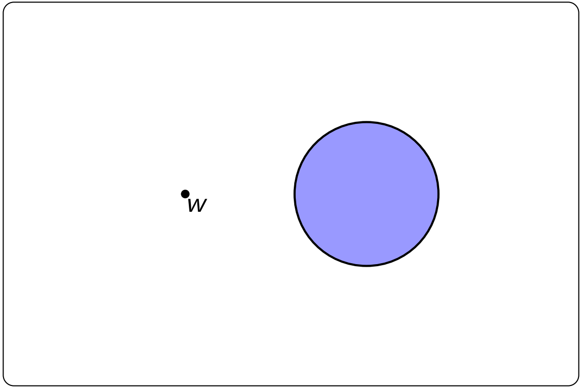
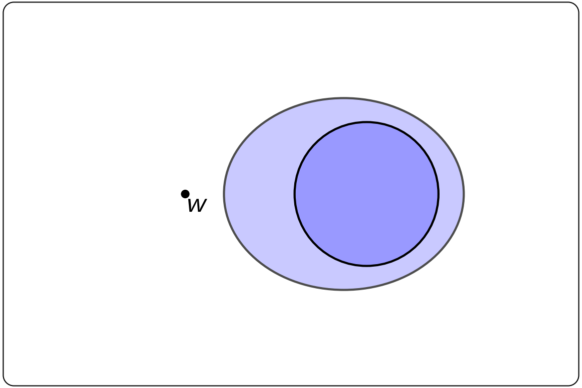
Moreover, all of the axioms of **KD45** and the (.2)-axiom $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$ are provable.

This means that we can take the logic of knowledge to be **S4.2** (the axioms K, T, 4 and .2) and *define* full belief as above (i.e., as the 'epistemic possibility of knowledge').

- ✓ Defining beliefs from *evidence*.
- ✓ Defining beliefs from *knowledge*.
  - Conditional beliefs
  - Strong belief and robust belief
- Paradoxes of belief
  - The Buriden-Burge Paradox
  - Can an ideally rational agent be modest about her beliefs?
  - Prior's Theorem
  - Problems with Agglomeration: The Preface Paradox and The Lottery Paradox



- The agent's beliefs (soft information—-the states consistent with what the agent believes)



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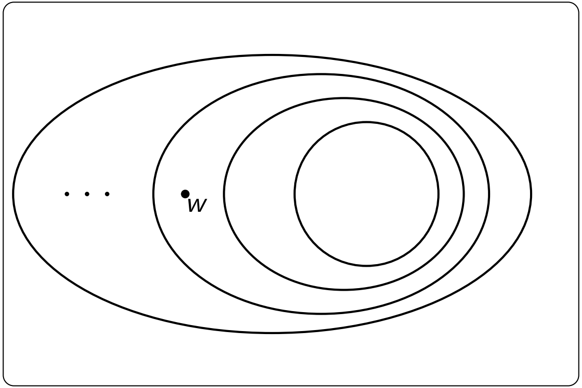

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Sphere Models

Let W be a set of states, A set $\mathcal{F} \subseteq \wp(W)$ is called a system of spheres provided:

- For each $S, S' \in \mathcal{F}$, either $S \subseteq S'$ or $S' \subseteq S$
- For any $P \subseteq W$ there is a smallest $S \in \mathcal{F}$ (according to the subset relation) such that $P \cap S \neq \emptyset$
- The spheres are non-empty $\bigcap \mathcal{F} \neq \emptyset$ and cover the entire information cell $\bigcup \mathcal{F} = W$

Let $\mathcal{F}$ be a system of spheres on W: for $w, v \in W$, let

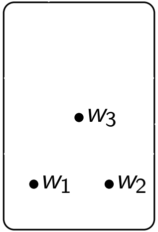
$$w \preceq_{\mathcal{F}} v \text{ iff for all } S \in \mathcal{F}, \text{ if } v \in S \text{ then } w \in S$$

Then, $\preceq_{\mathcal{F}}$ is reflexive, transitive, and well-founded.

$w \preceq_{\mathcal{F}} v$ means that no matter what the agent learns in the future, as long as world v is still consistent with her beliefs and w is still epistemically possible, then w is also consistent with her beliefs.

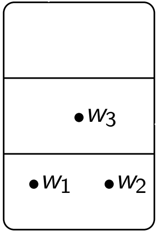
Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$



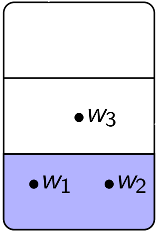
Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$
- $w_1 \preceq w_2$ and $w_2 \preceq w_1$ (w_1 and w_2 are equi-plausbile)
- $w_1 \prec w_3$ ($w_1 \preceq w_3$ and $w_3 \not\preceq w_1$)
- $w_2 \prec w_3$ ($w_2 \preceq w_3$ and $w_3 \not\preceq w_2$)

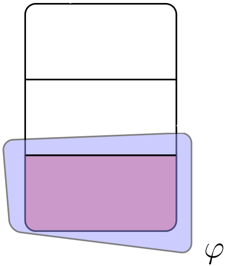


Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$
- $w_1 \preceq w_2$ and $w_2 \preceq w_1$ (w_1 and w_2 are equi-plausbile)
- $w_1 \prec w_3$ ($w_1 \preceq w_3$ and $w_3 \not\preceq w_1$)
- $w_2 \prec w_3$ ($w_2 \preceq w_3$ and $w_3 \not\preceq w_2$)
- $\{w_1, w_2\} \subseteq Min_{\preceq}([w_i])$



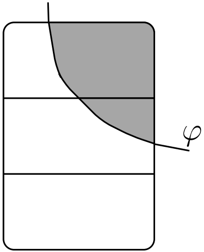
Belief Revision via Plausibility



Belief: $B\varphi$

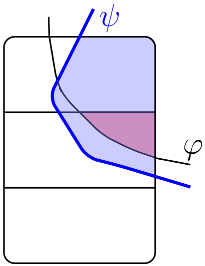
$$Min_{\preceq}(W) \subseteq [\![\varphi]\!]^{\mathcal{M}}$$

Belief Revision via Plausibility



Conditional Belief: $B^{\varphi}\psi$

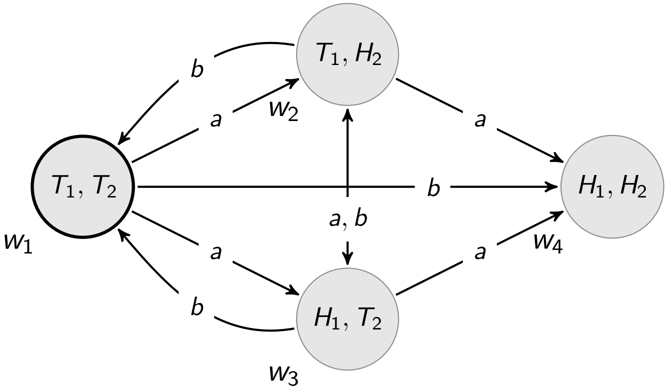
Belief Revision via Plausibility



Conditional Belief: $B^{\varphi}\psi$

$$Min_{\preceq}(\llbracket\varphi\rrbracket^{\mathcal{M}}) \subseteq \llbracket\psi\rrbracket^{\mathcal{M}}$$

Example



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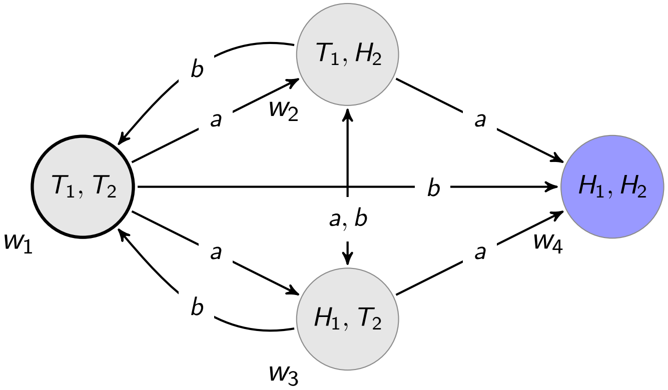
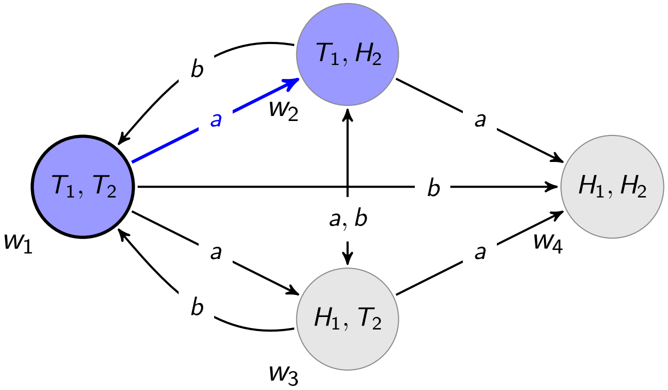


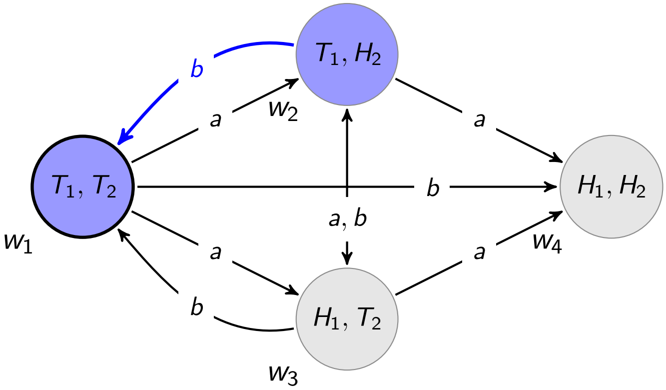
- $w_1 \models B_a(H_1 \wedge H_2) \wedge B_b(H_1 \wedge H_2)$

Example



- $w_1 \models B_a(H_1 \wedge H_2) \wedge B_b(H_1 \wedge H_2)$
- $w_1 \models B_a^{T_1} H_2$

Example



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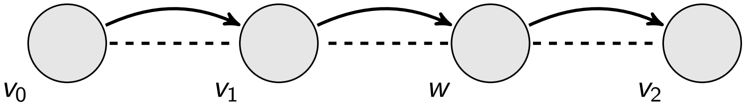
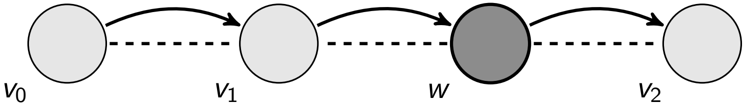
Grades of Doxastic Strength

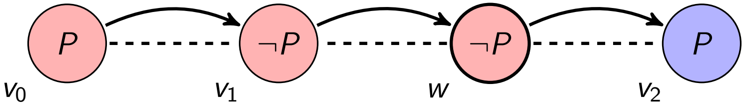


Grades of Doxastic Strength



Suppose that w is the current state.

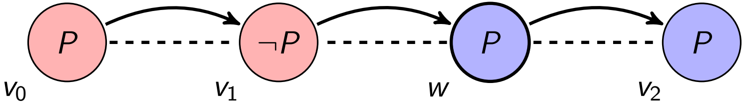
Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)

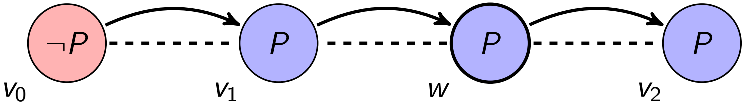
Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)
- **Robust Belief** ($[\preceq]P$)

Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)
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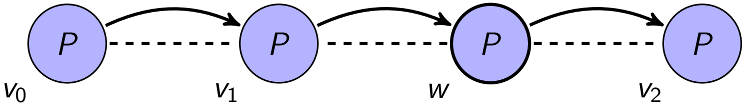
- **Strong Belief** (B^sP)

Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)
- **Robust Belief** ($[\preceq]P$)
- **Strong Belief** ($B^s P$)
- **Knowledge** (KP)

Is $B\varphi \rightarrow B^{\psi}\varphi$ valid?

Is $B\varphi \rightarrow B^{\psi}\varphi$ valid?

Is $B^{\alpha}\varphi \rightarrow B^{\alpha\wedge\beta}\varphi$ valid?

Is $B\varphi \to B^{\psi}\varphi$ valid?

Is $B^{\alpha}\varphi \to B^{\alpha \wedge \beta}\varphi$ valid?

Is $B\varphi \to B^{\psi}\varphi \vee B^{\neg\psi}\varphi$ valid?

Is $B\varphi \to B^{\psi}\varphi$ valid?

Is $B^{\alpha}\varphi \to B^{\alpha \wedge \beta}\varphi$ valid?

Is $B\varphi \to B^{\psi}\varphi \vee B^{\neg\psi}\varphi$ valid?

Exercise: Prove that B, B^{φ} and B^{s} are definable in the language with K (a universal modality) and $[\preceq]$ (a modality for the plausibility ordering).

$\mathcal{M}, w \models B^{\varphi}\psi$ if for each $v \in Min_{\preceq}([\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$.

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Core Logical Principles:

1. $B^{\varphi}\varphi$
2. $B^{\varphi}\psi \rightarrow B^{\varphi}(\psi \vee \chi)$
3. $(B^{\varphi}\psi_1 \wedge B^{\varphi}\psi_2) \rightarrow B^{\varphi}(\psi_1 \wedge \psi_2)$
4. $(B^{\varphi_1}\psi \wedge B^{\varphi_2}\psi) \rightarrow B^{\varphi_1 \vee \varphi_2}\psi$
5. $(B^{\varphi}\psi \wedge B^{\psi}\varphi) \rightarrow (B^{\varphi}\chi \leftrightarrow B^{\psi}\chi)$

J. Burgess. *Quick completeness proofs for some logics of conditionals*. Notre Dame Journal of Formal Logic 22, 76 – 84, 1981.

Types of Beliefs: Logical Characterizations

- $\mathcal{M}, w \models K_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ
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  i robustly believes φ iff i continues to believe φ given any true formula.

- $\mathcal{M}, w \models B_i^s\varphi$ iff $\mathcal{M}, w \models B_i\varphi$ and $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \neg K_i(\psi \rightarrow \neg\varphi)$.
  i strongly believes φ iff i believes φ and continues to believe φ given any evidence (truthful or not) that is not known to contradict φ.

- ✓ Defining beliefs from *evidence*.
- ✓ Defining beliefs from *knowledge*.
  - ✓ Conditional beliefs
  - ✓ Strong belief and robust belief
- ▶ Paradoxes of belief
  - ▶ The Buriden-Burge Paradox
  - ▶ Can an ideally rational agent be modest about her beliefs?
  - ▶ Prior's Theorem
  - ▶ Problems with Agglomeration: The Preface Paradox and The Lottery Paradox

Buridan-Burge Paradox I

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1. Suppose $\neg B_a q$. Then by the 5 axiom ($\neg B_a \varphi \to B_a \neg B_a \varphi$), we have that $B_a \neg B_a q$. But since q is $\neg B_a q$, we have $B_a q$. Contradiction.

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2. Suppose $B_a q$. By by the 4 axiom $(B_a \varphi \to B_a B_a \varphi)$, we have that $B_a B_a q$. By the D axioms $(B_a \varphi \to \neg B_a \neg \varphi)$, we have that $\neg B_a \neg B_a q$. But since $\neg B_a q$ is q, we have $\neg B_a q$. Contradiction.

Tyler Burge (1984). *Epistemic paradox*. Journal of Philosophy, 81(1), pp. 5 - 29.

Buridan-Burge Paradox II

Of course, "q is the statement that $\neg B_a q$" is not a sentence of the modal logic of beliefs.

What we have shown is that $\neg B_a(q \leftrightarrow \neg B_a q)$ is a theorem of **KD45**.

This is a paradox only if it should be possible for an ideally rational agent to believe that $q \leftrightarrow \neg B_a q$.

Wolfgang Lenzen (1981). *Doxastic Logic and the Burge-Buridan-Paradox*. Philosophical Studies, 39(1), pp. 43 - 49.

Michael Caie (2012). *Belief and indeterminacy*. The Philosophical Review, 121(1), pp. 1 - 54.

Propositional Quantifiers

While we naturally quantify over propositions in both ordinary and philosophical discussion of beliefs, the addition of propositional quantifiers is not given much attention in the literature.

Propositional Quantifiers

While we naturally quantify over propositions in both ordinary and philosophical discussion of beliefs, the addition of propositional quantifiers is not given much attention in the literature. Consider the following examples:

- "One believes that everything one believes is true": $B\forall p(Bp \to p)$
- "If no matter what p stands for, one believes that φ, then one believes that no matter what p stands for, φ": $\forall pB\varphi \to B\forall p\varphi$
- "There is a proposition that the agent takes to be consistent and to settle everything": $\exists q(Bq \wedge \forall p(B(q \to p) \vee B(q \to \neg p))$

See the course by Peter Fritz.

Immodest Beliefs

Immod: "One believes that everything one believes is true": $B\forall p(Bp \to p)$

- Even for idealized agents or idealized beliefs, as axiomatized by **KD45**, it seems that Immod should not be included in a logic of belief.

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Consider an agent who has credences about a real number x randomly generated from the interval $[0, 1]$. For all measurable $X \subseteq [0, 1]$, then the agent's credence that $x \in X$ is just the measure of X. Suppose that the agent outright believes precisely those propositions with credence 1. Then, for all $a \in [0, 1]$, the agent believes that $x \in [0, 1] \setminus \{a\}$ since $[0, 1] \setminus \{a\}$ is measure 1. However, the agent does not believe that for all $a \in [0, 1]$, $x \in [0, 1] \setminus \{a\}$ since $\bigcap_{a\in[0,1]}([0, 1] \setminus \{a\}) = \varnothing$, which is not measure 1. Hence the agent in this situation does not believe that all her beliefs are true.

Yifeng Ding (2021). *On the Logic of Belief and Propositional Quantification*. Journal of Philosophical Logic, 50, pp. 1143 - 1198.

In any possible world semantics for **KD45**, $B\forall p(Bp \rightarrow p)$ is valid on any frame. So, any logic validating **KD45** must validate Immod. Algebraic semantics is needed for logics with do not validate Immod.

Yifeng Ding (2021). *On the Logic of Belief and Propositional Quantification*. Journal of Philosophical Logic, 50, pp. 1143 - 1198.

Also, see:

Jeremy Goodman (2020). *I'm mistaken*. manuscript.