Neighborhood Semantics for Modal Logic
Lecture 1

Eric Pacuit, University of Maryland

August 5, 2024

https://pacuit.org/esslli2024/neighborhood-semantics/

Course Plan

1. **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Neighborhood Structures, Motivating Weak Modal Logics/Neighborhood Semantics (Monday, Tuesday)

2. **Core Theory**: Non-Normal Modal Logic, Completeness, Decidability, Complexity, Incompleteness, Relationship with Other Semantics for Modal Logic, Model Theory (Tuesday, Wednesday, Thursday)

3. **Extensions**: Inquisitive Logic on Neighborhood Models; First-Order Modal Logic, Subset Spaces, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods (Thursday, Friday)

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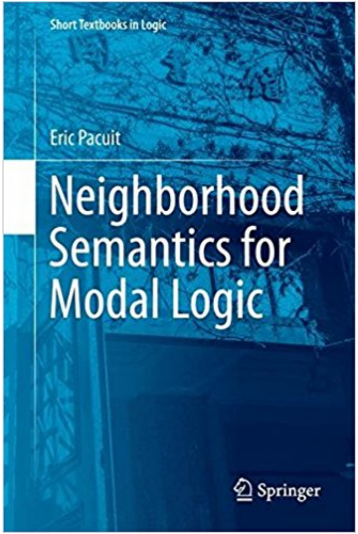
Short Textbooks in Logic
Eric Pacuit
Neighborhood Semantics for Modal Logic
Springer

Plan for Today

- Background: Relational semantics for modal logic, Logics of belief
- From Relational to Neighborhood Semantics
- Validities and Non-Validities
- Interpretation of Neighborhood Models: Evidence Models
- Neighborhood Frames/Models

Background: Relational Semantics

Suppose that At is a set of atomic propositions. The basic modal language $\mathcal{L}$ based on At is generated by the following grammar:

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi$$

where $p \in$ At. The Boolean connectives are defined as usual.

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where $p \in \mathsf{At}$. The Boolean connectives are defined as usual.

Relational Semantics

- Relational Frame: $\mathcal{F} = \langle W, R \rangle$ where $W \neq \emptyset$ and $R \subseteq W \times W$
- Relational Model: $\mathcal{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a frame and $V : \mathsf{At} \to \wp(W)$

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Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. Truth of a formula $\varphi \in \mathcal{L}$ at a state in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$ is defined as follows:

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- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

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- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
- Let $\Diamond\varphi$ be $\neg\Box\neg\varphi$
  $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that $w\,R\,v$ and $\mathcal{M}, v \models \varphi$

Background: Relational Semantics

Given a model $\mathcal{M} = \langle W, R, V \rangle$, let $[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$ be the map where:

$$[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$$

$$\begin{aligned}
&[\![p]\!]_{\mathcal{M}} = V(p) \\
&[\![\neg\varphi]\!]_{\mathcal{M}} = W \setminus [\![\varphi]\!]_{\mathcal{M}} \\
&[\![\varphi \vee \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cup [\![\psi]\!]_{\mathcal{M}} \\
&[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} \\
&[\![\Box\varphi]\!]_{\mathcal{M}} = \{w \mid \text{for all } x, \text{ if } w\, R\, x, \text{ then } x \in [\![\varphi]\!]_{\mathcal{M}}\}
\end{aligned}$$

Background: Doxastic Logic

Model: $\langle W, R, V \rangle$

States/possible worlds: $W \neq \emptyset$

Quasi-partitions: $R \subseteq W \times W$ is serial, transitive and Euclidean

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- *serial*: for all $w \in W$, there is a $v \in W$ such that $w\,R\,v$
- *transitive*: for all $w, v, u \in W$, if $w\,R\,v$ and $v\,R\,u$, then $w\,R\,u$
- *Euclidean*: for all $w, v, u \in W$, if $w\,R\,v$ and $w\,R\,u$, then $v\,R\,u$

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Valuation function: $V : \mathrm{At} \to \wp(W)$, where At is a set of atomic propositions.

Background: Doxastic Logic

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Boolean connectives:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
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Belief operators: $\mathcal{M}, w \models B\varphi$ iff for all v, if $w\, R\, v$, then $\mathcal{M}, v \models \varphi$.

Background: Doxastic Logic (**KD45**)

K	$B(\varphi \to \psi) \to (B\varphi \to B\psi)$
D	$B\varphi \to \neg B \neg \varphi$
4	$B\varphi \to BB\varphi$
5	$\neg B\varphi \to B\neg B\varphi$

Background: Doxastic Logic (**KD45**)

$$
\begin{array}{ll}
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\end{array}
$$

The logic **KD45** adds the above axiom schemes to an axiomatization of classical propositional logic with the rules Modus Ponens, Substitution of Equivalents, and Necessitation (from φ infer $B\varphi$).

KD45 is sound and strongly complete with respect to all quasi-partition frames.

Background: Doxastic Logic

Exercise: Show that the following axiom schemes and rules are valid on quasi-partition models and are theorems of **KD45**:

- agglomeration: $(B\varphi \wedge B\psi) \to B(\varphi \wedge \psi)$
- consistency: $\neg B\bot$
- monotonicity: From $\varphi \to \psi$ infer $B\varphi \to B\psi$

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- consistency: $\neg B\bot$
- monotonicity: From $\varphi \rightarrow \psi$ infer $B\varphi \rightarrow B\psi$
- $B(B\varphi \rightarrow \varphi)$
- correctness of own beliefs:
  $B\neg B\varphi \rightarrow \neg B\varphi$
  $BB\varphi \rightarrow B\varphi$

From Relational to Neighborhood Models

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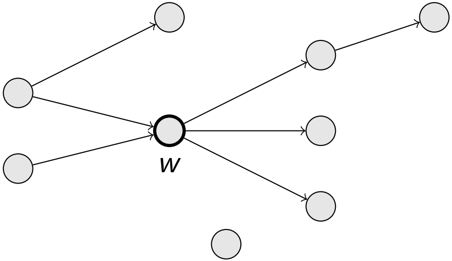

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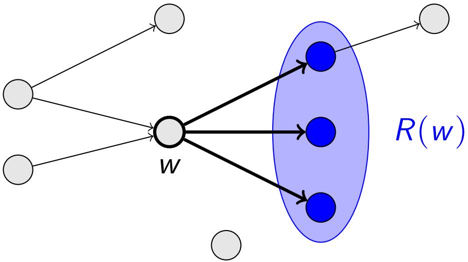

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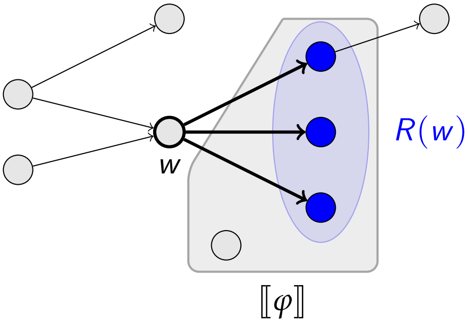

$w \models \Box\varphi$ iff for all v, if $w\,R\,v$, then $v \models \varphi$ (i.e., $R(w) \subseteq [\![\varphi]\!]$)

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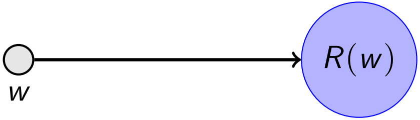
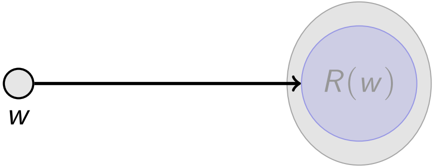


A relation $R \subseteq W \times W$ can be viewed as a function $R : W \to \wp(W)$

From Relational to Neighborhood Models



$w \models \Box\varphi$ iff **the** set of accessible worlds is contained in the truth-set of φ

$R(w) \subseteq [\![\varphi]\!]$

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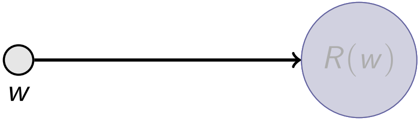
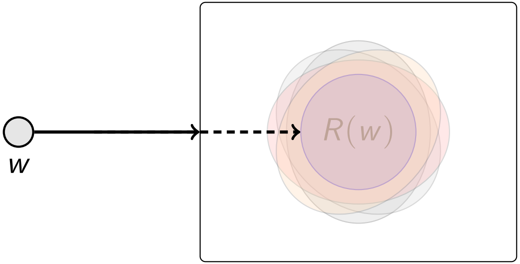


$w \models \boxplus\varphi$ iff **the** set of accessible worlds is equal to the truth-set of φ

$R(w) = [\![\varphi]\!]$

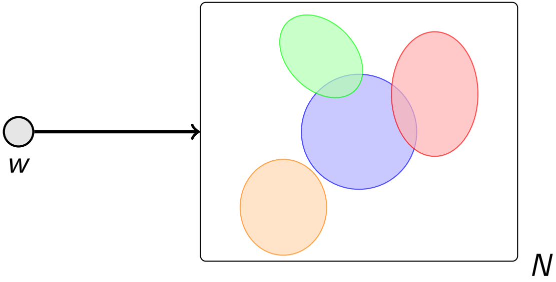
From Relational to Neighborhood Models



$w \models \Box\varphi$ iff **the** neighborhood of w is contained in the truth-set of φ

$R(w) \subseteq [\![\varphi]\!]$

From Relational to Neighborhood Models



$w \models \Box\varphi$ iff **a** neighborhood of w

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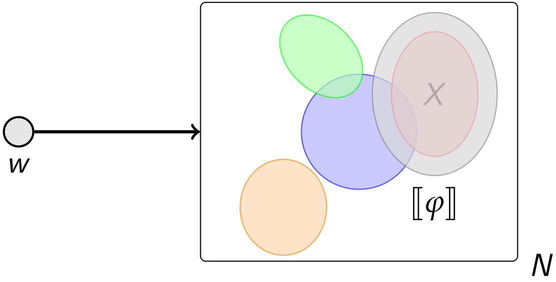

$w \models \Box\varphi$ iff **a** neighborhood of w is contained in the truth-set of φ

there is a $X \in N(w)$ such that $X \subseteq [\![\varphi]\!]$

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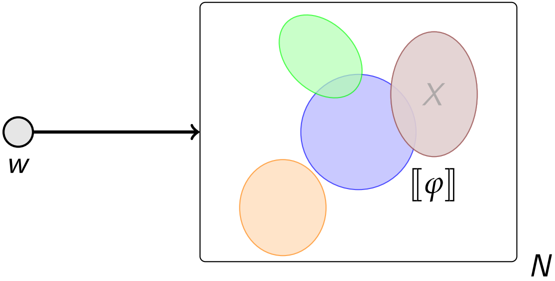

$w \models \Box\varphi$ iff the truth-set of φ **is** a neighborhood of w

there is a $X \in N(w)$ such that $X = [\![\varphi]\!]$ (i.e., $[\![\varphi]\!] \in N(w)$)

From Relational to Neighborhood Models

Relational model: $\langle W, R, V\rangle$ where $R : W \to \wp(W)$

- $w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]$

Neighborhood model: $\langle W, N, V\rangle$ where $N : W \to \wp(\wp(W))$

- $w \models \Box\varphi$ iff $[\![\varphi]\!] \in N(w)$
- $w \models \langle\ ]\varphi$ iff there is a $X \in N(w)$ such that $X \subseteq [\![\varphi]\!]$

Scott-Montague Semantics

> To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.
>
> (Montague, pg. 73)

R. Montague (1970). *Pragmatics and Intentional Logic*. Synthese, 22, pp. 68 - 94.

D. Scott (1970). *"Advice on modal logic", in Philosophical Problems in Logic*. Reidel.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

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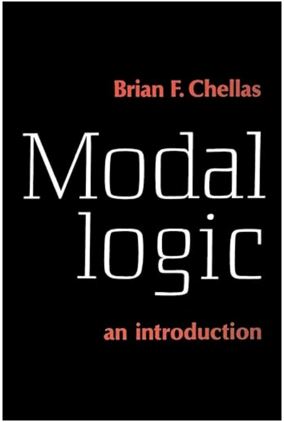
K. Segerberg. *An Essay on Classical Modal Logic*. Uppsala Technical Report, 1970.

> This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.
>
> (pg. 1)

Minimal Models for Modal Logic (Part III)



B. Chellas (1980). *Modal Logic: An Introduction*. Cambridge University Press.

Validities and Non-Validities

Some Validities

(M)	$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$(Dual)$	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Some Validities

(M) $\quad \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

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(K) $\quad \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

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(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

$$(RM) \quad \frac{\vdash \varphi \rightarrow \psi}{\vdash \Box\varphi \rightarrow \Box\psi}$$

...on Neighborhood Models

~~(M)~~ $\cancel{\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi}$

~~(C)~~ $\cancel{\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)}$

~~(N)~~ $\cancel{\Box\top}$

~~(K)~~ $\cancel{\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)}$

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Agglomeration Axiom

$$(C) \qquad (\Box\varphi \land \Box\psi) \rightarrow \Box(\varphi \land \psi)$$

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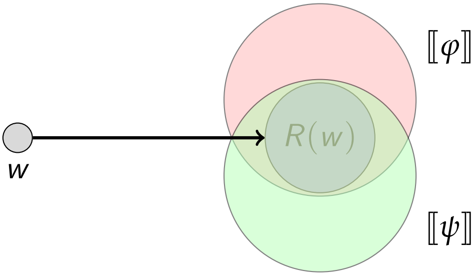

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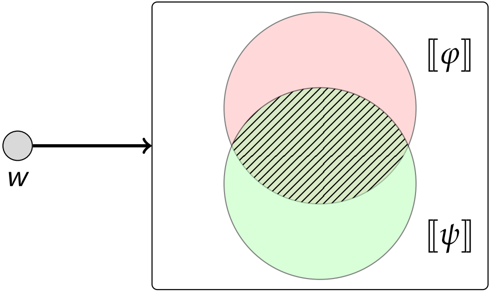

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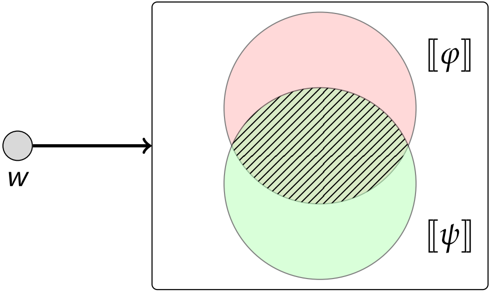

- Logics of high-probability
- Rational belief
- Ability
- Weakly aggregative logics

Logics of High Probability

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Claim: $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is not valid.

H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

J. Hawthorne (2009). *The Lockean Thesis and the Logic of Belief*. In *Degrees of belief*, Franz Huber & Christoph Schmidt-Petri (eds.), Springer, pp. 49 - 74.

Preface Paradox

D. Makinson (1965). *The Paradox of the Preface*. Analysis, 25, pp. 205 - 207.

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$$B(\neg(s_1 \wedge s_2 \wedge \cdots \wedge s_n))$$

But $(B(s_1) \wedge \cdots \wedge B(s_n)) \rightarrow B(s_1 \wedge \cdots \wedge s_n)$ is valid.

Preface Paradox: The Problem

“The author of the book is being rational even though inconsistent. More than this: he is being rational even though he believes each of a certain collection of statements, which *he knows* are logically incompatible....this appears to present a living and everyday example of a situation which philosophers have commonly dismissed as absurd; that it is sometimes rational to hold incompatible beliefs.”

D. Makinson. *The Paradox of the Preface*. Analysis, 25, 205 - 207, 1965.

Defining Beliefs from Knowledge

R. Stalnaker (2006). *On logics of knowledge and belief*. Philosophy Studies,128, pp. 169 - 199.

A. Baltag, N. Bezhanishvili, A. Özgün, and S. Smets (2019). *A Topological Approach to Full Belief*. Journal of Philosophical Logic, 48(2), pp. 205 - 244.

A. Bjorndahl and A. Özgün (2020). *Logic and Topology for Knowledge, Knowability, and Belief*. The Review of Symbolic Logic, 13(4), pp. 748-775.

Defining Beliefs from Knowledge

Stalnaker bases his analysis on a conception of belief as ‘subjective certainty’: From the point of the agent in question, her belief is subjectively indistinguishable from her knowledge.

Defining Beliefs from Knowledge

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Bi-modal language of knowledge and belief: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid B\psi$
Define $\langle K \rangle \varphi$ as $\neg K \neg \varphi$ and $\langle B \rangle \varphi$ as $\neg B \neg \varphi$

Defining Beliefs from Knowledge

K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
CB	$B\varphi \to \neg B \neg \varphi$

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KB	$K\varphi \to B\varphi$
FB	$B\varphi \to BK\varphi$

Defining Beliefs from Knowledge

Proposition (Stalnaker). The following equivalence is a theorem of the propositional modal logic that contains the previous axiom schemas (with Modus Ponens and Necessitation for both K and B):

$$B\varphi \leftrightarrow \langle K \rangle K\varphi$$

Defining Beliefs from Knowledge

Proposition (Stalnaker). The following equivalence is a theorem of the propositional modal logic that contains the previous axiom schemas (with Modus Ponens and Necessitation for both K and B):

$$B\varphi \leftrightarrow \langle K\rangle K\varphi$$

Moreover, all of the axioms of **KD45** and the (.2)-axiom $\langle K\rangle K\varphi \to K\langle K\rangle\varphi$ are provable.

Defining Beliefs from Knowledge

This means that we can take the logic of knowledge to be **S4.2** (the axioms K, T, 4 and .2) and *define* full belief as the 'epistemic possibility of knowledge'.

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K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
.2	$\langle K \rangle K\varphi \to K\langle K \rangle\varphi$
Nec	From φ infer $K\varphi$
DefKB	$B\varphi \leftrightarrow \langle K \rangle K\varphi$

Defining Beliefs from Knowledge

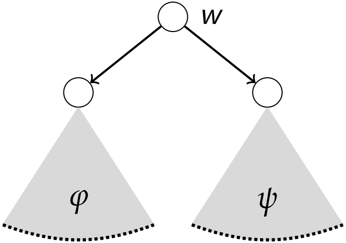
This means that we can take the logic of knowledge to be **S4.2** (the axioms K, T, 4 and .2) and *define* full belief as the 'epistemic possibility of knowledge'.

$$
\begin{array}{ll}
K & K(\varphi \to \psi) \to (K\varphi \to K\psi) \\
T & K\varphi \to \varphi \\
4 & K\varphi \to KK\varphi \\
.2 & \langle K \rangle K\varphi \to K\langle K \rangle \varphi \\
Nec & \text{From } \varphi \text{ infer } K\varphi \\
 & \\
DefKB & B\varphi \leftrightarrow \langle K \rangle K\varphi
\end{array}
$$

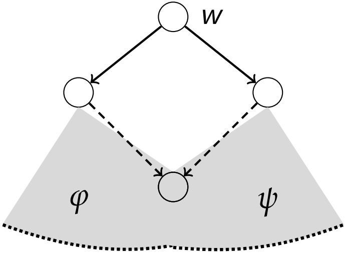
Claim. B validates all of the **KD45** axioms. In particular,

$$(B\varphi \wedge B\psi) \to B(\varphi \wedge \psi)$$

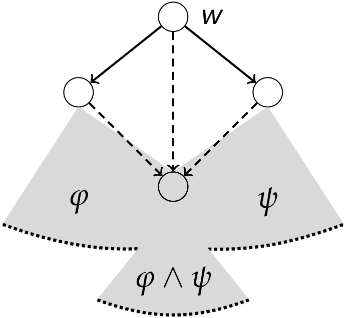
is valid/derivable.



- $w \models \langle K \rangle K\varphi \wedge \langle K \rangle K\psi$



- $w \models \langle K \rangle K\varphi \wedge \langle K \rangle K\psi$
- $\langle K \rangle K\varphi \to K\langle K \rangle\varphi$
  convergence: for all x, y, z if $x\,R\,y$ and $x\,R\,z$, then there is a v such that $y\,R\,v$ and $z\,R\,v$.



- $w \models \langle K \rangle K\varphi \wedge \langle K \rangle K\psi$
- $\langle K \rangle K\varphi \to K\langle K \rangle\varphi$
  convergence: for all x, y, z if $x\,R\,y$ and $x\,R\,z$, then there is a v such that $y\,R\,v$ and $z\,R\,v$.
- $K\varphi \to KK\varphi$
  transitivity: for all x, y, z if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$.
- $w \models \langle K \rangle K(\varphi \wedge \psi)$

$(\langle K \rangle K\varphi \wedge \langle K \rangle K\psi) \rightarrow \langle K \rangle K(\varphi \wedge \psi)$ is not valid if we drop convergence and/or transitivity.

D. Klein, N. Gratzl, and O. Roy (2015). *Introspection, normality and agglomeration*. Logic, Rationality, and Interaction, 5th Workshop, LORI 2015, pp. 195 - 206.

Ability

$Abl_i \varphi$: i is able to *guarantee* that φ is true
(alternatively, i has the ability to *see to it that* φ)

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Are the following valid?

1. $(Abl_i\varphi \wedge Abl_i\psi) \rightarrow Abl_i(\varphi \wedge \psi)$
2. $Abl_i(\varphi \vee \psi) \rightarrow (Abl_i\varphi \vee Abl_i\psi)$

$$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$$

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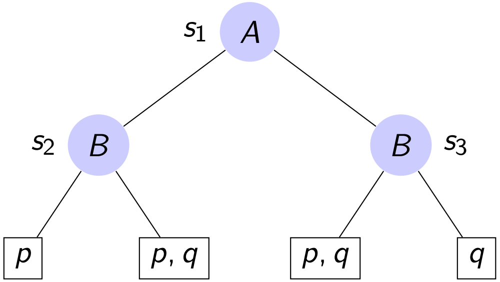
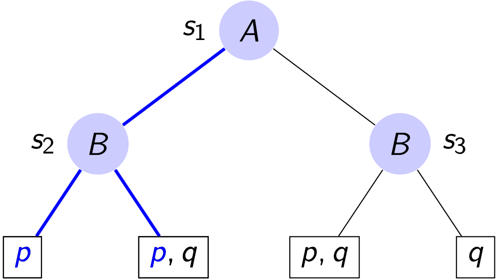
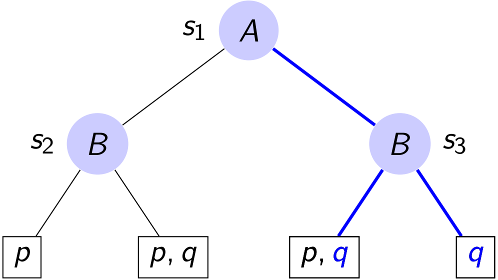


$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



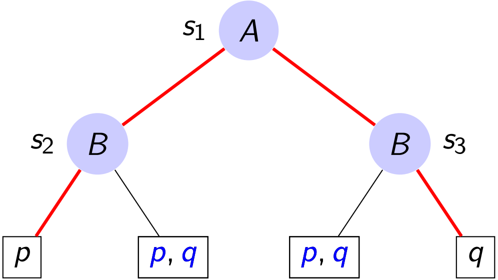
$$s_1 \models Abl_A p$$

$(Abl_i\varphi \wedge Abl_i\psi) \not\rightarrow Abl_i(\varphi \wedge \psi)$



$$s_1 \models Abl_A p \wedge Abl_A q$$

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



$$s_1 \models Abl_A p \wedge Abl_A q \wedge \neg Abl_A(p \wedge q)$$

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

See the course by Val Goranko on Logics for Strategic Reasoning!

R. Parikh (1985). *The Logic of Games and its Applications.* Annals of Discrete Mathematics.

M. Pauly and R. Parikh (2003). *Game Logic — An Overview.* Studia Logica.

J. van Benthem (2014). *Logic in Games.* The MIT Press.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

However, intuitively, Ann does not have the ability to hit the top half of the dart board, and does not have the ability to hit the bottom half of the dart board.

Weakly Aggregative Logics

Weakly Aggregative Logics

Suppose that the instructors at ESSLLI are divided into groups of two and you are told the following: For each pair,

- at least one person in the pair teaches a course on logic; and
- at least one person in the pair teaches a course on language.

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Does this imply that there is at least one person from each pair that teaches a course on logic *and* language?

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- at least one person in the pair teaches a course on logic; and
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Does this imply that there is at least one person from each pair that teaches a course on logic *and* language? No!

Weakly Aggregative Logics

Suppose that the instructors at ESSLLI are divided into groups of two and you are told the following: For each pair,

- at least one person in the pair teaches a course on logic;
- at least one person in the pair teaches a course on language; and
- at least one person in the pair teaches a course on computation.

What can you conclude now?

Weakly Aggregative Logics

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- at least one person in the pair teaches a course on logic;
- at least one person in the pair teaches a course on language; and
- at least one person in the pair teaches a course on computation.

What can you conclude now? For each pair, there must be one person that:

1. teaches a course on logic and language,
2. teaches a course on logic and computation, or
3. teaches a course on language and computation.

Weakly Aggregative Logics

$\Box p$: For each pair of people, at least one person in the pair has property p

Weakly Aggregative Logics

$\Box p$: For each pair of people, at least one person in the pair has property p

- $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ is not valid
- $(\Box p \wedge \Box q \wedge \Box r) \to \Box((p \wedge q) \vee (p \wedge r) \vee (q \wedge r))$ is valid

Weakly Aggregative Logics: n-ary Relational Model

An n-**ary relational model** is a tuple $\langle W, R, V\rangle$ where W is a non-empty set and $R \subseteq W^{n+1}$ is an $(n+1)$-ary relation and $V : \mathrm{At} \to \wp(W)$ is a valuation function.

Weakly Aggregative Logics: n-ary Relational Model

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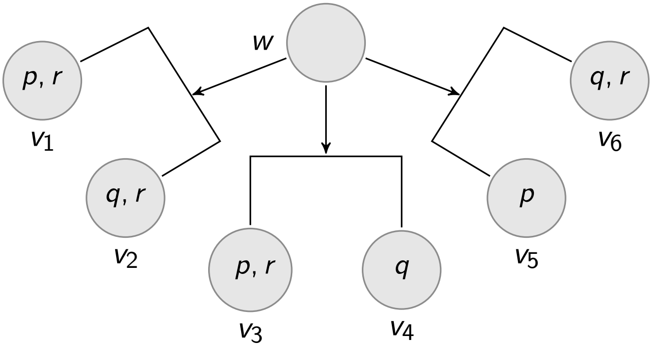
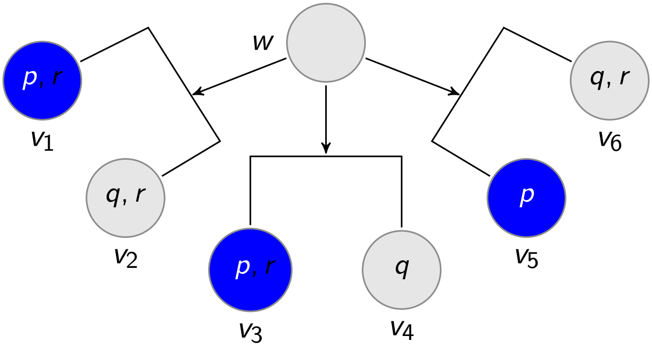
Suppose that $\mathcal{M}^n = \langle W, R, V \rangle$ is an n-relational model an φ is a formula. Truth is defined as follows:

- The atomic propositions and Boolean connectives are defined as usual.
- $\mathcal{M}^n, w \models \Box\varphi$ iff for all $v_1, \ldots, v_n \in W$, if $(w, v_1, \ldots, v_n) \in R$, then there exists i such that $1 \leq i \leq n$ and $\mathcal{M}^n, v_i \models \varphi$.

Weakly Aggregative Logics

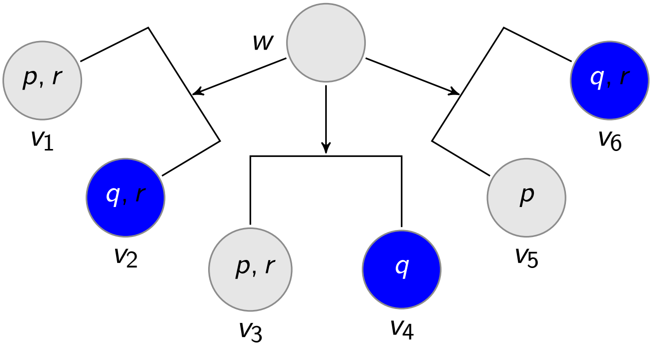


Weakly Aggregative Logics



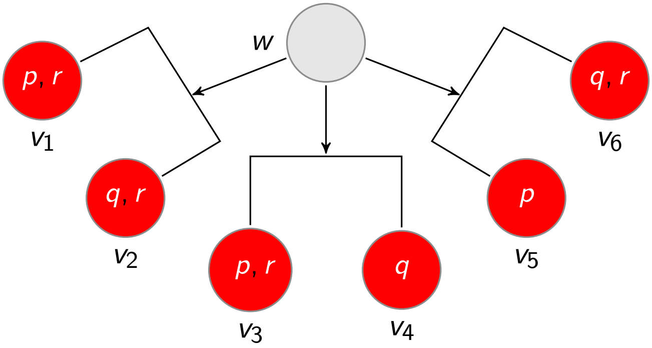
- $\mathcal{M}^2, w \models \Box p$ (and $\mathcal{M}^2, w \models \Box\neg p$)

Weakly Aggregative Logics



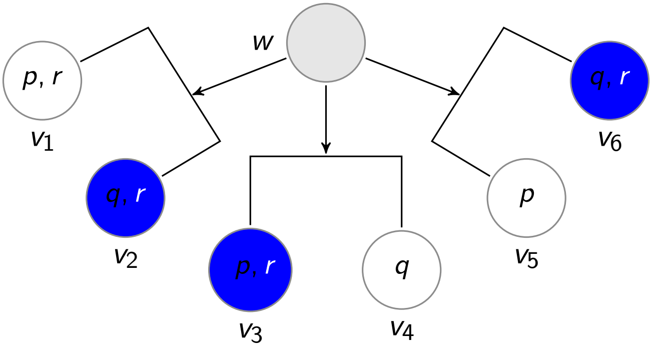
- $\mathcal{M}^2, w \models \Box p$ (and $\mathcal{M}^2, w \models \Box \neg p$)
- $\mathcal{M}^2, w \models \Box q$ (and $\mathcal{M}^2, w \models \Box \neg q$)

Weakly Aggregative Logics



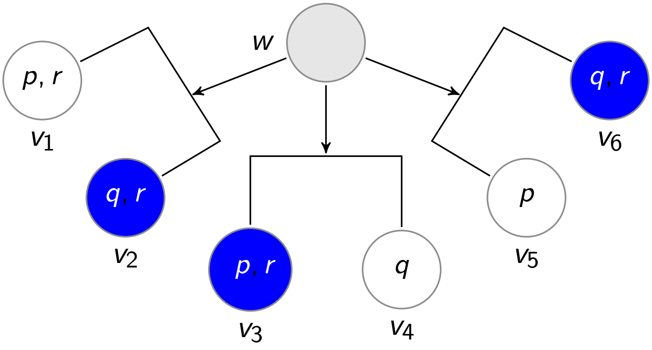
- $\mathcal{M}^2, w \models \Box p$ (and $\mathcal{M}^2, w \models \Box\neg p$)
- $\mathcal{M}^2, w \models \Box q$ (and $\mathcal{M}^2, w \models \Box\neg q$)
- $\mathcal{M}^2, w \not\models \Box(p \wedge q)$

Weakly Aggregative Logics



- $\mathcal{M}^2, w \models \Box p$ (and $\mathcal{M}^2, w \models \Box\neg p$)
- $\mathcal{M}^2, w \models \Box q$ (and $\mathcal{M}^2, w \models \Box\neg q$)
- $\mathcal{M}^2, w \not\models \Box(p \wedge q)$
- $\mathcal{M}^2, w \models \Box r$

Weakly Aggregative Logics



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- $\mathcal{M}^2, w \not\models \Box(p \wedge q)$
- $\mathcal{M}^2, w \models \Box r$
- $\mathcal{M}^2, w \models \Box((p \wedge r) \vee (q \wedge r))$

Weakly Aggregative Logics

$$(C) \qquad (\Box\varphi_0 \wedge \Box\varphi_1) \to \Box(\varphi_0 \wedge \varphi_1)$$

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$$(C^n) \qquad (\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_n) \to \Box \bigvee_{0 \leq i < j \leq n} (\varphi_i \wedge \varphi_j)$$

Weakly Aggregative Logics

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$$(C^n) \qquad (\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_n) \rightarrow \Box \bigvee_{0 \leq i < j \leq n} (\varphi_i \wedge \varphi_j)$$

Example: $(\Box\varphi_0 \wedge \Box\varphi_1 \wedge \Box\varphi_2) \rightarrow \Box((\varphi_0 \wedge \varphi_1) \vee (\varphi_1 \wedge \varphi_2) \vee (\varphi_0 \wedge \varphi_2))$

Weakly Aggregative Logics

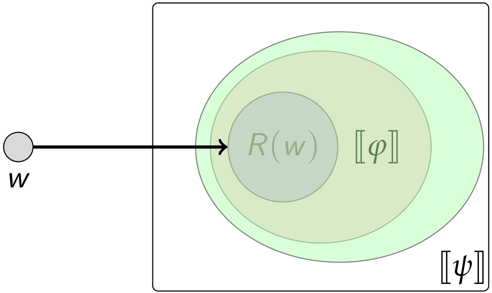
P. Schotch and R. Jennings (1980). *Modal logic and the theory of modal aggregation*. Philosophia, 9(2), pp. 265 - 278.

T. Nicholson and M. Allen (2003). *Aggregative combinatorics: an introduction*. In: Proceedings of the Student Session, 2nd North American Summer School in Logic, Language, and Information (NASSLLI-03), pp. 15 - 25.

Yifeng Ding, Jixin Liu, and Yanjing Wang (2022). *Model Theoretical Aspects of Weakly Aggregative Modal Logic*. Journal of Logic, Language and Information, 31, pp. 261 - 286.

Monotonicity

$$(M) \quad \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$$



- equivalent to the inference rule: from $\varphi \rightarrow \psi$ infer $\Box\varphi \rightarrow \Box\psi$

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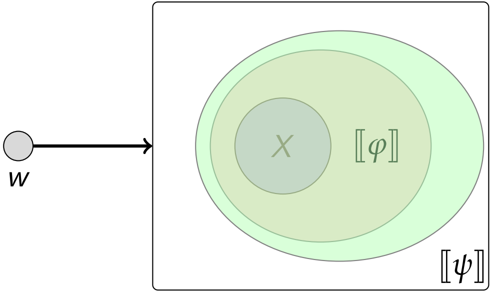
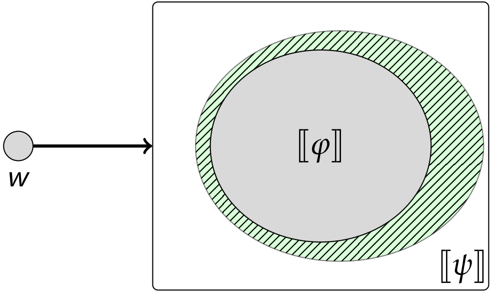


- equivalent to the inference rule: from $\varphi \rightarrow \psi$ infer $\Box\varphi \rightarrow \Box\psi$
- $\langle\ ]$ is monotonic

Monotonicity

$$(M) \quad \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$$



- equivalent to the inference rule: from $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$
- $\langle\ ]$ is monotonic
- $\Box$ is not monotonic
  - Deontic logic
  - A minimal logic of ability

Deontic Logic

$\Box\varphi$ means "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith

James William Forrester (1984). *Gentle Murder, Or The Adverbial Samaritan*. The Journal of Philosophy, 81(4), pp. 193 - 197.

L. Goble (1991). *Murder Most Gentle: The Paradox Deepens*. Philosophical Studies, 64(2), pp. 217 - 227.

Deontic Logic

$\Box\varphi$ means "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

James William Forrester (1984). *Gentle Murder, Or The Adverbial Samaritan*. The Journal of Philosophy, 81(4), pp. 193 - 197.

L. Goble (1991). *Murder Most Gentle: The Paradox Deepens*. Philosophical Studies, 64(2), pp. 217 - 227.

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2. Jones ought not to murder Smith

✓ If Jones murders Smith, then Jones ought to murder Smith gently

4. Jones ought to murder Smith gently

James William Forrester (1984). *Gentle Murder, Or The Adverbial Samaritan*. The Journal of Philosophy, 81(4), pp. 193 - 197.

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Deontic Logic

$\Box\varphi$ means "*it is obliged that* φ."

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4. Jones ought to murder Smith gently

$\Rightarrow$ If Jones murders Smith gently, then Jones murders Smith.

James William Forrester (1984). *Gentle Murder, Or The Adverbial Samaritan*. The Journal of Philosophy, 81(4), pp. 193 - 197.

L. Goble (1991). *Murder Most Gentle: The Paradox Deepens*. Philosophical Studies, 64(2), pp. 217 - 227.

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(RM) If Jones ought to murder Smith gently, then Jones ought to murder Smith

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L. Goble (1991). *Murder Most Gentle: The Paradox Deepens*. Philosophical Studies, 64(2), pp. 217 - 227.

Deontic Logic

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✓ Jones ought to murder Smith gently

5. If Jones murders Smith gently, then Jones murders Smith.

✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith

7. Jones ought to murder Smith

James William Forrester (1984). *Gentle Murder, Or The Adverbial Samaritan*. The Journal of Philosophy, 81(4), pp. 193 - 197.

L. Goble (1991). *Murder Most Gentle: The Paradox Deepens*. Philosophical Studies, 64(2), pp. 217 - 227.

Deontic Logic

$\Box\varphi$ means "*it is obliged that* φ."

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✗ Jones ought not to murder Smith

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4. Jones ought to murder Smith gently
5. If Jones murders Smith gently, then Jones murders Smith.
6. If Jones ought to murder Smith gently, then Jones ought to murder Smith

✗ Jones ought to murder Smith

James William Forrester (1984). *Gentle Murder, Or The Adverbial Samaritan*. The Journal of Philosophy, 81(4), pp. 193 - 197.

L. Goble (1991). *Murder Most Gentle: The Paradox Deepens*. Philosophical Studies, 64(2), pp. 217 - 227.

A Minimal Logic of Abilities

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, Vol. 17, pp. 1 - 26, 1988.

D. Elgesem. *The modal logic of agency*. Nordic Journal of Philosophical Logic 2(2), 1 - 46, 1997.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

1. All propositional tautologies
2. $\neg C\top$
3. $(E\varphi \wedge E\psi) \to E(\varphi \wedge \psi)$
4. $E\varphi \to \varphi$
5. $E\varphi \to C\varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E\varphi \leftrightarrow E\psi$ and from $\varphi \leftrightarrow \psi$ infer $C\varphi \leftrightarrow C\psi$

A Minimal Logic of Abilities: Non-Monotonicity

RM for E is inconsistent with $\neg E\top$ assuming that the agent performs at least one action.

Axioms

$\cancel{(M)}$ $\cancel{\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi}$

$\cancel{(C)}$ $\cancel{\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)}$

$\cancel{(N)}$ $\cancel{\Box\top}$

$\cancel{(K)}$ $\cancel{\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)}$

$(Dual)$ $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

$\cancel{(Nec)}$ ~~from $\vdash \varphi$ infer $\vdash \Box\varphi$~~

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$