Neighborhood Semantics for Modal Logic
Lecture 3

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Plan for Today

- Completeness
- Incompleteness
- Simulating non-normal modal logics

Logical consequence

Suppose that Γ is a set of formulas and $\mathbb{F}$ is a set of frames. We write $\mathcal{M}, w \models \Gamma$ iff $\mathcal{M}, w \models \alpha$ for all $\alpha \in \Gamma$.

$\Gamma \models_{\mathbb{F}} \varphi$ iff for all frames $\mathcal{F} \in \mathbb{F}$, for all models $\mathcal{M}$ based on $\mathcal{F}$ and all states w in $\mathcal{M}$, $\mathcal{M}, w \models \Gamma$ implies $\mathcal{M}, w \models \varphi$.

Soundness and Completeness

- A logic **L** is **sound** with respect to $\mathbb{F}$, provided $\vdash_{\mathbf{L}} \varphi$ implies $\models_{\mathbb{F}} \varphi$.
- A logic **L** is **weakly complete** with respect to a class of frames $\mathbb{F}$, if $\models_{\mathbb{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$.
- A logic **L** is **strongly complete** with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$.

A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas.

The **L-proof set** of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Canonical model

Let **L** be a logic and $\varphi, \psi \in \mathcal{L}$. Then

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. For any maximally **L**-consistent set Γ, if $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$
7. For any maximally **L**-consistent set Γ, If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$

Lindenbaum's Lemma. For any consistent set of formulas Γ, there exists a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

Canonical Model

Definition

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- $W = \{\Gamma \mid \Gamma$ is a maximally **L**-consistent set $\}$

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- for all $\varphi \in \mathcal{L}$ and $\Gamma \in W$, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\varphi \in \Gamma$

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- for all $\varphi \in \mathcal{L}$ and $\Gamma \in W$, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\varphi \in \Gamma$
- for all $p \in \mathrm{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathcal{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,

$$N_{\mathbf{L}}^{min}(\Gamma) = \{|\varphi|_{\mathbf{L}} \mid \Box\varphi \in \Gamma\}.$$

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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathcal{M}_{\mathbf{L}}^{max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{max}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,

$$N_{\mathbf{L}}^{max}(\Gamma) = N_{\mathbf{L}}^{min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin P_{\mathbf{L}}\}$$

The canonical model works...

Lemma

For any logic $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \to \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Lemma (Truth Lemma)

For any consistent classical modal logic $\mathbf{L}$ *and any consistent formula* φ, *if* $\mathcal{M}$ *is canonical for* $\mathbf{L}$,

$$[\![\varphi]\!]_{\mathcal{M}} = |\varphi|_{\mathbf{L}}$$

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The Proofs

Theorem

The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

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Lemma

If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem

The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

The Proofs

Fact: $\langle M_{\mathbf{EM}}, N_{\mathbf{EM}}^{min} \rangle$ is not closed under supersets.

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Fact: $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}} \rangle$ is not closed under supersets.

Lemma

Suppose that $\mathcal{M} = sup(\mathcal{M}^{min}_{\mathbf{EM}})$. *Then* $\mathcal{M}$ *is canonical for* **EM**.

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

The Proofs

Theorem

The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem

The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

What about the logic **EK**?

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$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$X \in N(w)$ and $\overline{X} \cup Y \in N(w)$ then $Y \in N(w)$.

Frederik van de Putte and Paul McNamara (2022). *Neighbourhood Canonicity for* **EK**, **ECK**, *and Relatives*. The Review of Symbolic Logic, 15(3), pp. 607-623.

$\mathcal{M}, w \models \Box(\psi_1, \ldots, \psi_k; \varphi)$ iff there is an $X \in N(w)$ such that

- for all $x \in X$, $\mathcal{M}, x \models \varphi$ and
- for all $i \in \{1, \ldots, k\}$ there is a $x_i \in X$ such that $\mathcal{M}, x_i \models \psi_i$

Johan van Benthem, Nick Bezhanishvili, Sebastian Enqvist, and Junhua Yu (2017). *Instantial Neighbourhood Logic*. The Review of Symbolic Logic 10(1), pp. 116 - 144.

Axiomatization

R-Mon $\Box(\gamma_1, \ldots, \gamma_j; \psi) \to \Box(\gamma_1, \ldots, \gamma_j; \psi \lor \chi)$

L-Mon $\Box(\gamma_1, \ldots, \gamma_j, \varphi; \psi) \to \Box(\gamma_1, \ldots, \gamma_j, \varphi \lor \chi; \psi)$

Axiomatization

R-Mon $\Box(\gamma_1, \dots, \gamma_j; \psi) \rightarrow \Box(\gamma_1, \dots, \gamma_j; \psi \vee \chi)$

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Inst $\Box(\gamma_1, \dots, \gamma_j, \varphi; \psi) \rightarrow \Box(\gamma_1, \dots, \gamma_j, \varphi \wedge \psi; \psi)$

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Norm $\neg\Box(\bot; \psi)$

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Norm $\neg\Box(\bot; \psi)$

Case $\Box(\gamma_1, \ldots, \gamma_j; \psi) \to \Box(\gamma_1, \ldots, \gamma_j, \delta; \psi) \vee \Box(\gamma_1, \ldots, \gamma_j; \psi \wedge \neg\delta)$

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Weak $\Box(\gamma_1, \dots, \gamma_j, \varphi, \delta_1, \dots, \delta_n; \psi) \to \Box(\gamma_1, \dots, \gamma_j, \delta_1, \dots, \delta_n, \varphi; \psi)$

Dupl $\Box(\gamma_1, \dots, \gamma_j, \varphi, \delta_1, \dots, \delta_n, \varphi; \psi) \to \Box(\gamma_1, \dots, \gamma_j, \delta_1, \dots, \delta_n, \varphi; \psi)$, provided $\varphi \in \{\gamma_1, \dots, \gamma_j, \delta_1, \dots, \delta_n\}$

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Weak $\Box(\gamma_1, \ldots, \gamma_j, \varphi, \delta_1, \ldots, \delta_n; \psi) \to \Box(\gamma_1, \ldots, \gamma_j, \delta_1, \ldots, \delta_n, \varphi; \psi)$

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MP From $\alpha \to \beta$ and α infer β

RE From $\alpha \leftrightarrow \beta$ and φ infer $\varphi[\alpha/\beta]$, where $\varphi[\alpha/\beta]$ is the result of possibly replacing some occurrences of α with β.

Theorem (Soundness and Weak Completeness)

For any formula φ*,* $\vdash \varphi$ *if, and only if,* $\models \varphi$*.*

Johan van Benthem, Nick Bezhanishvili, Sebastian Enqvist, and Junhua Yu (2017). *Instantial Neighbourhood Logic*. The Review of Symbolic Logic 10(1), pp. 116 - 144.

Incompleteness

φ is **globally true** in a Kripke model $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for all $w \in \mathcal{M}$

φ is **valid** in a Kripke frame $\mathcal{F}$, written $\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for all $\mathcal{M}$ based on $\mathcal{F}$

φ is **valid over a class $\mathbb{F}$ of frames** if for all $\mathcal{F} \in \mathbb{F}$, $\mathcal{F} \models \varphi$

For a class $\mathbb{F}$ of frames, let $Log(\mathbb{F}) = \{\varphi \mid \mathcal{F} \models \varphi \text{ for all } \mathcal{F} \in \mathbb{F}\}$

A logic **L** is **Kripke complete** if there is a class $\mathbb{F}$ of Kripke frames for which $\mathsf{L} = Log(\mathbb{F})$. Otherwise, it is **Kripke incomplete**

Let $Fr(\mathbf{L}) = \{\mathcal{F} \mid \mathcal{F} \models \varphi \text{ for all } \varphi \in \mathbf{L}\}$

For Kripke complete logics **L**, $\mathbf{L} = Log(Fr(\mathbf{L}))$

For a Kripke incomplete logic **L**, $\mathbf{L} \subsetneq Log(Fr(\mathbf{L}))$

Theorem (Thomason 1972; Fine 1975, Thomason 1974). There are Kripke incomplete logics.

Lattice

A **lattice** is an algebra $\mathcal{A} = (A, \wedge, \vee)$ where A is a set (called the *carrier set* or the *domain*) and $\wedge$ and $\vee$ are binary operators (i.e., functions mapping pairs of elements from A to elements of A) satisfying the following equations: for all $x, y, z \in A$:

$$\begin{array}{llll}
\text{(1a)} & x \vee x = x & \text{(1b)} & x \wedge x = x \\
\text{(2a)} & x \vee y = y \vee x & \text{(2b)} & x \wedge y = y \wedge x \\
\text{(3a)} & x \vee (y \vee z) = (x \vee y) \vee z & \text{(2b)} & x \wedge (y \wedge z) = (x \wedge y) \wedge z \\
\text{(4a)} & x \vee (x \wedge y) = x & \text{(4b)} & x \wedge (x \vee y) = x
\end{array}$$

Boolean Algebra

$\mathcal{A} = (A, \wedge, \vee)$ is a **distributive lattice** if $\mathcal{A}$ is a lattice and the following equations are satisfied: for all $x, y, z \in A$

(5a) $x \wedge (x \vee y) = (y \wedge z) \vee (x \wedge z)$ (5b) $x \vee (x \wedge y) = (y \vee z) \wedge (x \vee z)$

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A (distributive) lattice $\mathcal{A}$ is **bounded** if there are $0 \in A$ and $1 \in A$ such that: for all $x \in A$,

$$\begin{array}{llll} (6a) & x \vee 1 = 1 & (6b) & x \wedge 1 = x \\ (7a) & x \vee 0 = x & (7b) & x \wedge 0 = 0 \end{array}$$

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The structure $\mathcal{A} = (A, \wedge, \vee, -)$ is **Boolean algebra** if $(A, \wedge, \vee)$ is a bounded distributive lattice, $-$ is a unary operator on A satisfying the following equations: for all $x \in A$,

$$\text{(8a)} \quad x \vee -x = 1 \qquad \text{(8b)} \quad x \wedge -x = 0$$

Examples of Boolean Algebras

- $\mathbf{2} = (\{0,1\}, \wedge, \vee, -)$ where $0 \leq 1$ is a Boolean algebra, $-0 = 1$ and $-1 = 0$.

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- Suppose that $\mathcal{S} \subseteq \wp(W)$ is closed under $\cap$, $\cup$ and $\bar{\cdot}$. Then $(\mathcal{S}, \cap, \cup, \varnothing, W)$ is a Boolean algebra. It is a **subalgebra** of $\mathbf{2}^W$.

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- Suppose that $\mathcal{S} \subseteq \wp(W)$ is closed under $\cap$, $\cup$ and $\bar{\cdot}$. Then $(\mathcal{S}, \cap, \cup, \varnothing, W)$ is a Boolean algebra. It is a **subalgebra** of $\mathbf{2}^W$.
- Let $\mathcal{S} = \{X \subseteq \mathbb{N} \mid X \text{ is finite or } \mathbb{N} \setminus X \text{ is finite}\}$. Then $(\mathcal{S}, \cup, \cap, -, \varnothing, \mathbb{N})$ is a Boolean algebra.

Examples of Boolean Algebras

- Let At be a countable set of propositional variables, and $\mathit{Form}(\mathrm{At})$ the propositional formulas generated from At, $\wedge$, $\vee$ and $\neg$. Then, $(\mathit{Form}(\mathrm{At}), \wedge, \vee, \neg)$ is a Boolean algebra (called a **term algebra**)

Examples of Boolean Algebras

- Lindenbaum-Tarski algebra: Let At be a countable set of propositional variables, and $\mathsf{F} = \mathit{Form}(\mathsf{At})$ the propositional formulas generated from At, $\wedge$, $\vee$ and $\neg$.

  Suppose that $\vdash$ is derivability is some axiomatization of propositional logic. For $\varphi, \psi \in \mathit{Form}(\mathsf{At})$, write $\varphi \equiv \psi$ when $\vdash \varphi \leftrightarrow \psi$.

  Then $\equiv$ is an equivalence relation and a **congruence** on $(\mathit{Form}(\varphi), \wedge, \vee, \neg)$.

  The Lindenbaum-Tarski algebra is the quotient space, denoted $\mathsf{F}/{\equiv}$, is $(\{[\varphi] \mid \varphi \in \mathit{Form}(\mathsf{At}), \wedge, \vee, \neg)$ where $[\varphi] \vee [\psi] = [\varphi \vee \psi]$, $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$, and $\neg[\varphi] = [\neg\varphi]$.

  It is not hard to see that $\mathsf{F}/{\equiv}$ is a Boolean algebra.

Boolean Algebra with Operators

A *BAO* is a **Boolean algebra together with one more more unary operators** f such that $f(x \vee y) = f(x) \vee f(y)$ and for the bottom element of the algebra 0, $f(0) = 0$.

We often denote the operator f by '$\Diamond$'. So, a BAO is a tuple $\langle A, \wedge, \vee, \neg, 0, 1, \Diamond \rangle$ where A is a set and all the axioms 1a-8a, 1b-8b are all satisfied and $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ and $\Diamond 0 = 0$.

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Theorem. Every normal modal logic is sound and complete with respect to a BAO: The Lindenbaum-Tarski algebra of the logic

General Frames

General frames/models: $\langle W, R, \mathcal{A}\rangle$ where $\langle W, R\rangle$ is a frame, and $\mathcal{A} \subseteq \wp(W)$ is a BAO: Boolean algebra closed under the operator $R^{-1} : \wp(W) \to \wp(W)$: where for all X, $R^{-1}(X) = \{w \mid \text{there is a } v \in X \text{ with } w\,R\,v\}$.

A general model is a structure $\langle W, R, \mathcal{A}, V\rangle$, where $\langle W, R, \mathcal{A}\rangle$ is a general frame and for all $p \in \mathrm{At}$, $V(p) \in \mathcal{A}$.

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A general model is a structure $\langle W, R, \mathcal{A}, V\rangle$, where $\langle W, R, \mathcal{A}\rangle$ is a general frame and for all $p \in \mathrm{At}$, $V(p) \in \mathcal{A}$.

Theorem. Every consistent modal logic is sound and complete with respect to some class of general frames.

A Kripke frame $\mathcal{F} = \langle W, R\rangle$ is associated with its dual $\mathcal{F}^+ = \langle \wp(W), \cap, \cup, -, R^{-1}\rangle$.

Let $\mathfrak{A} = (A, \wedge, \vee, -, \bot, \top, \Diamond)$ be a BAO.

- $\mathcal{C}$: For all $X \subseteq A$, $\bigvee X$ exists and is an element of A
- $\mathcal{A}$: Any non-bottom element is above an **atom**, i.e., minimal non-bottom element (if $a \neq \bot$, then there is a $b \neq \bot$ such that $a > b$ and for all c if $b > c$, then $c = \bot$)
- $\mathcal{V}$: For all $X \subseteq A$, if $\bigvee X$ exists, then

$$\Diamond \bigvee X = \bigvee \{\Diamond x \mid x \in X\}$$

For every Kripke frame $\mathcal{F}$, $\mathcal{F}^+$ is a $\mathcal{CAV}$-*BAO*

Taking any Kripke frame/$\mathcal{CAV}$-*BAO*, converting it into its dual $\mathcal{CAV}$-*BAO*/Kripke frame, and then going back produces an output isomorphic to the original input. Therefore, Kripke completeness is just $\mathcal{CAV}$-completeness.

The fact that a normal modal logic is not the logic of any class of Kripke frames means that it is not the logic of any class of $\mathcal{CAV}$-*BAO*.

Let $\mathfrak{A} = (A, \wedge, \vee, -, \bot, \top, f)$ be a BAO and let $\theta : \mathsf{At} \to A$, then define $\hat{\theta}(p) = \theta(p)$, $\hat{\theta}(\neg\varphi) = -\hat{\theta}(\varphi)$; $\hat{\theta}(\varphi \vee \psi) = \hat{\theta}(\varphi) \vee \hat{\theta}(\psi)$; and $\hat{\theta}(\Diamond\varphi) = f\hat{\theta}(\varphi)$

The *BAO* $\mathfrak{A}$ validates a modal formula φ iff for all maps θ, $\hat{\theta}(\varphi) = \top$

$\Sigma \models_{\mathcal{X}} \varphi$ iff for every $\mathfrak{A} \in \mathcal{X}$, if $\mathfrak{A}$ validates σ for every $\sigma \in X$, then $\mathfrak{A}$ validates φ.

Let $\mathcal{X}$ be a class of *BAO*s and L a normal modal logic in a language with modal operators. We say that L is $\mathcal{X}$-complete if for all formulas φ, we have $\varphi \in \mathsf{L}$ iff $\mathsf{L} \models_{\mathcal{X}} \varphi$. Otherwise L is $\mathcal{X}$-incomplete.

$$(vB) \qquad \Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$$

Let vB be the smallest normal modal logic containing vB.

Theorem (van Benthem, 1979)

The logic vB *is incomplete.*

Lemma

Any Kripke frame that validates vB *also validates* $\Box\Diamond\top \to \Box\bot$.

Definition (van Benthem Frame)

Let $\mathcal{VB} = \langle W, R, \mathbb{W} \rangle$ where:

1. $W = \mathbb{N} \cup \{\infty, \infty + 1\}$;
2. $R = \{(\infty + 1, \infty), (\infty, \infty)\} \cup \{(\infty, n) \mid n \in \mathbb{N}\} \cup \{(m, n) \mid m, n \in \mathbb{N}, m > n\}$;
3. $\mathbb{W} = \{X \subseteq W \mid X \text{ is finite and } \infty \notin X\ \} \cup \{X \subseteq W \mid X \text{ is cofinite and } \infty \in X\ \}$

Lemma

$\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$ is valid over $\mathcal{VB}$ while $\Box\Diamond\top \to \Box\bot$ is not.

Given that the properties $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{V}$ are independent of each other, will arbitrary combinations of these three lead to distinct notions of completeness, each more general than Kripke completeness but less general than algebraic completeness? Or is the propositional modal language too coarse to care about differences between all or at least some of these semantics?

W. Holliday and T. Litak. *Complete Additivity and Modal Incompleteness*. The Review of Symbolic Logic, 2020.

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

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Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

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There are two logics **L** and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for **L** and $\mathbf{L}'$ respectively, but φ and φ' are not deducible in the respective logics).

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L$'$ is above **S4** (adapts Fine's incomplete logic)

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

Neighborhood completeness does not imply Kripke completeness

- extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

W. Holliday and T. Litak. *Complete Additivity and Modal Incompleteness*. The Review of Symbolic Logic, 2020.

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

W. Holliday and Y. Ding. *Another Problem in Possible World Semantics*. Proceedings of AiML, 2020.

Kaplan's Paradox

$$(A) \qquad \forall p \Diamond \forall q (Qq \leftrightarrow \Box (p \leftrightarrow q))$$

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For what sentential operators Q does (A) hold? As Kaplan writes:

"Perhaps, for every proposition, it is possible that it and only it is Queried [That is, it is asked whether it is the case that p....]. Or Perhaps not. It shouldn't really matter. There may be no operator expressible in English which satisfies (A). Still, *logic* shouldn't rule it out." (p. 43)

D. Kaplan (1995). *A problem in possible world semantics*. in: W. Sinnott-Armstrong, D. Raffman and N. Asher, editors, Modality, morality, and belief: essays in honor of Ruth Barcan Marcus, Cambridge University Press, Cambridge, pp. 41 - 52.

Two weaknesses as a problem for possible world semantics.

1. As (A) involves quantification over propositions in the object language, Kaplan's paradox does not pose a direct problem for possible world semantics for modal languages without propositional quantifiers.
2. Even if we want propositional quantification, on careful inspection (A) does not in fact target the *world* part of possible world semantics.

Basic modal language: $\varphi \quad := \quad p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi \mid Q\varphi$
where $p \in \mathsf{At}$

Frame: $\mathcal{M} = \langle W, N_\Box, N_Q \rangle$ where $W \neq \emptyset$, $N_\Box : W \to \wp(\wp(W))$ and $N_Q : W \to \wp(\wp(W))$

Model: $\mathcal{M} = \langle W, N_\Box, N_Q, V \rangle$ where $\langle W, N_\Box, N_Q \rangle$ is a frame and $V : \mathsf{At} \to \wp(W)$

Truth:

- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_\mathcal{M} \in N_\Box(w)$
- $\mathcal{M}, w \models Q\varphi$ iff $[\![\varphi]\!]_\mathcal{M} \in N_Q(w)$

A logic L is congruential if it contains all propositional tautologies, is closed under modus ponens, closed under uniforms substitution and closed under the congruence rule: if $\varphi \leftrightarrow \psi \in \mathsf{L}$, then $O\varphi \leftrightarrow O\psi \in \mathsf{L}$ (for each operator O).

$$(Split) \qquad p \to (\Diamond(p \land Qp) \land \Diamond(p \land \neg Qp))$$

Let S be the smallest congruential modal logic containing *Split* and $\Box\top$.

Theorem (Holliday and Ding, 2020)

- There is no neighborhood frame that validates S;
- If a *BAO* validates S, then it is atomless;
- The logic S is complete for a class of neighborhood *possibility* frames.

Proof

Suppose $\mathcal{F} = \langle W, N_\Box, N_Q \rangle$ validates S. Define a model $\mathcal{M} = \langle W, N_\Box, N_Q, V \rangle$, such that for some $w \in W$, $V(p) = \{w\}$.

Proof

Suppose $\mathcal{F} = \langle W, N_\Box, N_Q \rangle$ validates S. Define a model $\mathcal{M} = \langle W, N_\Box, N_Q, V \rangle$, such that for some $w \in W$, $V(p) = \{w\}$.

Since $\mathcal{F}$ validates *Split*, $\mathcal{M}, w \models \Diamond(p \wedge Qp) \wedge \Diamond(p \wedge \neg Qp)$.
$[\![\neg(p \wedge Qp)]\!]_{\mathcal{M}} \notin N_\Box(w)$ and $[\![\neg(p \wedge \neg Qp)]\!]_{\mathcal{M}} \notin N_\Box(w)$

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Since $V(p)$ is a singleton, $[\![p \wedge Qp]\!]_\mathcal{M} = \varnothing$ or $[\![p \wedge \neg Qp]\!]_\mathcal{M} = \varnothing$

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$[\![\neg(p \wedge Qp)]\!]_{\mathcal{M}} = W$ or $[\![\neg(p \wedge \neg Qp)]\!]_{\mathcal{M}} = W$. This implies $W \notin N_\Box(w)$, contradicting the validity of $\Box\top$.

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General Neighborhood Frames

A **general neighborhood frame** is a tuple $\mathcal{F}^g = \langle W, N, \mathcal{A} \rangle$ where $\langle W, N \rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

A valuation $V : \mathsf{At} \rightarrow \wp(W)$ is admissible for a general frame if for each $p \in \mathsf{At}$, $V(p) \in \mathcal{A}$.

Suppose that $\mathcal{F}^g = \langle W, N, \mathcal{A} \rangle$ is a general neighborhood frame. A **general modal** based on $\mathcal{F}^g$ is a tuple $\mathcal{M}^g = \langle W, N, \mathcal{A}, V \rangle$ where V is an admissible valuation.

General Neighborhood Frames

Lemma

Let $\mathcal{M}^g = \langle W, N, \mathcal{A}, V \rangle$ *be an general neighborhood model. Then for each* $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathcal{M}^g} \in \mathcal{A}$.

Lemma

Let **L** *be any logic extending* **E**. *Then a general canonical frame for* **L** *validates* **L**.

Corollary

Any modal logic extending **E** *is strongly complete with respect to some class of general frames.*

Summary

For any consistent modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of *general frames*

Summary

For any consistent modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of *general frames*

There are modal logics showing that

- neighborhood completeness does not imply Kripke completeness
- algebraic completeness does not imply neighborhood completeness

We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

Definition

Given a neighborhood model $\mathcal{M} = \langle W, N, V\rangle$, define a Kripke model $\mathcal{M}^\circ = \langle V, R_N, R_{\not\ni}, R_N, Pt, V\rangle$ as follows:

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- $Pt = W$

Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \mathrm{Pt}$$

where $p \in \mathrm{At}$ and Pt is a unary modal operator.

Define $ST : \mathcal{L} \to \mathcal{L}'$ as follows

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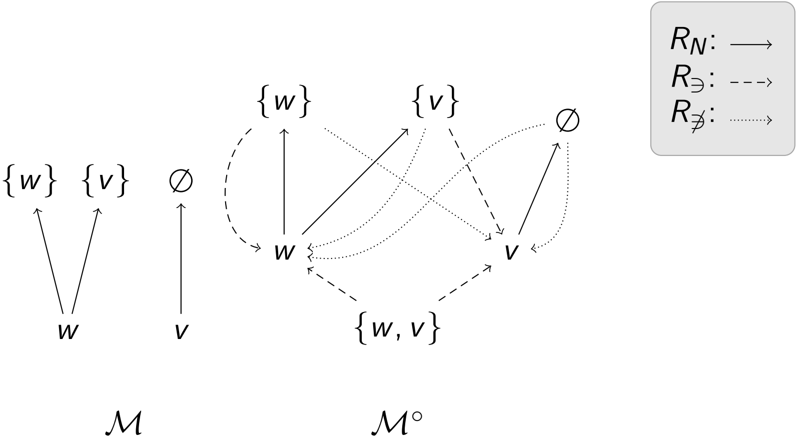
Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V\rangle$ *and each formula* $\varphi \in \mathcal{L}$, *for any* $w \in W$,

$$\mathcal{M}, w \models \varphi \textit{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$



$\mathcal{M}, w \models \Box p$ and $\mathcal{M}, v \models \Box\bot$.

- $\mathcal{M}^\circ, w \models \langle N\rangle([\ni]p \land [\not\ni]\neg p)$ and $\mathcal{M}^\circ, v \not\models \langle N\rangle([\ni]p \land [\not\ni]\neg p)$
- $\mathcal{M}^\circ, v \models \langle N\rangle([\ni]\bot \land [\not\ni]\top)$ and $\mathcal{M}^\circ, w \not\models \langle N\rangle([\ni]\bot \land [\not\ni]\top)$

Monotonic Models

Lemma

On Monotonic Models $\langle N\rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N\rangle([\ni]ST(\varphi))$

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. in Proof Theory of Modal Logic, Kluwer, pgs. 293 - 311, 1996.

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*. The Journal of Symbolic Logic, 64:1, pgs. 99 - 138, 1999.