Neighborhood Semantics for Modal Logic
Lecture 4

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Plan for Today

- ✓ Completeness
- ✓ Incompleteness
- ▶ Simulating non-normal modal logics
- ▶ Brief discussion of decidability and complexity
- ▶ Bisimulations
- ▶ Neighborhood semantics for inquisitive logic

General Neighborhood Frames

A **general neighborhood frame** is a tuple $\mathcal{F}^g = \langle W, N, \mathcal{A} \rangle$ where $\langle W, N \rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

A valuation $V : \mathsf{At} \to \wp(W)$ is admissible for a general frame if for each $p \in \mathsf{At}$, $V(p) \in \mathcal{A}$.

Suppose that $\mathcal{F}^g = \langle W, N, \mathcal{A} \rangle$ is a general neighborhood frame. A **general modal** based on $\mathcal{F}^g$ is a tuple $\mathcal{M}^g = \langle W, N, \mathcal{A}, V \rangle$ where V is an admissible valuation.

General Neighborhood Frames

Lemma

Let $\mathcal{M}^g = \langle W, N, \mathcal{A}, V \rangle$ be an general neighborhood model. Then for each $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathcal{M}^g} \in \mathcal{A}$.

Lemma

Let **L** *be any logic extending* **E***. Then a general canonical frame for* **L** *validates* **L***.*

Corollary

Any modal logic extending **E** *is strongly complete with respect to some class of general frames.*

Summary

For any consistent modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of *general frames*

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There are modal logics showing that

- neighborhood completeness does not imply Kripke completeness
- algebraic completeness does not imply neighborhood completeness

Non-Normal Modal Logic with a Universal Modality

(*A-K*)	$A(\varphi \to \psi) \to (A\varphi \to A\psi)$
(*A-T*)	$A\varphi \to \varphi$
(*A-4*)	$A\varphi \to AA\varphi$
(*A-B*)	$E\varphi \to AE\varphi$
(*A-Nec*)	From φ infer $A\varphi$
($\langle\,]$-*RM*)	From $\varphi \to \psi$ infer $\langle\,]\varphi \to \langle\,]\psi$
($\langle\,]$-*Cons*)	$\neg\langle\,]\bot$
(*A-N*)	$A\varphi \to \langle\,]\varphi$
(*Pullout*)	$\langle\,](\varphi \wedge A\psi) \leftrightarrow (\langle\,]\varphi \wedge A\psi)$

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$(Pullout)$	$\langle\,](\varphi \wedge A\psi) \leftrightarrow (\langle\,]\varphi \wedge A\psi)$

Theorem. The logic EMA is sound and strongly complete with respect to neighborhood frames that are consistent, non-trivial and monotonic.

We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

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Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \mathrm{Pt}$$

where $p \in$ At and Pt is a unary modal operator.

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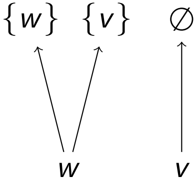
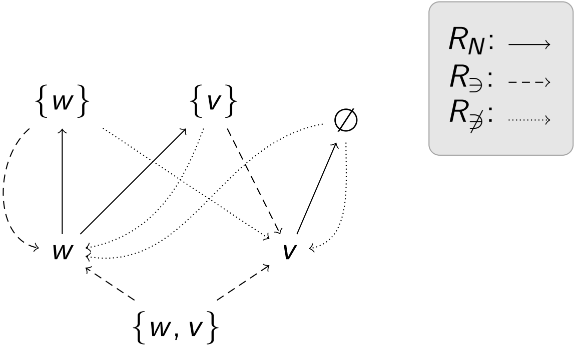
Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V\rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,

$$\mathcal{M}, w \models \varphi \textit{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$



$\mathcal{M}$



$\mathcal{M}^\circ$

$\mathcal{M}, w \models \Box p$ and $\mathcal{M}, v \models \Box\bot$.

- $\mathcal{M}^\circ, w \models \langle N\rangle([\ni]p \wedge [\not\ni]\neg p)$ and $\mathcal{M}^\circ, v \not\models \langle N\rangle([\ni]p \wedge [\not\ni]\neg p)$
- $\mathcal{M}^\circ, v \models \langle N\rangle([\ni]\bot \wedge [\not\ni]\top)$ and $\mathcal{M}^\circ, w \not\models \langle N\rangle([\ni]\bot \wedge [\not\ni]\top)$

Monotonic Models

Lemma

On Monotonic Models $\langle N \rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N \rangle([\ni]ST(\varphi))$

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. in Proof Theory of Modal Logic, Kluwer, pgs. 293 - 311, 1996.

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*. The Journal of Symbolic Logic, 64:1, pgs. 99 - 138, 1999.

Filtrations

Let $\mathcal{M} = \langle W, N, V \rangle$ be a neighborhood model and suppose that Σ is a set of sentences from $\mathcal{L}$.

For each $w, v \in W$, we say $w \sim_\Sigma v$ iff for each $\varphi \in \Sigma$, $w \models \varphi$ iff $v \models \varphi$.

For each $w \in W$, let $[w]_\Sigma = \{v \mid w \sim_\Sigma v\}$ be the equivalence class of $\sim_\Sigma$.

If $X \subseteq W$, let $[X]_\Sigma = \{[w] \mid w \in X\}$.

Filtrations

Definition

Let $\mathcal{M} = \langle W, N, V \rangle$ be a neighborhood model and Σ a set of sentences closed under subformulas. A filtration of $\mathcal{M}$ through Σ is a model $\mathcal{M}^f = \langle W^f, N^f, V^f \rangle$ where

1. $W^f = [W]$
2. For each $w \in W$
   2.1 for each $\Box\varphi \in \Sigma$, $[\![\varphi]\!]_{\mathcal{M}} \in N(w)$ iff $[[\![\varphi]\!]_{\mathcal{M}}] \in N^f([w])$
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Theorem

Suppose that $\mathcal{M}^f = \langle W^f, N^f, V^f \rangle$ is a filtration of $\mathcal{M} = \langle W, N, V \rangle$ through (a subformula closed) set of sentences Σ. Then for each $\varphi \in \Sigma$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^f, [w] \models \varphi$

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3. For each $p \in \mathrm{At}$, $V(p) = [V(p)]$

Corollary

E *has the finite model property. I.e., if* φ *has a model then there is a finite model.*

A Few Comments on Complexity

Logics without C (eg., $\mathbf{E}, \mathbf{EM}, \mathbf{E} + (\neg\Box\bot), \mathbf{E} + (\Box\varphi \to \Box\Box\varphi)$) are in NP.

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Logics with C are in PSPACE.

M. Vardi. *On the Complexity of Epistemic Reasoning*. IEEE (1989).

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J. Halpern and L. Rego. *Characterizing the NP-PSPACE gap in the satisfiability problem for modal logic*. Journal of Logic and Computation, 17:4, pgs. 795-806, 2007.

Background: Bismulations for Relational Semantics

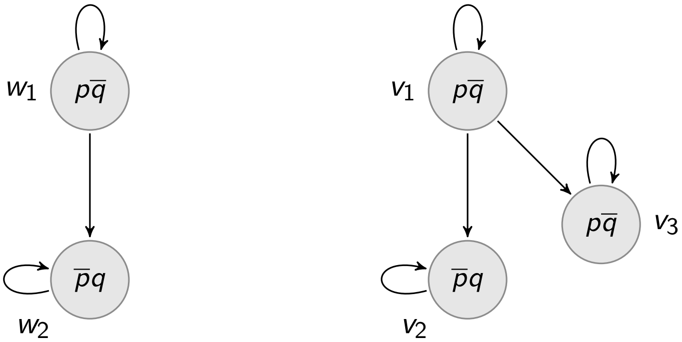
Relational Models: $\mathcal{M} = \langle W, R, V \rangle$ where $W \neq \varnothing$, $R \subseteq W \times W$, and $V : \mathsf{At} \to \wp(W)$

Truth:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$ if wRv then $\mathcal{M}, v \models \varphi$

  $\mathcal{M}, w \models \Box\varphi$ iff $\{v \mid w\,R\,v\} = R(w) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Distinguishing States



What is the difference between states w_1 and v_1?

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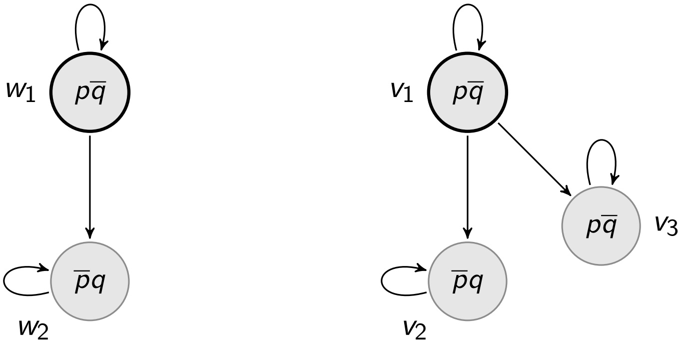

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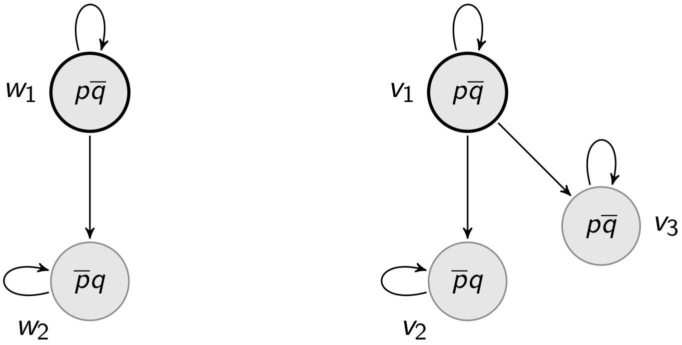

Is there a modal formula true at w_1 but not at v_1?

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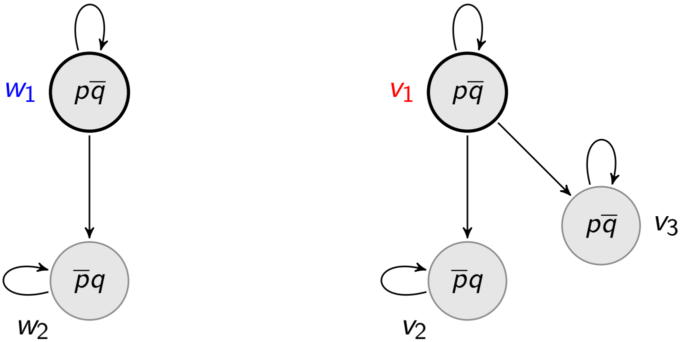

$$w_1 \models \Box\Diamond\neg p \text{ but } v_1 \not\models \Box\Diamond\neg p.$$

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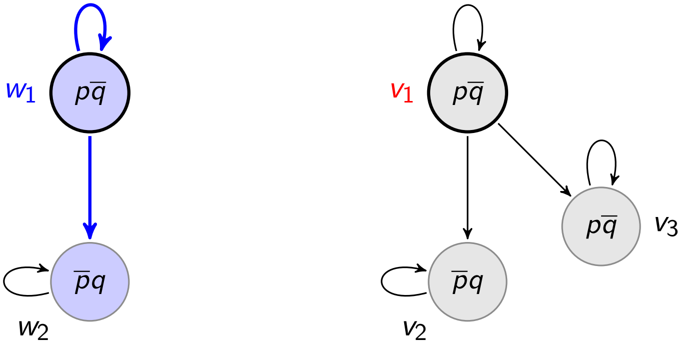

$w_1 \models \Box\Diamond\neg p$ but $v_1 \not\models \Box\Diamond\neg p$.

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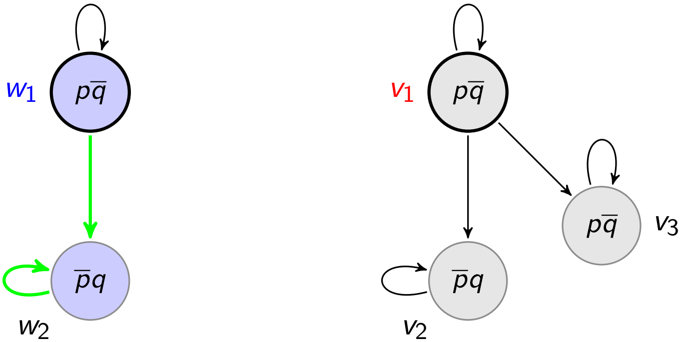

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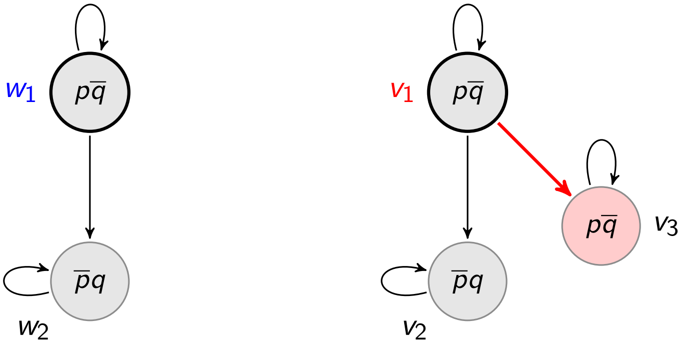

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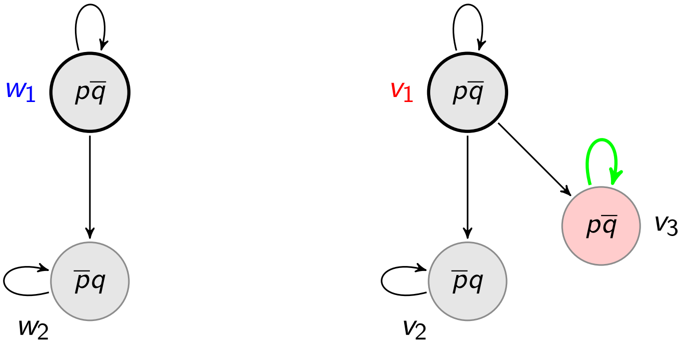

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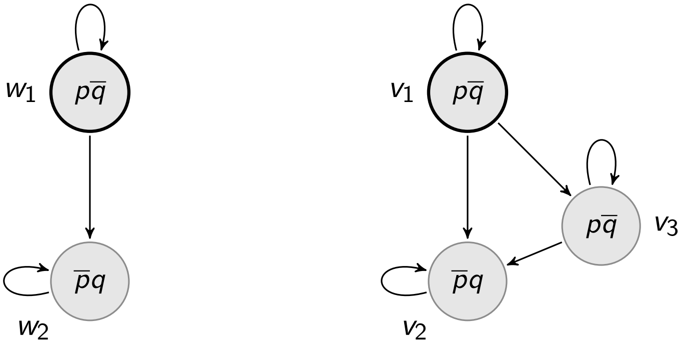

What about now? Is there a modal formula true at w_1 but not v_1?

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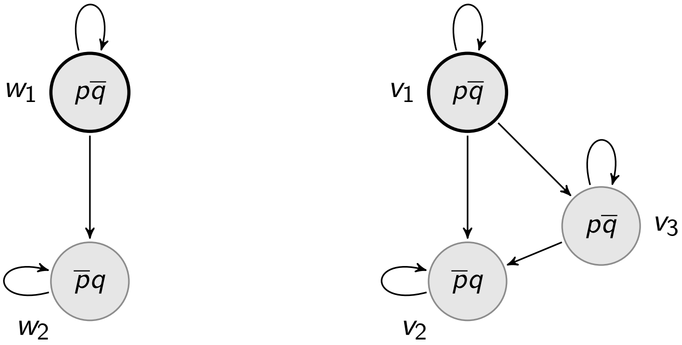

No modal formula can distinguish w_1 and v_1!

Consider the following modalities:

- $\mathcal{M}, w \models A\varphi$ iff for all $w \in W$, $\mathcal{M}, w \models \varphi$
- $\mathcal{M}, w \models \Diamond^{\leftarrow}\varphi$ iff there is a $v \in W$, vRw and $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models \Diamond_n\varphi$ iff there are $v_1, \ldots, v_n$ such that for all $1 \leq j \neq k \leq n$, $v_j \neq v_k$, for all $j = 1, \ldots, n$, wRv_j and for all $j = 1, \ldots, n$, $\mathcal{M}, v_j \models \varphi$.

  For instance, $\Diamond_2\varphi$ is true at a state if there are at least two accessible states that satisfy φ.
- $\mathcal{M}, w \models \circlearrowleft$ iff wRw

Are these modalities definable using the basic modal language? Intuitively, the answer is "no", but how do we *prove* this?

Bisimulation

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \mathrm{At}$, $w \in V(p)$ iff $w' \in V'(p)$
Zig: if wRv, then $\exists v' \in W'$ such that vZv' and $w'R'v'$
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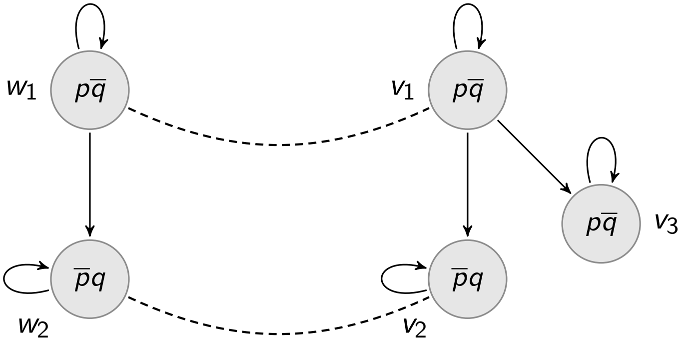

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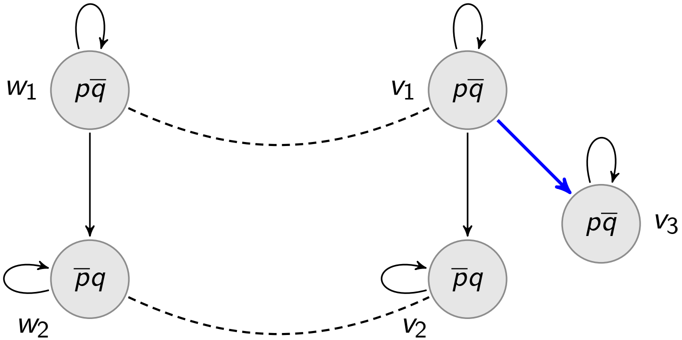

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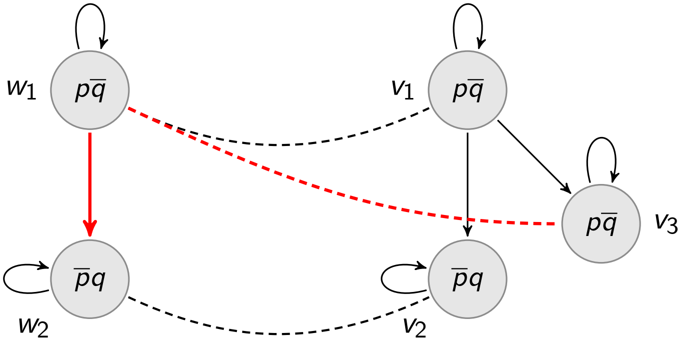

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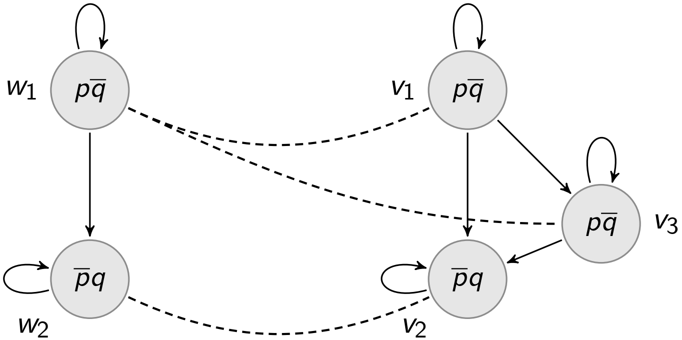

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- We write $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ if there is a Z such that wZw'.
- We write $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ iff for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

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- **Lemma** If $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$.
- **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Bisimulation for Relational Models

A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If wRv, then there is a $v' \in W'$ such that

$$w'R'v' \text{ and } vZv'$$

Zag: If $w'R'v'$ then there is a $v \in W$ such that

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Monotonic Bisimulation for Neighborhood Models

A bisimulation between $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

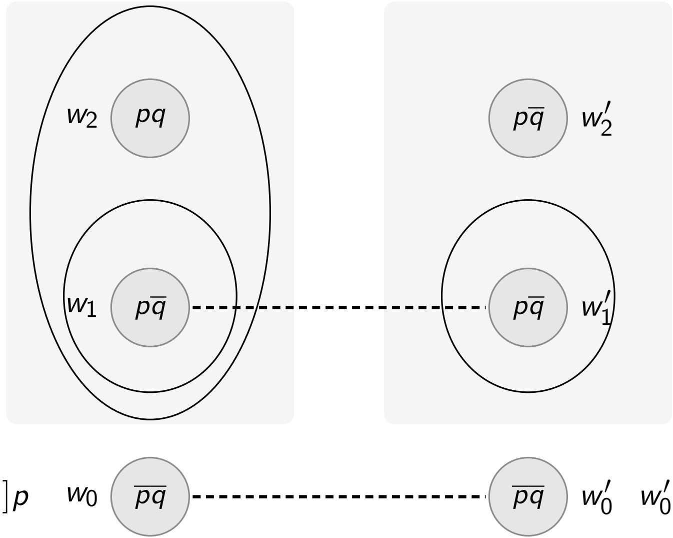
Zig: If $X \in N(w)$ then there is an $X' \subseteq W'$ such that

$X' \in N'(w')$ and $\forall x' \in X' \, \exists x \in X$ such that xZx'

Zag: If $X' \in N'(w')$ then there is an $X \subseteq W$ such that

$X \in N(w)$ and $\forall x \in X \, \exists x' \in X'$ such that xZx'

Example



$w_0 \models \Box p \wedge \langle\ ]p$

$w_0' \models \neg\Box p \wedge \langle\ ]p$

- We write $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}} \mathcal{M}', w'$ if there is a monotonic bisimulation Z such that wZw'.
- We write $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ iff for all φ in the language with the modality $\langle\ ]$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

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- **Lemma** If $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$.
- **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

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- **Lemma** If $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$.
- **Lemma** On finite models, if $\mathcal{M}, w \leftrightsquigarrow \mathcal{M}', w'$ then $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

M. Pauly. *Bisimulation for Non-normal Modal Logic*. Manuscript, 1999.

H. Hansen. *Monotonic Modal Logic*. Masters Thesis, ILLC, 2003.

H. Hansen, C. Kupke, EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5(2:2), pp. 1 - 38, 2009.

Locally Core-Finite Models

Suppose that $\mathcal{F}$ is a monotonic collection of subsets of W. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F} \text{ and for all } X' \subseteq W, \text{ if } X' \subseteq X, \text{ then } X' \notin \mathcal{F}\}.$$

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

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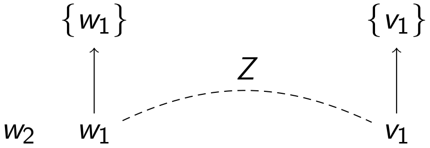
Question: Is every monotonic collection core-complete?

Locally Core-Finite Models

A neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ is **locally core-finite** provided that $\mathcal{M}$ is core-complete and for each $w \in W$, $N^{nc}(w)$ is finite, and for all $X \in N^{nc}(w)$, X is finite.

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are monotonic, locally core-finite models. Then, for all $w \in W$, $w' \in W'$, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Do monotonic bisimulations work when we drop monotonicity? No!



$\mathcal{M}$ $\quad$ $\mathcal{M}'$

$V(p) = \{w_1, w_2\}$ $\quad$ $V'(p) = \{v_1\}$

Bounded Morphisms

If $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models, and $f : W_1 \to W_2$ is a function, then f is a **(frame) bounded morphism** if

$$\text{for all } X \subseteq W_2 \text{, we have } f^{-1}[X] \in N_1(w) \text{ iff } X \in N_2(f(w));$$

and for all $p \in \mathsf{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

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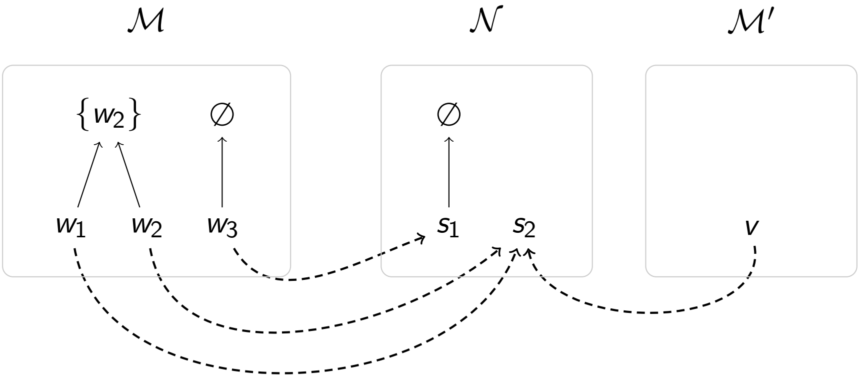
and for all $p \in \mathsf{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

Lemma Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models and $f : \mathcal{M}_1 \to \mathcal{M}_2$ a bounded morphism. For each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathcal{M}_1, w \models \varphi$ iff $\mathcal{M}_2, f(w) \models \varphi$.

Behavioral Equivalence

Definition

Two points w_1 from $\mathcal{M}_1$ and w_2 from $\mathcal{M}_2$ are **behaviorally equivalent** provided there is a neighborhood frame $\mathcal{F}$ and bounded morphisms $f : \mathcal{F}_1 \to \mathcal{F}$ and $g : \mathcal{F}_2 \to \mathcal{F}$ such that $f(w_1) = g(w_2)$.


$\mathcal{M}$
$\mathcal{N}$
$\mathcal{M}'$
$\{w_2\}$
$\emptyset$
$\emptyset$
w_1
w_2
w_3
s_1
s_2
v

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are two neighborhood models. If states $w \in W$ and $w' \in W'$ are behaviorally equivalent, then for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

Theorem

Over the class **N** *(of neighborhood models), the following are equivalent:*

- $\alpha(x)$ *is equivalent to the translation of a modal formula*
- $\alpha(x)$ *is invariant under behavioural equivalence.*

H. Hansen, C. Kupke and EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5(2:2), pgs. 1 - 38, 2009.

The Language $\mathcal{L}_2$

The language $\mathcal{L}_2$ is built from the following grammar:

$$x = y \mid u = v \mid \mathsf{P}_i x \mid x\mathsf{N}u \mid u\mathsf{E}x \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x \varphi \mid \exists u \varphi$$

$\mathfrak{M} = \langle D, \{P_i \mid i \in \omega\}, N, E \rangle$ where

- $D = D^{\mathrm{s}} \cup D^{\mathrm{n}}$ (and $D^{\mathrm{s}} \cap D^{\mathrm{n}} = \varnothing$),
- $P_i \subseteq D^{\mathrm{s}}$,
- $N \subseteq D^{\mathrm{s}} \times D^{\mathrm{n}}$ and
- $E \subseteq D^{\mathrm{n}} \times D^{\mathrm{s}}$.

The Language $\mathcal{L}_2$

Definition

Let $\mathcal{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathcal{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

The *standard translation* of the basic modal language are functions $st_x : \mathcal{L} \to \mathcal{L}_2$ defined as follows as follows: $st_x(p_i) = \mathrm{P}_i x$, st_x commutes with boolean connectives and

$$st_x(\Box\varphi) = \exists u(x\mathrm{R}_N u \wedge (\forall y(u\mathrm{R}_\ni y \leftrightarrow st_y(\varphi)))$$

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Lemma

Let $\mathcal{M}$ be a neighbourhood structure and $\varphi \in \mathcal{L}$. For each $s \in S$, $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}^\circ \models st_x(\varphi)[s]$.

$\mathbf{N} = \{\mathfrak{M} \mid \mathfrak{M} \cong \mathcal{M}^\circ \text{ for some neighbourhood model } \mathcal{M}\}$

(A1) $\exists x(x = x)$

(A2) $\forall u \exists x(x \mathrm{R}_N u)$

(A3) $\forall u, v(\neg(u = v) \to \exists x((u\mathrm{R}_\ni x \wedge \neg v\mathrm{R}_\ni x) \vee (\neg u\mathrm{R}_\ni x \wedge v\mathrm{R}_\ni x)))$

Theorem

Suppose $\mathfrak{M}$ is an $\mathcal{L}_2$-structure. Then there is a neighbourhood structure $\mathfrak{M}_\circ$ such that $\mathfrak{M} \cong (\mathfrak{M}_\circ)^\circ$.

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