Probabilistic Methods in Social Choice

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Plan

- ✓ Background on voting theory
- ⇒ Generating preference profiles
- ⇒ Quantitative analysis of voting methods
- ▶ Probabilistic voting methods
- ▶ Condorcet jury theorem and related results
- ▶ Aggregating probabilistic judgements

Models of voters behavior: IC (Impartial culture), IAC (Impartial anonymous culture), IANC (Impartial anonymous and neutral culture), Mallows models, Spatial models, Structured Preferences (e.g., Single Peaked models)

`http://preflib.org`

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$$Pr(m, n) = \frac{\text{“the number of preference profiles that generate a Condorcet cycle”}}{\text{“the total number of preference profiles”}}$$

(A **preference profile** is a list of preferences, one for each voter.)

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$$Pr(3,3) = \frac{12}{216} = 0.0555555\ldots$$

The probability of a Condorcet paradox, II

Number of candidates		Number of voters: 3	5	7	9	11	⋯	∞
	3	.056	.069	.075	.078	.080		.088
	4	.111	.139	.150	.156	.160		.176
	5	.160	.200	.215				.251
	6	.212						.315
	⋮							⋮
	∞	1.00	1.00	1.00	1.00	1.00		1.00

(Source: W. Riker, *Liberalism Against Populism*, pg. 122)

How bad is this?

Is there *empirical evidence* that Condorcet cycles have shown up in real elections?

W. Riker. *Liberalism against Populism*. Waveland Press, 1982.

G. Mackie. *Democracy Defended*. Cambridge University Press, 2003.

Against the IC model

"...changing the distribution in *any fashion* (whether we call it 'realistic' or not) away from an impartial culture over linear orders will automatically have the effect of reducing the probability of majority cycles in infinite samples..." (pg., 28, 29)

This means that assuming an impartial culture is a *worst case analysis*.

M. Regenwetter, B. Gromfan, A. Marley, and I. Tsetlin. *Behavioral Social Choice*. Cambridge University Press, 2006.

See, also,

W. Gehrlein. *Condorcet's Paradox*. Springer, 2006.

W. Gehrlein, D. Lepelley and H. Smaoui. *The Condorcet Efficiency of Voting Rules with Mutually Coherent Voter Preferences: A Borda Compromise*. Annals of Economics and Statistics, Number 101/102, 2011.

Quantitative Analysis of Voting Methods

- What is the frequency of voting paradoxes?
- How should we use the frequency of voting paradoxes to compare voting methods?

F. Plassmann and T. N. Tideman. *How frequently do different voting rules encounter voting paradoxes in three-candidate elections?*. Social Choice and Welfare, 42, pp. 31 - 75, 2014.

We are interested in voting methods that:

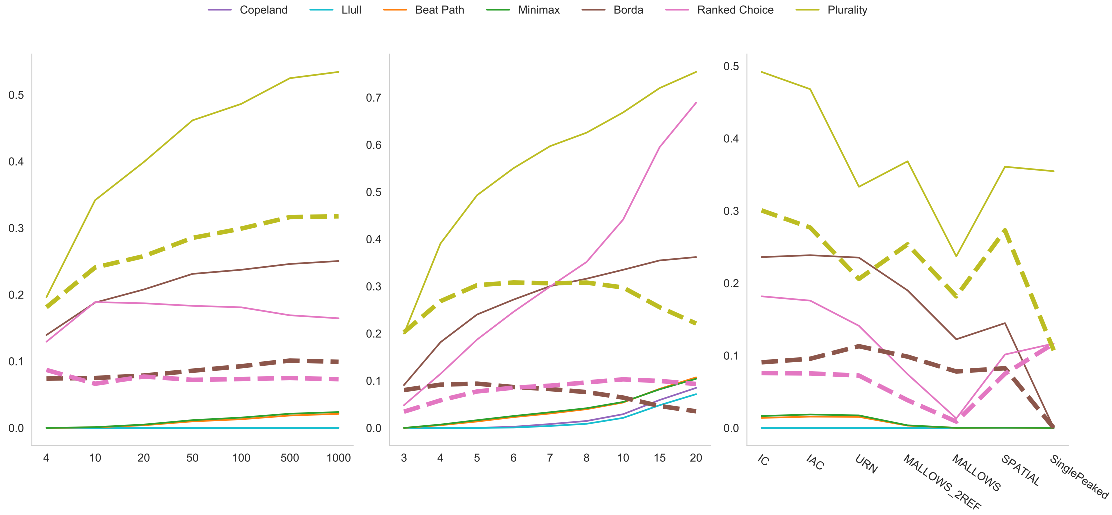
1. respond in a reasonable way to **new candidates** joining the election (Stability for Winners, Immunity of Spoilers);
2. respond in a reasonable way to **new voters** joining the election.

Frequencies of violation: Stability for Winners

How *often* do other methods violate, say, Stability for Winners?

Stability for Winners: For all profiles $\boldsymbol{P}$ and $a, b \in X(\boldsymbol{P})$, if $a \in F(\boldsymbol{P}_{-b})$ and $Margin_{\boldsymbol{P}}(a, b) > 0$, then $a \in F(\boldsymbol{P})$.

5 Candidates, (100, 101) Voters



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If we wish to use an axiom to discriminate between one method F_1 that satisfies the axiom and another method F_2 that violates the axiom, we should ask:

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- *in the profiles in which F_1 and F_2 disagree*, with what frequency does F_2 violate the axiom?

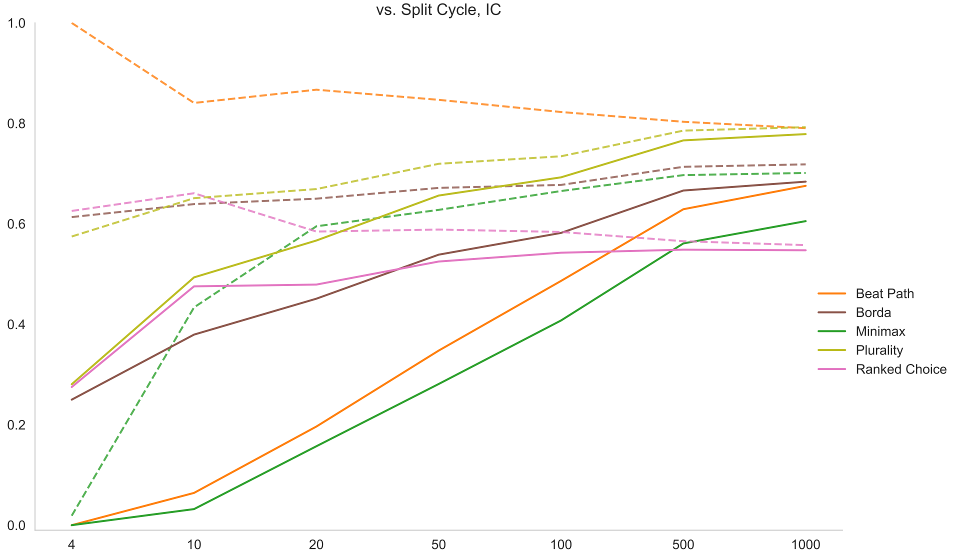

Figure: 5 candidates; on x-axis, # of voters; on y-axis, probability of F violating SW (solid) or SSW (dashed) conditional on F disagreeing with SC, according to IC.

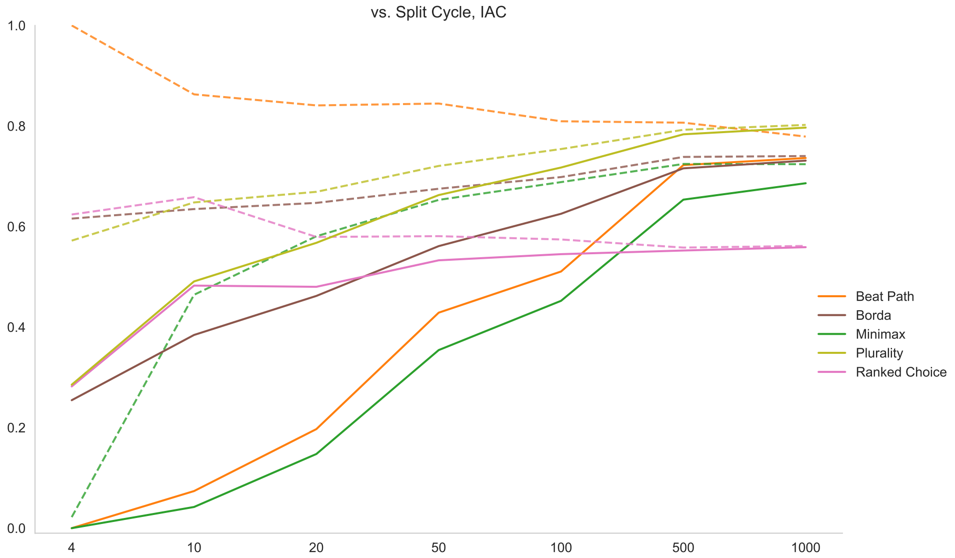

Figure: 5 candidates; on x-axis, # of voters; on y-axis, probability of F violating SW (solid) or SSW (dashed) conditional on F disagreeing with SC, according to IAC.

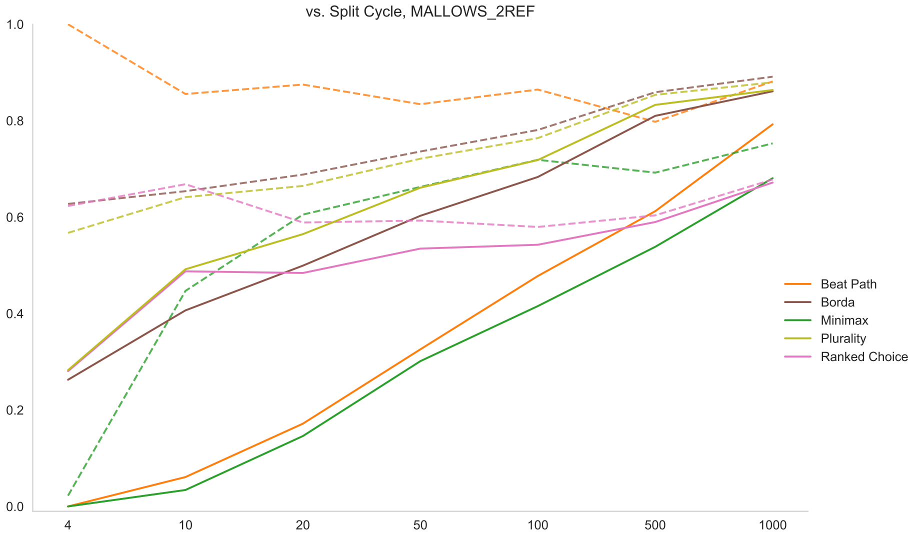

Figure: 5 candidates; on x-axis, # of voters; on y-axis, probability of F violating SW (solid) or SSW (dashed) conditional on F disagreeing with SC, according to Mallow's model with 2 reference rankings, inverses of each other, with $\phi = .08$.

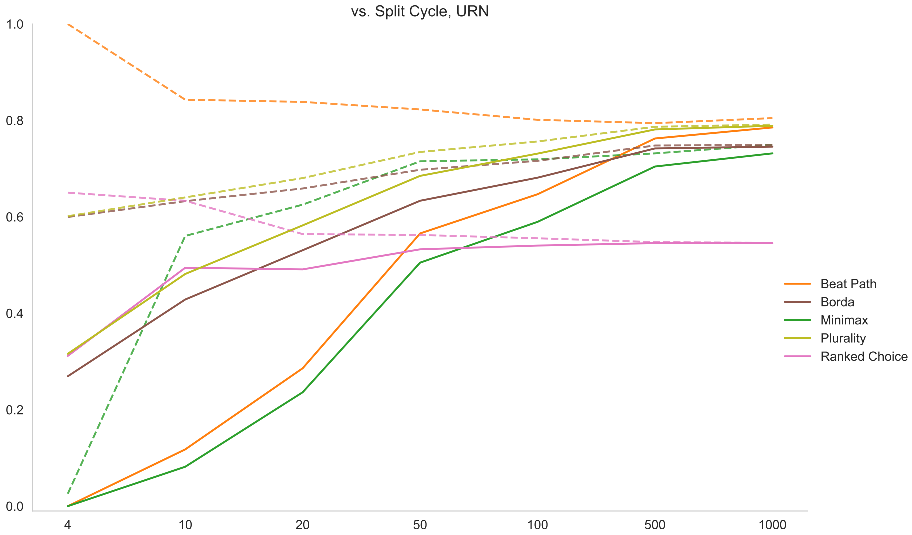

Figure: 5 candidates; on x-axis, # of voters; on y-axis, probability of F violating SW (solid) or SSW (dashed) conditional on F disagreeing with SC, according to the Pólya-Eggenberger urn model with $\alpha = 10$.

Methods that satisfy Expansion: Top Cycle, Uncovered Set, Split Cycle

Methods that satisfy Binary Expansion but violate Expansion: Banks

Methods that violate Binary Expansion: Plurality, Borda, Instant Runoff, Copeland, Minimax, Ranked Pairs, Beat Path, ...

Quasi-Resoluteness

Several of the methods violating Binary Expansion satisfy:

Asymptotic Resolvability: in the limit as the number of voters goes to infinity, the proportion of profiles with a unique winner before any tiebreaking (e.g., runoff election, lottery, etc.) goes to 1.

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For margin-based methods, asymptotic resolvability is equivalent to:

Quasi-Resoluteness: for any profile $\boldsymbol{P}$, if there are no ties in the margins of any candidates in $\boldsymbol{P}$, then $|F(\boldsymbol{P})| = 1$.

The equivalence follows from results in

M. Harrison Trainor. *An Analysis of Random Elections with Large Numbers of Voters*. arXiv:2009.02979.

Violations of Quasi-Resoluteness

The known methods that satisfy Binary Expansion violate Asymptotic Resolvability/Quasi-Resoluteness.

Voting Method	3	4	5	6	7	8	9	10	20	30
Split Cycle	1	1.01	1.03	1.06	1.08	1.11	1.14	1.16	1.42	1.62
Uncovered Set	1.17	1.35	1.53	1.71	1.9	2.09	2.26	2.46	4.56	6.82
Top Cycle	1.17	1.44	1.8	2.21	2.72	3.31	3.94	4.68	13.55	22.94

Figure: Estimated **average sizes of winning sets** for profiles with a given number of candidates (top row) in the limit as the number of voters goes to infinity, obtained using the Monte Carlo simulation technique in M. Harrison Trainor, "An Analysis of Random Elections with Large Numbers of Voters," arXiv:2009.02979.

The Cost of Quasi-Resoluteness

Theorem (W. Holliday, EP and S. Zahedian)

There is no Anonymous and Neutral voting method that satisfies Binary Expansion and Quasi-Resoluteness.

Moral: Making room for tiebreaking (runoff, lottery, etc.) is necessary and sufficient to find voting methods that satisfy Binary Expansion.