Conditionals in Game Theory

Ilaria Canavotto, University of Maryland
Eric Pacuit, University of Maryland

Lecture 1

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"[O]ne really cannot discuss rationality, or indeed decision making, without substantive conditionals and counterfactuals. Making a decision means choosing among alternatives. Thus one must consider hypothetical situations what would happen if one did something different from what one actually does. [I]n interactive decision making—games—you must consider what other people would do if you did something different from what you actually do." (p. 15)

R. Aumann. *Backward induction and common knowledge of rationality*. Games and Economic Behavior, 8: 6 - 19, 1995.

Prisoner's Dilemma

Ann \ Bob	c	d
c	3,3	0,4
d	4,0	1,1

Symmetric Games

Player 1 \ Player 2	c	d
c	R, R	S, T
d	T, S	P, P

Both players rewarded (c, c: R, R)

One player tempted; the other is a sucker (c, d: S, T)

One player tempted; the other is a sucker (d, c: T, S)

Both players punished (d, d: P, P)

Symmetric Games

Both players rewarded

One player tempted; the other is a sucker

		Player 2	
		c	d
Player 1	c	R, R	S, T
	d	T, S	P, P

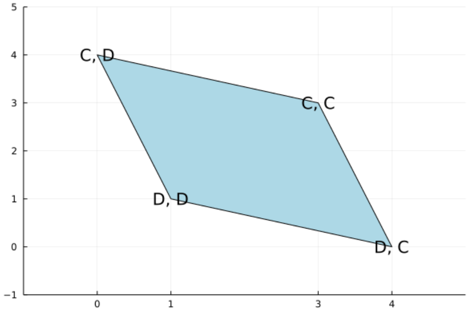
One player tempted; the other is a sucker

Both players punished

Symmetric games are classified in terms of the relationship between R (reward), T (temptation), S (sucker) and P (punishment):

Prisoner's Dilemma

		Bob	
		c	*d*
Ann	*c*	3,3	0,4
	d	4,0	1,1



If $T > R > P > S$, then the game is a Prisoner's Dilemma.

d strictly dominates c

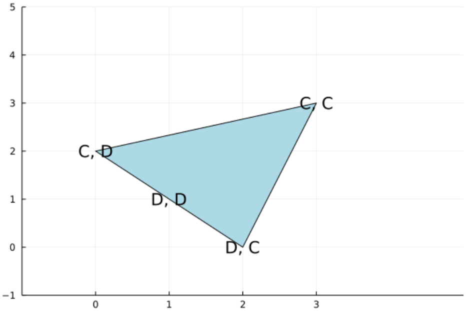
(c, c) Pareto dominates (d, d)

(d, d) is the unique Nash equilibrium

Stag Hunt

Bob

Ann	c	d
c	3,3	0,2
d	2,0	1,1



If $R > T$ and $P > S$, then the game is called Stag Hunt.

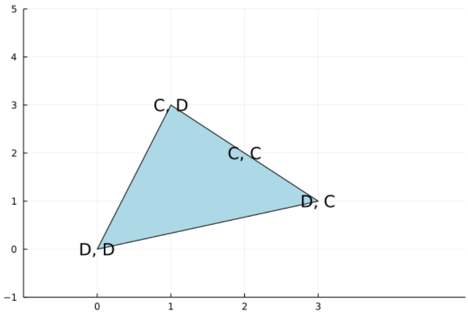
d is a less "risky" option than c

(c, c) Pareto dominates (d, d)

(c, c) and (d, d) are both Nash equilibria

Chicken

		Bob *c*	Bob *d*
Ann	*c*	2,2	1,3
Ann	*d*	3,1	0,0



If $T > R$ and $S > P$, then the game is called Chicken (or Hawk-Dove).

c is a less "risky" option than d

(c, c) Pareto dominates (d, d)

(c, d) and (d, c) are both Nash equilibria

Game in Normal Form

A **game in normal form** is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where:

- N is a finite set of players.
- For each $i \in N$, S_i is a (finite) set of actions, or strategies, for player i.
- For each $i \in N$, $u_i : \Pi_{i \in N} S_i \to \mathbb{R}$

Notation

Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game in strategic form.

- For $s \in \Pi_{i \in N} S_i$, s_i is the ith component of s and $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$ is the tuple of all strategies except s_i
- For $i \in N$, let $S = \Pi_{i \in N} S_i$ be the set of **strategy profiles**, also called the outcomes of G.
- For $i \in N$, let $S_{-i} = \Pi_{j \in N, j \neq i} S_j$.

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- For $i \in N$, let $S_{-i} = \Pi_{j \in N, j \neq i} S_j$.
- For a set X, let $\Delta(X)$ be the set of probability measures on X.
- $m \in \Delta(S_i)$ is called a **mixed strategy** for player i.
- A mixed strategy profile is an element of $\Pi_{i \in N} \Delta(S_i)$.

Expected Utility, Best Response

Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game in strategic form.

For $a \in S_i$ and $p \in \Delta(S_{-i})$ the **expected utility of** a **with respect to** p is

$$EU_i(a, p) = \sum_{t \in S_{-i}} p(t) u_i(a, t)$$

Expected Utility, Best Response

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$$EU_i(a, p) = \sum_{t \in S_{-i}} p(t) u_i(a, t)$$

For $X \subseteq \Delta(S_{-i})$, the **best response set for player** i, $BR_i : X \to \wp(S_i)$, is defined as follows: for $p \in X$,

$$BR_i(p) = \{a \mid a \in S_i \text{ and } \forall a' \in S_i : EU_i(a, p) \geqslant EU_i(a', p)\}$$

Identify S_{-i} with the set $\{p \mid p \in \Delta(S_{-i}), p(s) = 1 \text{ for some } s \in S_{-i}\}$,

A strategy profile $s \in \Pi_{i \in N} S_i$ is a (pure strategy) **Nash equilibrium** provided that for all $i \in N$, $s_i \in BR_i(s_{-i})$

Mixed Extension

Suppose that $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a game in strategic form.

The **mixed extension of** G is the tuple $\langle N, (\Delta(S_i))_{i\in N}, (U_i)_{i\in N}\rangle$, where for $m \in \Pi_{i\in N}\Delta(S_i)$

$$U_i(m) = \sum_{s\in S} u_i(s)\Pi_{i\in N} m_i(s_i)$$

A **mixed strategy Nash equilibrium** in G is a tuple $m \in \Pi_{i\in N}\Delta(S_i)$ that is a Nash equilibrium in the mixed extension of G.

Correlation

Players can improve their expected value by correlating their choices on an "outside signal".

Correlated Strategies

		Player 2 l	Player 2 r
Player 1	u	2, 1	0, 0
Player 1	d	0,0	1,2

- Three Nash equilibria:
  - (u, l): the payoff is $(2, 1)$
  - (d, r): the payoff is $(1, 2)$
  - Mixed Nash Equilibrium: $([\frac{2}{3} : u, \frac{1}{3} : d], [\frac{1}{3} : l, \frac{2}{3} : r])$: the payoff is $(\frac{2}{3}, \frac{2}{3})$

Correlated Strategies

Player 2

Player 1		l	r
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- Mixed Strategies: Each player conducts a private, independent lottery to choose their strategy.

Correlated Strategies

		Player 2	
		l	r
Player 1	u	2, 1	0, 0
	d	0,0	1,2

	l	r
u	0.5	0
d	0	0.5

- Mixed Nash Equilibrium: $([\frac{2}{3}: u, \frac{1}{3}: d], [\frac{1}{3}: l, \frac{2}{3}: r])$: the payoff is $(\frac{2}{3}, \frac{2}{3})$
- Mixed Strategies: Each player conducts a private, independent lottery to choose their strategy.
- Conduct a *public* lottery: flip a fair coin and follow the strategy $(H \Rightarrow (u, l),\ T \Rightarrow (d, r))$. The expected payoff is $(1.5, 1.5)$.

Two extremes:

1. Completely private, independent lotteries
2. A single, completely public lottery

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1. Completely private, independent lotteries
2. A single, completely public lottery

What about: a public lottery, but reveal only partial information about the outcome to each of the players?

		Player 2	
		c	d
Player 1	c	6, 6	2, 7
	d	7, 2	0, 0

	c	d
c	1/3	1/3
d	1/3	0

- Three Nash equilibria:
  - (c, d): the payoff is $(2, 7)$; (d, c): the payoff is $(7, 2)$
  - $([\frac{2}{3} : c, \frac{1}{3} : d], [\frac{2}{3} : c, \frac{1}{3} : d])$: the payoff is $(4\frac{2}{3}, 4\frac{2}{3})$

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- After conducting the lottery, an outside observer provides Ann with a recommendation to play the first component of the profile that was chosen, and Bob the second component.

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- After conducting the lottery, an outside observer provides Ann with a recommendation to play the first component of the profile that was chosen, and Bob the second component.
- The expected payoff is $\frac{1}{3}(6, 6) + \frac{1}{3}(2, 7) + \frac{1}{3}(7, 2) = (5, 5)$

Correlated Equilibrium

Let $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game.

A correlated strategy n-tuple in G is a function from a finite probability space Γ into $S = S_1 \times \cdots \times S_n$. That is, f is a *random variable* whose values are n-tupels of actions.

Chance (according to the probability space Γ) chooses an element $\gamma \in \Gamma$, then each player is recommended to take action $f_i(\gamma)$.

Correlated Equilibrium: A correlated equilibrium in G is a correlated strategy n-tuple f such that

$$EU_i(f) \geqslant EU_i(g_i, f_{-i})$$

Nash equilibrium is the outcome that results from assuming that each of the following are *common knowledge* among the players:

1. The game's payoff structure.
2. The Bayesian rationality of the players.
3. The players' beliefs about each other.
4. Players regard their opponents strategies as independent.
5. The players' beliefs must be *consistent*.

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Rationalizability is the outcome that results from assuming each of the following are *common knowledge* among the players:

1. The game's payoff structure.
2. The Bayesian rationality of the players.
3. ~~The players' beliefs about each other.~~
4. ~~Players regard their opponents strategies as independent.~~
5. ~~The players' beliefs must be consistent.~~

Game Models

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- A **model of a game** is a completion of the partial specification of the Bayesian decision problems *and* a representation of a particular play of the game.

Game Model: Possible Worlds and Strategy Function

$G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game in strategic form.

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- W is a set of *possible worlds* (possible outcomes of the game)
- σ is a function $\sigma : W \to \Pi_{i \in N} S_i$
  write $\sigma_i(w)$ for $\sigma(w)_i$: the ith component of $\sigma(w)$
  write $\sigma_{-i}(w)$ for $\sigma(w)_{-i}$

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- $\sigma_i(w) = a \in S_i$ means that "player i chooses strategy a in state w.

  The exact meaning of 'choosing' is not elaborated further in the literature: does it mean that player i has *actually played* a or that she *will play* a or that a is the *output of her deliberation process*?

Game Model: Possible Worlds and Strategy Function

- Each $s \in \Pi_{i \in N} S_i$ is associated with the following events:
  - $[s_i] = \{w \mid \sigma(w)_i = s_i\}$ is the event that i chooses s_i
  - $[s_{-i}] = \{w \mid \sigma(w)_{-i} = s_{-i}\} = \bigcap_{j \in N, j \neq i} [s_i]$ is the event that all players except i choose their strategies in s_{-i}
  - $[s] = \{w \mid \sigma(w) = s\} = \bigcap_{i \in N} [s_i]$ is the event that all players choose their strategies in s

Game Model

Given a strategic-form game $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a model of G is a triple

$$\langle W, (I_i)_{i\in N}, \sigma\rangle$$

where W is a non-empty set of states, $\sigma : W \to \Pi_{i\in N} S_i$, and:

For each $i \in N$, $I_i : W \to \wp(W)$ is player i's **information correspondence**.

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- Relations are often used instead of correspondences: $R_i \subseteq W \times W$ where $w \, R_i \, v$ iff $v \in I_i(w)$
- For $E \subseteq W$, let $\Box_i(E) = \{w \mid I_i(w) \subseteq E\}$

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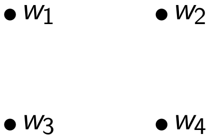
For each $i \in N$, $I_i : W \to \wp(W)$ is player i's **information correspondence**.

- Standard assumptions:
  - Truth: For all $w \in W$, $w \in I_i(w)$
  - Consistency: For all $w \in W$, $I_i(w) \neq \emptyset$
  - Fully Introspective: For all $w, v \in W$, if $v \in I_i(w)$, then $I_i(w) = I_i(v)$

Player 2

Player 1		l	r
	u	3,3	0,4
	d	4,0	1,1

Game G



Model of G

$$\langle W, (I_1, I_2), \sigma \rangle$$

$$W = \{w_1, w_2, w_3, w_4\}$$

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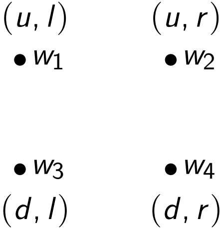

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Player 1 \ Player 2	l	r
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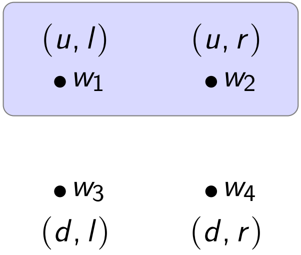

Model of G

$$\langle W, (I_1, I_2), \sigma \rangle$$

$$s = (u, l)$$

$$[s_1] = [u] = \{w_1, w_2\} = [s_{-2}]$$

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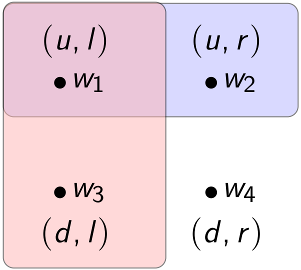

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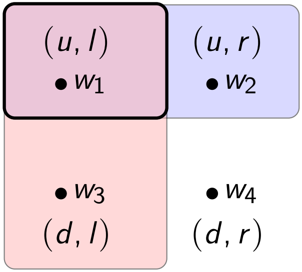

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$$[s] = [s_1] \cap [s_2] = [u] \cap [l] = \{w_1, w_2\} \cap \{w_1, w_3\} = \{w_1\}$$

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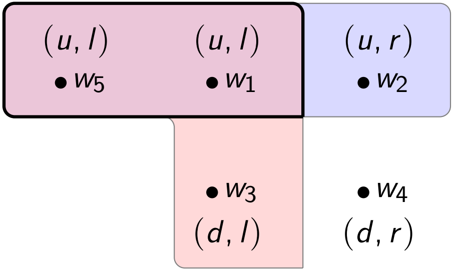

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$$[s] = [s_1] \cap [s_2] = [u] \cap [l] = \{w_1, w_2, w_5\} \cap \{w_1, w_3, w_5\} = \{w_1, w_5\}$$

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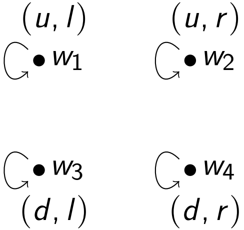

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	w_1	w_2	w_3	w_4
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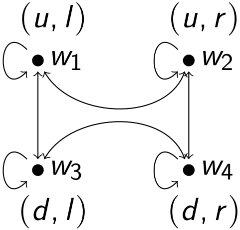

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I_2	$\{w_1, w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$

Knowledge of own action

For all $i \in N$, for all $w \in W$, $I_i(w) \subseteq [\sigma_i(w)]$

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		l	r
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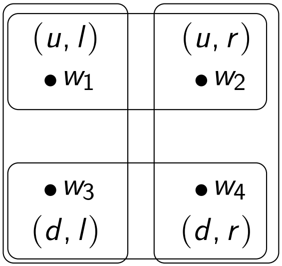
Game G



Model of G

$$\langle W, (I_1, I_2), \sigma \rangle$$

	w_1	w_2	w_3	w_4
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I_1	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_3, w_4\}$	$\{w_3, w_4\}$
I_2	$\{w_1, w_3\}$	$\{w_2, w_4\}$	$\{w_1, w_3\}$	$\{w_2, w_4\}$

Beliefs

Given a strategic-form game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a model of G is a triple

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where W is a non-empty set of states, $\sigma : W \to \Pi_{i \in N} S_i$, and:

$$\text{For each } i \in N,\ I_i : W \to \wp(W).$$

- For all $w \in W$, $I_i(w) \subseteq [\sigma_i(w)]$
- For all $w \in W$ for all $v \in W$, if $v \in I_i(w)$, then $I_i(w) = I_i(v)$.

Beliefs

Given a strategic-form game $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a model of G is a triple

$$\langle W, (P_i)_{i\in N}, \sigma\rangle$$

where W is a non-empty set of states, $\sigma : W \to \Pi_{i\in N} S_i$, and:

$$\text{For each } i \in N,\ P_i : W \to \wp(W).$$

- For all $w \in W$, $P_i(w)([\sigma_i(w)]) = 1$.
- For all $w \in W$, $P_i(w)(\{v \mid P_i(v) = P_i(w)\}) = 1$.

Posterior beliefs: For each $w \in W$, let $p_{i,w} = P_i(w) \in \Delta(W)$.

Beliefs

Given a strategic-form game $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a model of G is a triple

$$\langle W, (I_i, p_i)_{i\in N}, \sigma\rangle$$

where W is a non-empty set of states, $\sigma : W \to \Pi_{i\in N} S_i$, and:

$$\text{For each } i \in N,\ I_i : W \to \wp(W) \text{ and } p_i \in \Delta(W).$$

- For all $w \in W$, $I_i(w) \subseteq [\sigma_i(w)]$.
- For all $w \in W$ for all $v \in W$, if $v \in I_i(w)$, then $I_i(w) = I_i(v)$.
- For all $w \in W$, then $p_i(I_i(w)) > 0$ (or we can assume $p_i(w) > 0$).

Posterior beliefs: For each $w \in W$, let $p_{i,w} = p_i(w \mid I_i(w)) \in \Delta(W)$.

Rational choice, I

Given a strategic-form game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a model of G is a triple

$$\langle W, (I_i)_{i \in N}, \sigma \rangle$$

Player i is **rational** at state w when there is no $a \in S_i$ such that

$$u_i(a, \sigma_{-i}(v)) > u_i(\sigma_i(w), \sigma_{-i}(v)) \text{ for all } v \in I_i(w)$$

Rational choice, II

Given a strategic-form game $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a model of G is a triple

$$\langle W, (P_i)_{i\in N}, \sigma\rangle \text{ or } \langle W, (I_i, p_i)_{i\in N}, \sigma\rangle$$

Player i is **Bayes rational** at w if, for all $a \in S_i$:

$$\sum_{s_{-i}\in S_{-i}} p_{i,w}([s_{-i}])u_i(\sigma_i(w), s_{-i}) \geqslant$$

$$\sum_{s_{-i}\in S_{-i}} p_{i,w}([s_{-i}])u_i(a, s_{-i})$$

An Example

Ann \ Bob	l	r
u	1,2	0,0
d	0,0	2,1

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u	1,2	0,0
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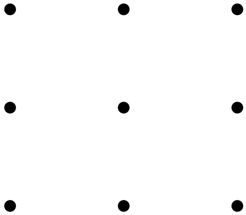
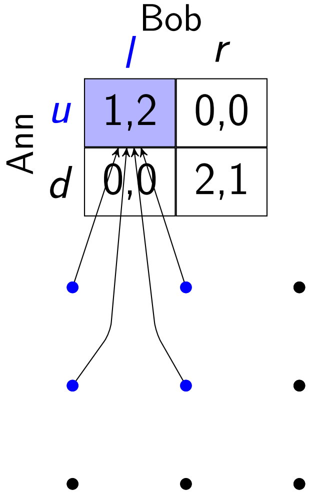
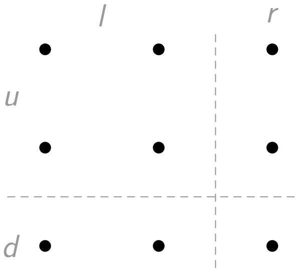


An Example



An Example

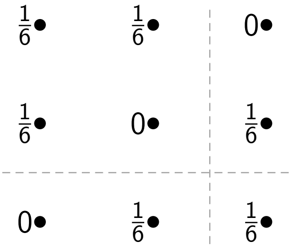
		Bob	
		l	r
Ann	u	1,2	0,0
	d	0,0	2,1



An Example

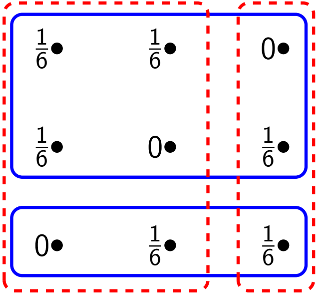
Bob

Ann	l	r
u	1,2	0,0
d	0,0	2,1



An Example

		Bob: l	Bob: r
Ann	u	1,2	0,0
	d	0,0	2,1



An Example

Bob

Ann		l	r
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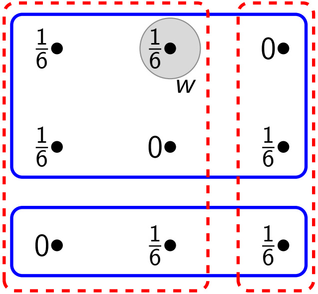

An Example

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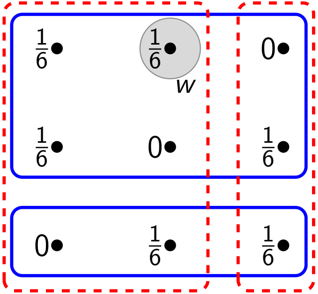
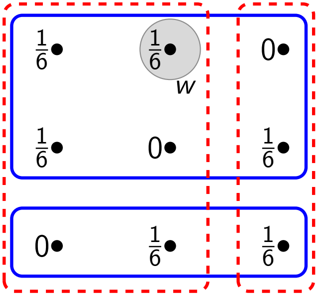


- Ann's choice is *rational* at w

An Example

Bob

Ann		l	r
	u	1,2	0,0
	d	0,0	2,1

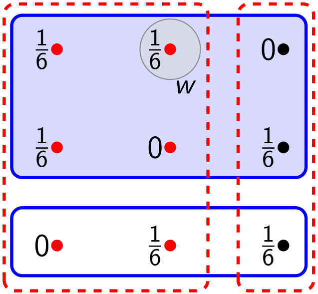


- Ann's choice is *rational* at w

$$1 \cdot p_{Ann,w}([l]) + 0 \cdot p_{Ann,w}([r]) \geqslant 0 \cdot p_{Ann,w}([l]) + 2 \cdot p_{Ann,w}([r])$$

An Example

Ann \ Bob	l	r
u	1,2	0,0
d	0,0	2,1



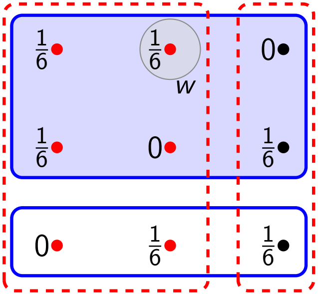
- Ann's choice is *rational* at w

$$1 \cdot p_{Ann,w}([l]) + 0 \cdot p_{Ann,w}([r]) \geqslant 0 \cdot p_{Ann,w}([l]) + 2 \cdot p_{Ann,w}([r])$$

An Example

Bob

Ann	l	r
u	1,2	0,0
d	0,0	2,1

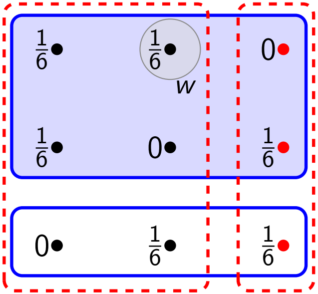


- Ann's choice is *rational* at w

$$1 \cdot \frac{3}{4} + 0 \cdot p_{Ann,w}([r]) \geqslant 0 \cdot \frac{3}{4} + 2 \cdot p_{Ann,w}([r])$$

An Example

Ann \ Bob	l	r
u	1,2	0,0
d	0,0	2,1

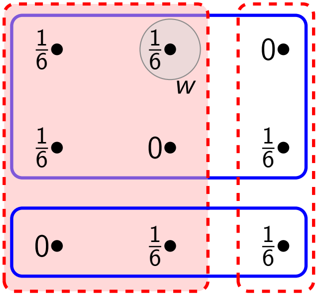


- Ann's choice is *rational* at w

$$1 \cdot \frac{3}{4} + 0 \cdot \frac{1}{4} \geqslant 0 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4}$$

An Example

	Bob: l	Bob: r
Ann: u	1,2	0,0
Ann: d	0,0	2,1

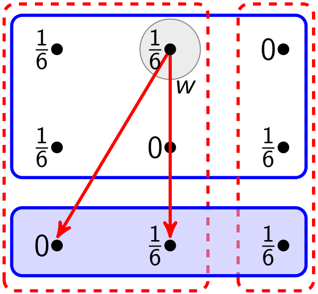


- Ann's choice is *rational* at w
- Bob's choice is *rational* at w

$$2 \cdot \frac{3}{4} + 0 \cdot \frac{1}{4} \geqslant 0 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4}$$

An Example

Ann \ Bob	l	r
u	1,2	0,0
d	0,0	2,1



- Ann's choice is *rational* at w
- Bob's choice is *rational* at w
- Bob *considers it possible* Ann is *irrational*

$$1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \not> 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}$$

R. Aumann (1987). *Correlated Equilibrium as an Expression of Bayesian Rationality*. Econometrica , 55:1, pp. 1-18.

Given a strategic-form game $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a model of G is a triple

$$\langle W, (I_i, p_i)_{i\in N}, \sigma\rangle$$

- W is a non-empty set of states and $\sigma : W \to \Pi_{i\in N} S_i$

  "The term "state of the world" implies a definite specification of all parameters that may be the object of uncertainty on the part of any player of G. In particular, each w includes a specification of which action is chosen by each player of G at that state w. Conditional on a given world, everybody knows everything; but in general, nobody knows which is really the true w." (pg. 6)

- For each $i \in N$, I_i is a partition of W and for all $i \in N$ and all $w \in W$, $I_i(w) \subseteq [\sigma_i(w)]$.

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- W is a non-empty set of states and $\sigma : W \to \Pi_{i\in N} S_i$
- For each $i \in N$, I_i is a partition of W and for all $i \in N$ and all $w \in W$, $I_i(w) \subseteq [\sigma_i(w)]$.
- Common Prior Assumption (CPA): There is a probability measure p on W such that

$$p_1 = p_2 = \cdots = p_n = p$$

Common priors

$\not\Rightarrow$ same posteriors!

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$\not\Rightarrow$ same posteriors! For the simple reason that agents can receive different private information.

- We play card together. Before the cards are dealt, our common prior belief that the other end up with a Joker is $0.037 = 2/54$.
- We get 5 card each (and don't show them to each other). I end up with the 2 Jokers.
  - My posterior belief that you have a Joker is 0.
  - Your posterior belief that I have a Joker is $0.04 = 2/49$.

Common Prior Assumption

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"Harsanyi doctrine" [Aumann, 1976].

R. Aumann (1976). *Agreeing to Disagree*. *Annals of Statistics*, Vol.4, No.6, 1976.

J.C. Harsanyi (1967-68). *Games with incomplete informations played by 'Bayesian' players*. *Management Science* 14:159–182, 320–334, 486–502.

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"Harsanyi doctrine" [Aumann, 1976].

- CPA not an innocuous assumption! (cf. Aumann's agreeing to disagree theorem)

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  - See Morris (1995) for a thorough discussion. One methodological observation:
    - Better to explain differences in posterior on the basis of identifiable differences in information or plausible errors in information processing.
    - Resorting on differences in priors often appears ad hoc (the resulting theory is "too permissive").

S. Morris (1995). *The Common Prior Assumption in Economic Theory*. *Economics and Philosophy*, 11(2): pp. 227- 253.

Theorem (Aumann, 1987). Suppose that $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategic form game and $\langle W, (I_i, p_i)_{i \in N}, \sigma \rangle$ is a model of G satisfying the common prior assumption and such that for all $i \in N$, I_i is a partition of W. If each player is Bayes rational at each state of the world, then the distribution of the action n-tuples σ is a correlated equilibrium.

		Player 2	
		c	d
Player 1	c	3,3	0,4
	d	4,0	1,1

Player 1: "I believe that if I play c then Player 2 will play c and that if I play d then Player 2 will play d. Thus, if I play c my payoff will be 3 and if I play d my payoff will be 1. Hence I have decided to play c."

O. Board (2006). *The Equivalence of Bayes and Causal Rationality in Games*. Theory and Decision, 61, pp. 1-19.

From Bayesian rationality to counterfactual rationality

[T]he various actions of each player might be inter-connected: my opponents' choices given that I play s_i might not be the same as they would have been had I chosen to play s'_i. Each player must consider what her opponents will do given her actual choice, and also what they would do if she were to choose something else. (p.8)

A causal expected utility calculus, then, depends on counterfactual sentences such as "if it were the case that player i chose strategy s_i, then it would be the case that her opponents chose strategy profile s_{-i}. (p.8)

Counterfactual rationality

- Player i is **Bayes rational** at w if, for all $a \in S_i$:

$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([s_{-i}])u_i(\sigma_i(w), s_{-i}) \geqslant$$

$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([s_{-i}])u_i(a, s_{-i})$$

- Player i is **counterfactually rational** at w if, for all $a \in S_i$:

$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([\sigma_i(w)] \,\Box\!\!\rightarrow [s_{-i}])u_i(\sigma_i(w), s_{-i}) \geqslant$$

$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([a] \,\Box\!\!\rightarrow [s_{-i}])u_i(a, s_{-i})$$

Counterfactual rationality

- Player i is **Bayes rational** at w if, for all $a \in S_i$:

$$\sum_{s_{-i} \in S_{-i}} p_i([s_{-i}] \mid I_i(w)) u_i(\sigma_i(w), s_{-i}) \geqslant$$

$$\sum_{s_{-i} \in S_{-i}} p_i([s_{-i}] \mid I_i(w)) u_i(a, s_{-i})$$

- Player i is **counterfactually rational** at w if, for all $a \in S_i$:

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$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([a] \mathrel{\Box\!\!\rightarrow} [s_{-i}]) u_i(a, s_{-i})$$

Counterfactual rationality

- Player i is **Bayes rational** at w if, for all $a \in S_i$:

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- Player i is **counterfactually rational** at w if, for all $a \in S_i$: **???**

$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([\sigma_i(w)] \mathrel{\Box\!\!\rightarrow} [s_{-i}]) u_i(\sigma_i(w), s_{-i}) \geqslant$$

$$\sum_{s_{-i} \in S_{-i}} p_{i,w}([a] \mathrel{\Box\!\!\rightarrow} [s_{-i}]) u_i(a, s_{-i})$$