Social Choice Theory and Machine Learning
Lecture 2

Eric Pacuit, University of Maryland

August 6, 2024

Plan for today

- ✓ A brief introduction to social choice theory
- ✓ A survey of voting methods
- Characterizing voting methods
- Splitting cycles and breaking ties
- ~~(time permitting) Probabilistic voting methods~~

Characterizing Voting Methods

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- Majority Rule for 2 candidates (May 1952; Asan and Sanver 2002)
- Plurality Rule (Richelson 1978; Ching 1996; Sekiguchi 2012)
- Borda (Young 1974; Nitzan and Rubinstein 1981; Maskin 2023)
- Instant Runoff Voting (Freeman, Brill, and Conitzer 2014)
- Any positional scoring rule (Young 1975)
- Copeland (Henriet 1985)
- Minimax for 3 candidates (Holliday and Pacuit, under submission, 2024)
- Split Cycle (Ding, Holliday, and Pacuit, forthcoming, 2024)

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$\Longrightarrow$ in two-alternative profiles, if the same number of voters rank a above b as b above a, then $F(\mathbf{P}) = \{a, b\}$.

- F satisfies **weak positive responsiveness** if for any profiles $\mathbf{P}$ and $\mathbf{P}'$, if
  1. $a \in F(\mathbf{P})$ and
  2. $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter who ranked a uniquely last in $\mathbf{P}$ switching to ranking a uniquely first in $\mathbf{P}'$,

  then $F(\mathbf{P}') = \{a\}$.

Voting on two alternatives

Theorem (May 1952)

Let F be a voting method on the domain of two-alternative profiles. Then the following are equivalent:

1. F satisfies anonymity, neutrality, and weak positive responsiveness;
2. F is Majority Voting.

Positional Scoring Rules

Theorem (Smith 1973, Young 1974)

A voting method satisfies Anonymity, Neutrality and **Reinforcement** if and only if F is a scoring rule.

For profiles $\mathbf{P}$ and $\mathbf{P}'$ with $V(\mathbf{P}) \cap V(\mathbf{P}') = \varnothing$, let $\mathbf{P} + \mathbf{P}'$ be the profile that combines the voters in $\mathbf{P}$ with the voters in $\mathbf{P}'$.

F satisfies **reinforcement** (also called **consistency**) provided that for all $\mathbf{P}$ and $\mathbf{P}'$ with $V(\mathbf{P}) \cap V(\mathbf{P}') = \varnothing$,

$$F(\mathbf{P} + \mathbf{P}') = F(\mathbf{P}) \cap F(\mathbf{P}')$$

Positional Scoring Rules

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A voting method satisfies Anonymity, Neutrality and **Reinforcement** if and only if F is a scoring rule.

Saari's argument, Balinski and Laraki (2010, pg. 77); Zwicker (2016, Proposition 2.5): Multiple districts paradox, f *cancels properly*. **No Condorcet consistent voting method satisfies reinforcement**:

2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c

- no Condorcet winner in the left profile
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

Borda

Theorem (Young 1975)

A voting method satisfies Anonymity, Neutrality and **Reinforcement**, **Faithfulness**, and **Cancels Properly** if and only if F is Borda.

F **cancels properly** if whenever $\mathbf{P}$ is a profile such that the number of voters preferring x to y equals the number preferring y to x for all $x, y \in X(\mathbf{P})$, then $F(\mathbf{P}) = X(\mathbf{P})$.

F is **faithful** if whenever $\mathbf{P}$ is a profile with a single voter, then $F(\mathbf{P}) = \{x\}$ where x is the candidate ranked first by the voter.

Splitting cycles and breaking ties

The Paradox of Voting

Consider an election with three candidates, a, b, and c and $3n$ voters, who rank the candidates as in the following preference profile:

n	n	n
a	b	c
b	c	a
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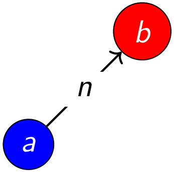

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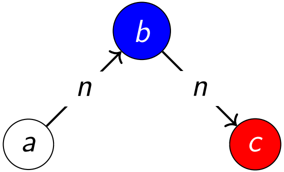

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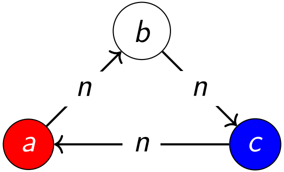

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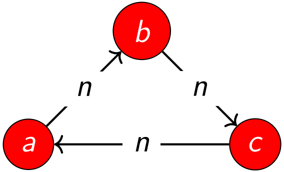

Applying majority rule, every candidate is defeated! So no one wins.

- $2n$ voters prefer a to b while n prefer b to a, so a **beats** b by a margin of n.
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This was discovered by Nicolas Condorcet in the 1780s.

Arrovian Social Choice

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Arrow wanted a collective choice rule (CCR), a function f that inputs a preference profile $\mathbf{P}$ and outputs a binary relation $f(\mathbf{P})$ on the set of candidates: we take $(a, b) \in f(\mathbf{P})$ to mean that a **defeats** b in $\mathbf{P}$ according to f.

An **undefeated candidate** can then be chosen as the winner.

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An **undefeated candidate** can then be chosen as the winner.

But as we have seen, as a CCR, majority voting can judge everyone to be defeated...

In the search for a better CCR, Arrow laid down a number of axioms.

Arrow's Theorem

Universal Domain: every profile is in the domain of the CCR.

Social Rationality: for any $\mathbf{P}$, $f(\mathbf{P})$ gives a ranking of the candidates (allowing ties).

Pareto: if all voters prefer x to y in $\mathbf{P}$, then $(x, y) \in f(\mathbf{P})$.

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Independence of Irrelevant Alternatives: the social ranking of x vs. y depends only on how voters rank x and y, i.e.,if there are two profiles $\mathbf{P}$, $\mathbf{P}'$ whose ballot-restrictions to x and y are equal, denoted $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then $(x, y) \in f(\mathbf{P})$ iff $(x, y) \in f(\mathbf{P}')$.

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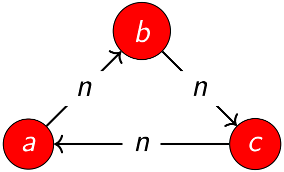
Theorem (Arrow 1951). If there are at least three candidates, the only CCRs satisfying the above axioms are **dictatorships**: there is some voter i such that for all profiles $\mathbf{P}$ and candidates x, y, if i prefers x to y in $\mathbf{P}$, then $(x, y) \in f(\mathbf{P})$.

n	n	n
a	b	c
b	c	a
c	a	b



This is a very special profile: it is perfectly **symmetrical**.

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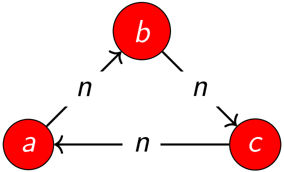

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The only perfectly symmetrical analogue in the two-candidate case is one where n voters prefer a over b and n voters prefer b over a, so there is a perfect tie.

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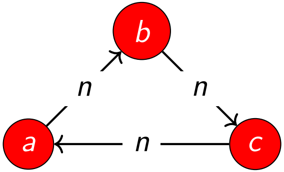

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The only perfectly symmetrical analogue in the two-candidate case is one where n voters prefer a over b and n voters prefer b over a, so there is a perfect tie.

Indeed, in May's setting where we pick a nonempty subset of the candidates, Anonymity and Neutrality imply that in such symmetrical profiles there must be a tie among all the candidates.

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The probability of such a tie goes to zero as the number of voters increases.

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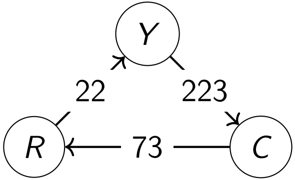

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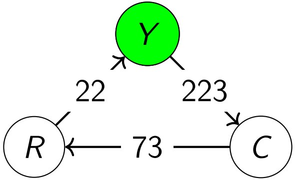

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Split Cycle

Generalizing the idea of the previous example leads to the **Split Cycle** voting method.

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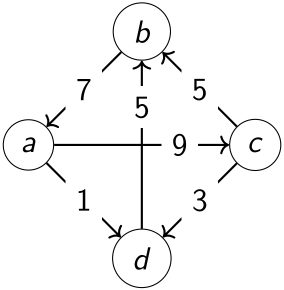
Split Cycle deals with the problem of majority cycles as follows:

1. In each majority cycle (if any), identify the head-to-head win(s) with the smallest margin of victory in that cycle.
2. After completing step 1 for all cycles, discard the identified wins. All remaining wins count as defeats of the losing candidates.

Example

Suppose an election produces the following majority margin graph:



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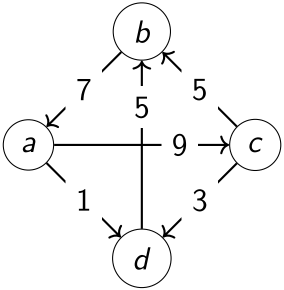
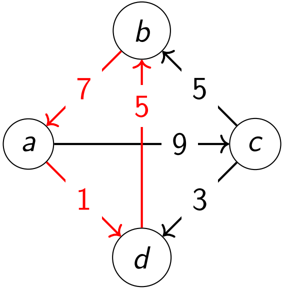


Our first step is to identify the cycles...

Example



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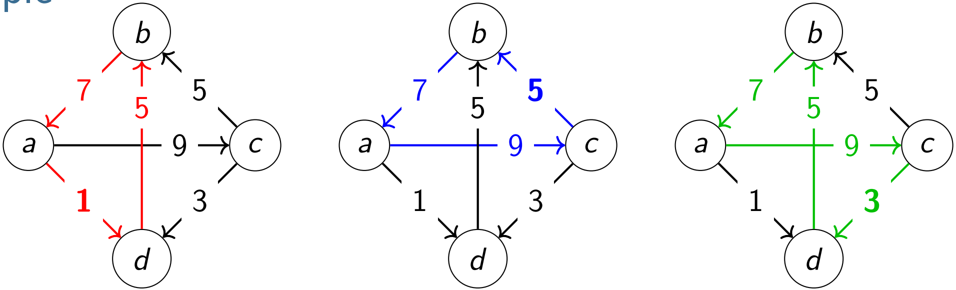
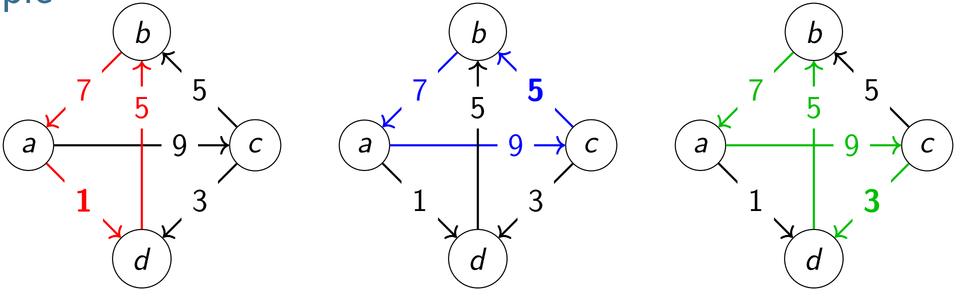


Next find the smallest margin in each cycle.

Example



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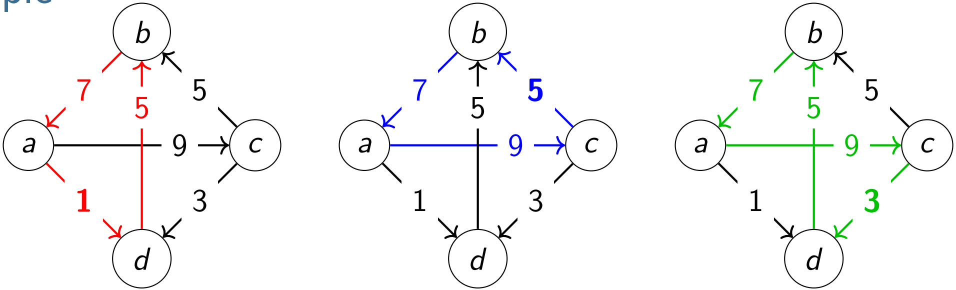
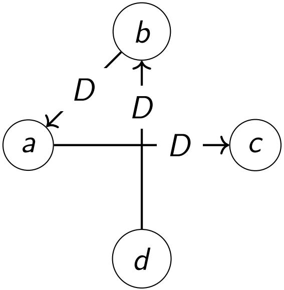


Next find the smallest margin in each cycle.
These edges cannot be defeats.
But all other edges are defeats:



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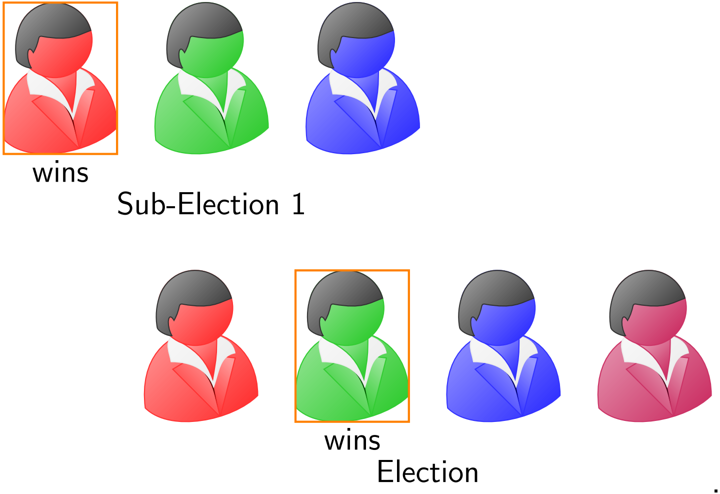
What makes Split Cycle special is the way it responds to the inclusion of **new candidates** and the inclusion of **new voters** in an election:

1. **Immunity to Spoilers**: if a wins in $\mathbf{P}_{-b}$ without b in the race, and more voters prefer a to b than vice versa, then it's not the case that both a and b lose in $\mathbf{P}$.
2. **Positive Involvement** (Saari 1995): if a would win in $\mathbf{P}$, and $\mathbf{P}'$ is obtained from $\mathbf{P}$ by adding one new voter who ranks a uniquely a uniquely first, then a still wins in $\mathbf{P}'$.

Spoilers

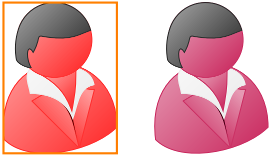
Candidate b is a **spoiler** for candidate a provided that a would win without b in the election, a would beat b in a head-to-head election, but neither a nor b wins in the election with b.

Spoilers

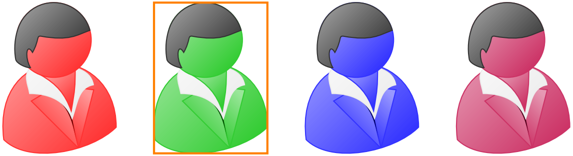


Spoilers

Sub-Election 1



Head-to-Head Election



Election

Spoilers

- **2000 Florida Presidential Election** (Plurality):

  Gore would have won had the election not included Nader, whom Gore (plausibly) beat head-to-head. But with Nader included, Bush won.

- **2007 Burlington Mayoral Election** (Instant Runoff):

  Montroll would have won had the election not included Wright, whom Montroll beat head-to-head. But with Wright included, Kiss won.

- **2022 Special Election for U.S. Rep. in Alaska** (Instant Runoff):

  Begich would have won had the election not included Palin, whom Begich beat heat-to-head. But with Palin included, Peltola won.

Positive Involvement

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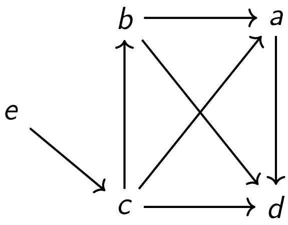
Positive Involvement

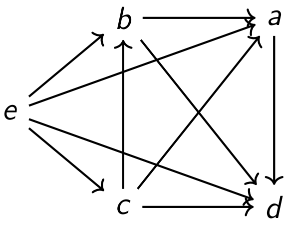
If *a* would win in **P**, and **P**′ is obtained from **P** by adding one new voter who ranks a uniquely *a* uniquely first, then *a* still wins in **P**′.

2	1	1
e	*c*	*a*
c	*b*	*d*
b	*a*	*b*
a	*d*	*e*
d	*e*	*c*



Copeland winners: $\{c\}$

2	1	1	1
e	*c*	*a*	*c*
c	*b*	*d*	*e*
b	*a*	*b*	*d*
a	*d*	*e*	*b*
d	*e*	*c*	*a*



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Note that we are now in a **variable election setting**: profiles can have different sets of candidates and voters, which is of course what happens in real life.

Split Cycle for Three Candidates

Theorem (Holliday and Pacuit, 2024) If a voting method satisfies May's axioms together with the following axioms, then it picks the same winners as Split Cycle in any three-candidate election in which there are no tied margins:

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Split Cycle for Three Candidates

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1. **Immunity to Spoilers**
2. **Positive Involvement**
3. **Homogeneity**: if a wins in $\mathbf{P}$, then a wins in $2\mathbf{P}$, where each voter is cloned;
4. **Block preservation**: if a wins in $\mathbf{P}$, and $\mathbf{P}'$ is obtained from $\mathbf{P}$ by adding exactly one voter submitting each linear order of the candidates, then a wins in $\mathbf{P}'$.

W. Holliday and E. Pacuit (2024). *An Extension of May's Theorem to Three Alternatives*. Under review, arXiv:2312.14256.

To axiomatize Split Cycle for more than 3 candidates, we must return to Arrow...

The Fallacy of IIA

Suppose x defeats y in a profile $\mathbf{P}$, and a profile $\mathbf{P}'$ is exactly like $\mathbf{P}$ with respect to how every voter ranks x vs. y. Should it follow that x defeats y in $\mathbf{P}'$?

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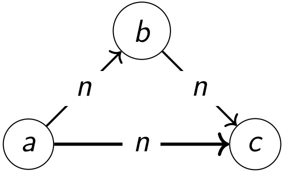
Arrow's Independence of Irrelevant Alternatives (IIA) says 'yes'.

We say 'no': if $\mathbf{P}'$ is sufficiently *incoherent*, we may need to suspend judgment on many defeat relations that could be coherently accepted in $\mathbf{P}$.

W. Holliday and E. Pacuit (2021). *Axioms for defeat in democratic elections*. Journal of Theoretical Politics, 33(4), pp. 475-524, https://doi.org/10.1177/09516298211043236.

In the context of the following perfectly coherent profile $\mathbf{P}$, the margin of n for a over b should be sufficient for a to defeat b:

n	n	n
$\boldsymbol{a}$	$\boldsymbol{b}$	c
$\boldsymbol{b}$	$\boldsymbol{a}$	$\boldsymbol{a}$
c	c	$\boldsymbol{b}$



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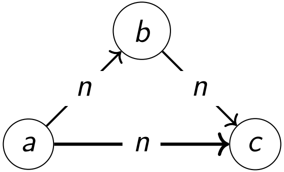
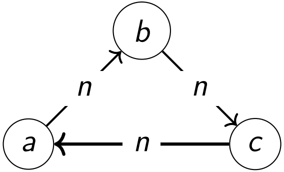


Yet in the following $\mathbf{P}'$ with $\mathbf{P}'_{|\{a,b\}} = \mathbf{P}_{|\{a,b\}}$, no VCCR satisfying Anonymity, Neutrality, and Availability can say that a defeats b:

n	n	n
a	b	c
b	c	a
c	a	b



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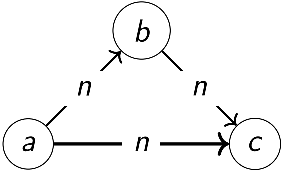
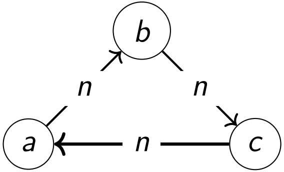


Yet in the following $\mathbf{P}'$ with $\mathbf{P}'_{|\{a,b\}} = \mathbf{P}_{|\{a,b\}}$, no VCCR satisfying Anonymity, Neutrality, and Availability can say that a defeats b:

n	n	n
a	b	c
b	c	a
c	a	b



This is a counterexample to IIA as a normative principle.

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Characterizing Split Cycle I

Theorem (Holliday and Pacuit 2021) In the variable election setting, Split Cycle is the most resolute CCR (i.e., locks in all the defeats of any CCR) satisfying **Coherent IIA** and the following axioms:

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1. **Anonymity** and **Neutrality**;
2. **Availability**: there is always some undefeated candidate;
3. **Upward homogeneity**: if a defeats b in $\mathbf{P}$, then a defeats b in $2\mathbf{P}$;
4. **Monotonicity**: if a wins in $\mathbf{P}$, and $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter moving a up in their ranking, keeping everything else the same, then a wins in $\mathbf{P}'$;
5. **Neutral Reversal**: if $\mathbf{P}'$ is obtained from $\mathbf{P}$ by adding exactly two voters who submit reversed linear orders of the candidates, then $f(\mathbf{P}) = f(\mathbf{P}')$.

W. Holliday and E. Pacuit (2021). *Axioms for defeat in democratic elections*. Journal of Theoretical Politics, 33(4), pp. 475-524.

Characterizing Split Cycle II

Theorem (Ding, Holliday, and Pacuit 2024) Any CCR satisfying the following axioms locks in all the defeats of Split Cycle:

1. **Coherent Defeat**: if more voters prefer a to b than vice versa, and there is no majority cycles with a followed by b, then a defeats b;
2. **Positive Involvement in Defeat**: if b doesn't defeat a in $\mathbf{P}$, and $\mathbf{P}'$ is obtained from $\mathbf{P}$ by adding one voter who ranks a above b, then b doesn't defeat a in $\mathbf{P}'$.

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Thus, Split Cycle is the unique CCR satisfying the above axioms, Coherent IIA, and the standard axioms from before.

Y. Ding, W. Holliday, and E. Pacuit (2024). *An Axiomatic Characterization of Split Cycle*. Forthcoming in Social Choice and Welfare.

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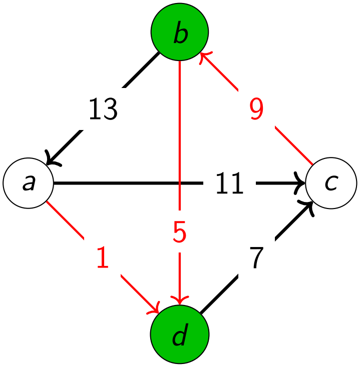

Split Cycle: $\{b, d\}$

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Definition

A Voting Method is **quasi-resolute** if there is a unique winner in any profile **P** that is *uniquely weighted*, i.e., any profile in which all the margins are unique.

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A Voting Method is **asymptotically resolvable** when, for a fixed number of candidates, the proportion of profiles with a tie goest to 0 as the number of voters goes to infinity.

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	3	4	5	6	7	8	9	10	20	30
Split Cycle	1	1.01	1.03	1.06	1.08	1.11	1.14	1.16	1.42	1.62
Copeland	1.17	1.26	1.29	1.3	1.31	1.31	1.31	1.31	1.28	1.25
Uncovered Set	1.17	1.35	1.53	1.71	1.9	2.09	2.26	2.46	4.56	6.82
Top Cycle	1.17	1.44	1.8	2.21	2.72	3.31	3.94	4.68	13.55	22.94

Figure: Estimated **average number of winners** for profiles with a given number of candidates (top row) in the limit as the number of voters goes to infinity, obtained using the Monte Carlo simulation technique in M. Harrison Trainor (2022). "An Analysis of Random Elections with Large Numbers of Voters," Mathematical Social Sciences, 116, pp. 68-84

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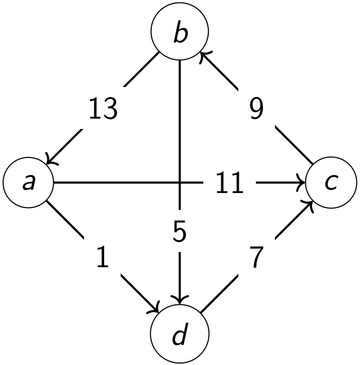

Minimax:	$\{d\}$
Copeland:	$\{a, b\}$
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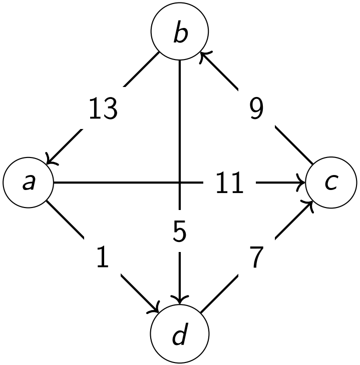

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Proposition (Holliday and Pacuit, 2023). Both Beat Path and Ranked Pairs are Quasi-Resolute refinements of Split Cycle.

Problem: Beat Path and Ranked Pairs both violate **Immunity to Spoilers** and a stronger property called **Stability for Winners**:

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Definition

A Voting Method F satisfies **Stability for Winners** (SW) if for every profile $\mathbf{P}$, any candidate who is stable for F in $\mathbf{P}$ wins in $\mathbf{P}$.

Impossibility Theorem (via SAT)

Theorem (Holliday, Norman, Pacuit, and Zahedian 2024) here is no Voting Method satisfying Anonymity, Neutrality, Stability for Winners, and Quasi-Resoluteness.

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Multiple claims based on stability

The basic problem is that inevitably there are profiles with *multiple* candidates who have the same kind of claim to winning based on Stability for Winners:

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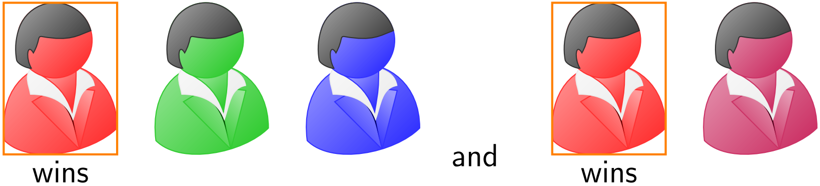

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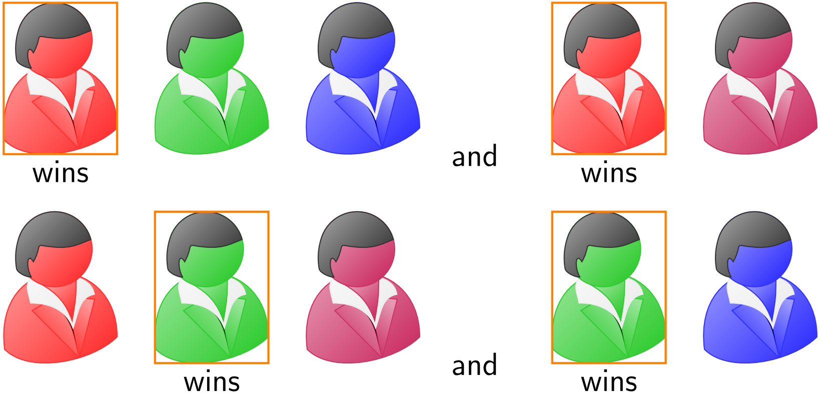

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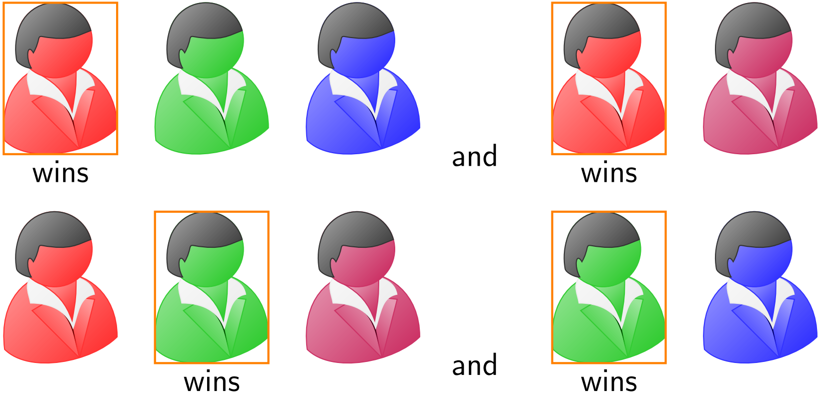

In such a situation—and only such a situation—it is legitimate to violate Stability for Winners for one of red or green in the name of tiebreaking between them.

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Is there a Voting Method that is Quasi-Resolute and satisfies Stability for Winners with Tiebreaking? Yes!