Neighborhood Semantics for Modal Logic
Lecture 2

Eric Pacuit, University of Maryland

August 6, 2024

Plan for Today (and tomorrow and Thursday)

- Interpretation of Neighborhood Models: Evidence Models
- Neighborhood Frames/Models
- Non-Normal Modal Logics
- Completeness
- Incompleteness
- Decidability and Complexity
- Bisimulation and Expressivity

Interpretation of Neighborhood Models: Evidence Models

Defining beliefs from evidence

J. van Benthem and EP. *Dynamic logics of evidence-based beliefs*. Studia Logica, 99(61), 2011.

J. van Benthem, D. Fernández-Duque and EP. *Evidence and plausibility in neighborhood structures*. Annals of Pure and Applied Logic, 165, pp. 106-133.

Evidence Models: Basic Assumptions

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1. Sources may or may not be *reliable*: a subset recording a piece of evidence need not contain the actual world. Also, agents need not know which evidence is reliable.
2. The evidence gathered from different sources (or even the same source) may be jointly inconsistent. And so, the intersection of all the gathered evidence may be empty.
3. Despite the fact that sources may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.

Evidential States

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Assumptions:

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In addition, much of the literature would suggest a 'monotonicity' assumption:

If the agent has evidence X and $X \subseteq Y$ then the agent has evidence Y.

Example: $W = \{w, v\}$ where p is true at w

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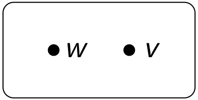
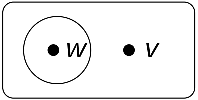
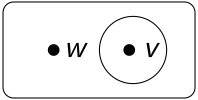


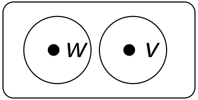
There is no evidence for or against p.



There is evidence that supports p.



There is evidence that rejects p.



There is evidence that supports p and also evidence that rejects p.

Evidence Model

Evidence model: $\mathcal{M} = \langle W, E, V \rangle$

- W is a non-empty set of worlds,
- $V : \text{At} \to \wp(W)$ is a valuation function, and
- $E \subseteq W \times \wp(W)$ is an evidence relation

  $E(w) = \{X \mid w\,E\,X\}$ and $X \in E(w)$: "the agent accepts X as evidence at state w".

Uniform evidence model (E is a constant function): $\langle W, \mathcal{E}, V \rangle, w$ where $\mathcal{E}$ is the fixed family of subsets of W related to each state by E.

Assumptions

(Cons) For each state w, $\emptyset \notin E(w)$.

(Triv) For each state w, $W \in E(w)$.

The Basic Language $\mathcal{L}$ of Evidence and Belief

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid B\varphi \mid A\varphi$$

- $\Box\varphi$: "the agent has evidence that φ is true" (i.e., "the agent has evidence for φ")
- $B\varphi$ says that "the agents believes that φ is true" (based on her evidence)
- $A\varphi$: "φ is true in all states" (for technical convenience/knowledge)

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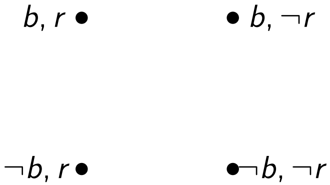
Example



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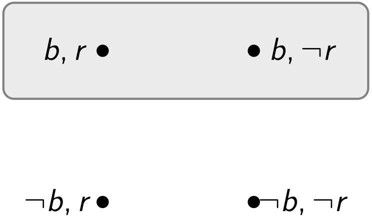

- Receive evidence that the animal is a bird

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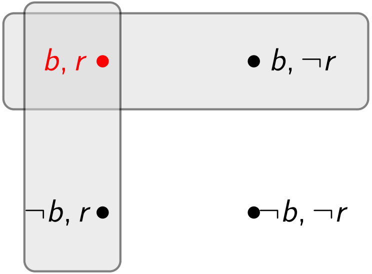

- Receive evidence that the animal is a bird
- Receive evidence that the animal is red
- $B(b \wedge r)$

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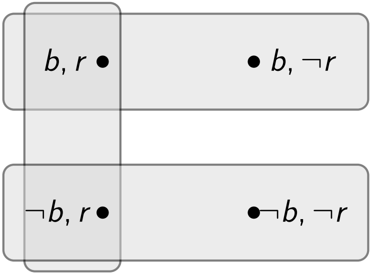

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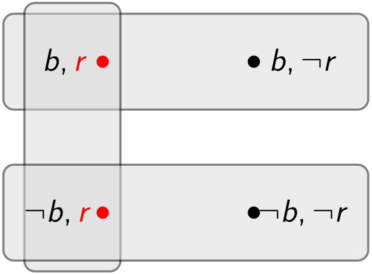

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Defining Beliefs

w-**scenario**: A maximal family of evidence sets $\mathcal{X} \subseteq E(w)$ that has the **finite intersection property** (f.i.p.: for each finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{X}$, $\bigcap_{1 \leq i \leq n} X_i \neq \emptyset$).

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An agent believes φ at w if each w-scenario implies that φ is true (i.e., φ is true at each point in the intersection of each w-scenario).

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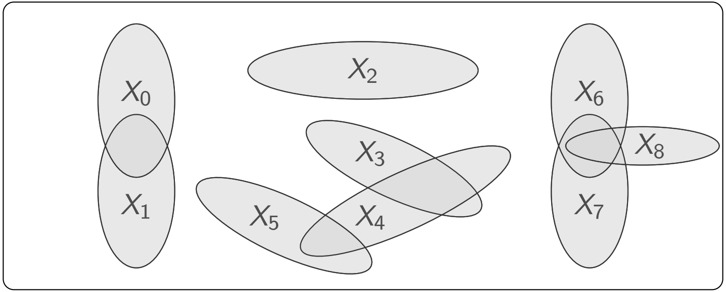

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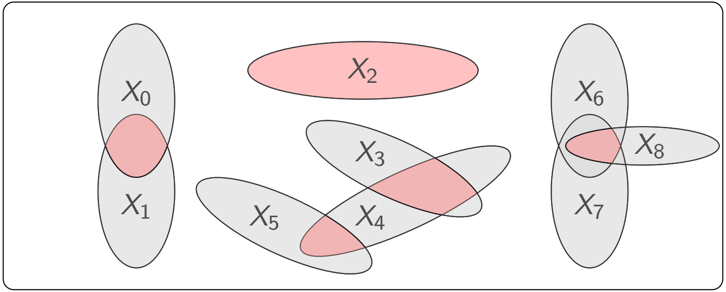

Our definition of belief is very conservative, many other definitions are possible (there exists a w-scenario, "most" of the w-scenarios,...)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ $\qquad (p \in \mathrm{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

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- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B\varphi$ for each maximal f.i.p. $\mathcal{X} \subseteq E(w)$ and for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$

Notation for the truth set: $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$

Flat Evidence Models

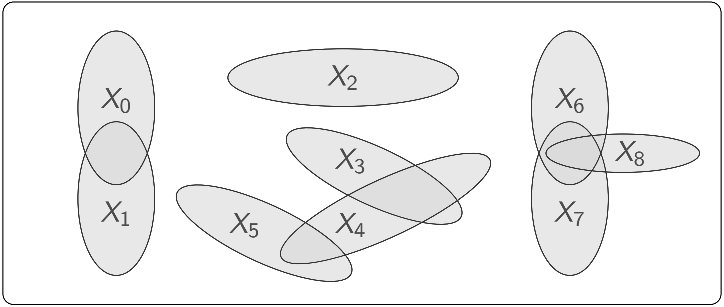
An evidence model $\mathcal{M}$ is **flat** if every scenario on $\mathcal{M}$ has non-empty intersection.

Proposition. The formula $\Box\varphi \to \langle B \rangle \varphi$ is valid on the class of flat evidence models, but not on the class of all evidence models.

Exercises

1. Prove that $(\Box\varphi \wedge A\psi) \leftrightarrow \Box(\varphi \wedge A\psi)$ is valid on all evidence models.
2. Prove that $B\varphi \to AB\varphi$ is valid on all uniform evidence models.

Conditional Beliefs on Evidence Models



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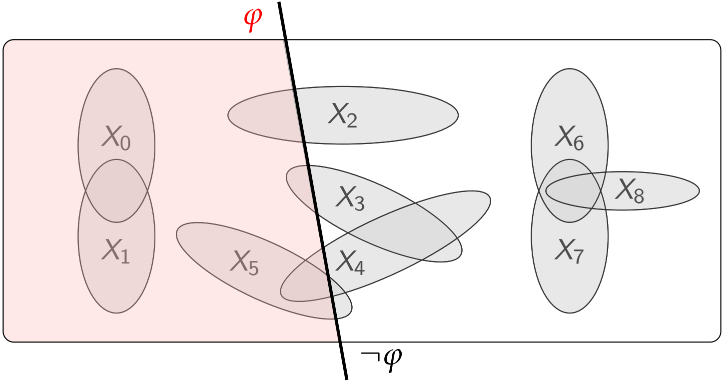

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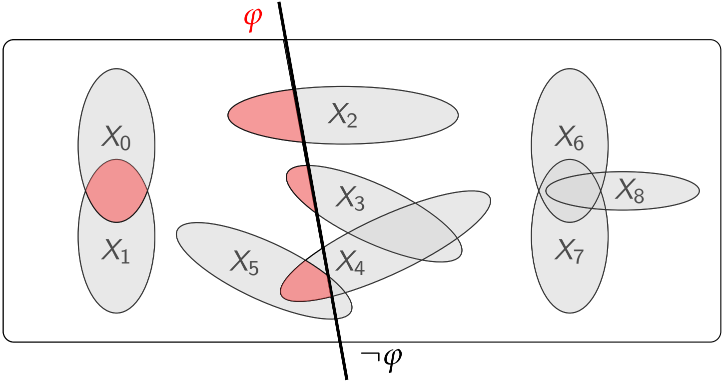

Conditional Beliefs on Evidence Models

$B^{\varphi}\psi$: "the agent believes ψ conditional on φ."

Main idea: Ignore the evidence that is inconsistent with φ.

Relativized w-**scenario**: Suppose that $X \subseteq W$. Given a collection $\mathcal{X} \subseteq \wp(W)$, let $\mathcal{X}^X = \{Y \cap X \mid Y \in \mathcal{X}\}$. We say that a collection $\mathcal{X}$ of subsets of W has the **finite intersection property relative to** X (X-f.i.p.) if, $\mathcal{X}^X$ as the f.i.p. and is maximal if $\mathcal{X}^X$ is.

- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^{\varphi}$, $\mathcal{M}, v \models \psi$

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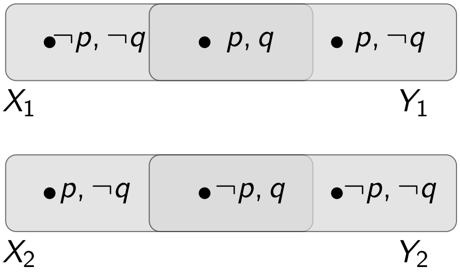

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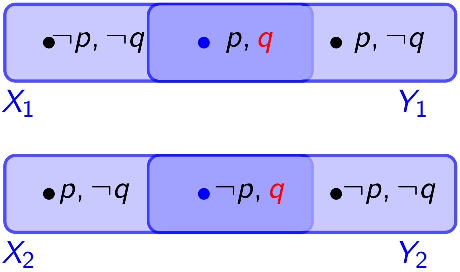

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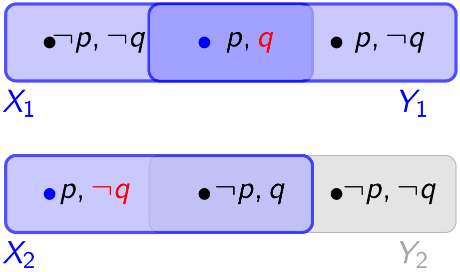

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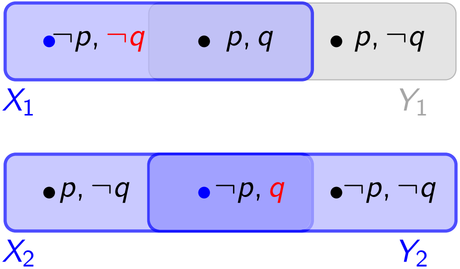

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Course Plan

✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Neighborhood Structures, Motivating Weak Modal Logics/Neighborhood Semantics
(Monday, Tuesday)

2. **Core Theory**: Non-Normal Modal Logic, Completeness, Decidability, Complexity, Incompleteness, Relationship with Other Semantics for Modal Logic, Model Theory

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Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathcal{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \mathsf{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
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- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(w)$
- $\mathcal{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathcal{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathcal{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathcal{M}} = W - [\![\varphi]\!]_{\mathcal{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
4. $[\![\Box\varphi]\!]_{\mathcal{M}} = m_N([\![\varphi]\!]_{\mathcal{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathcal{M}} = W - m_N(W - [\![\varphi]\!]_{\mathcal{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
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Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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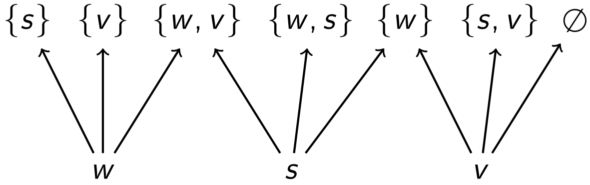

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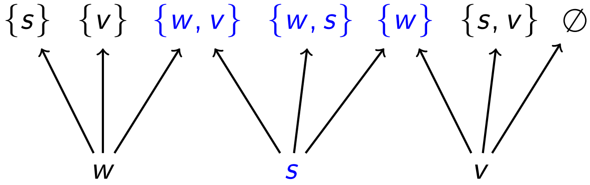

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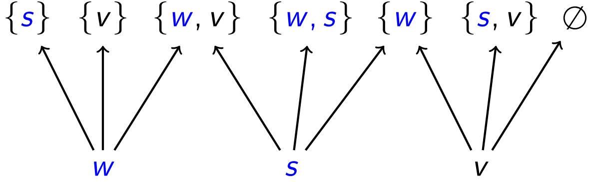

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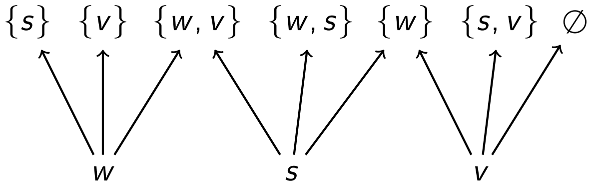

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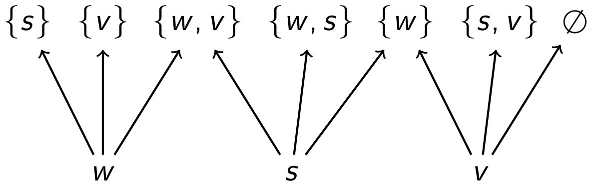

$$\mathcal{M}, s \models \Box p$$

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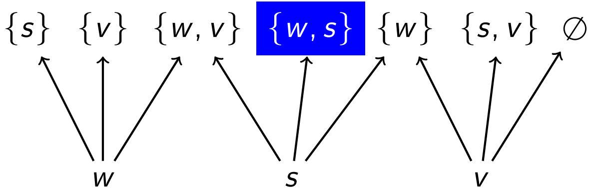

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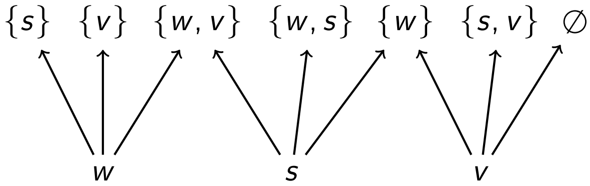

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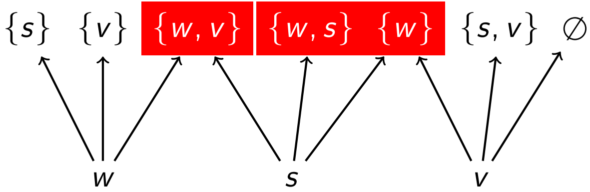

$$\mathcal{M}, s \models \Diamond p$$

$$[\![\neg p]\!]_{\mathcal{M}} = \{v\}$$

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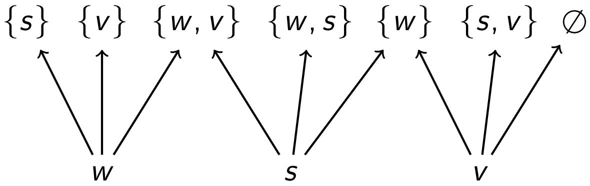

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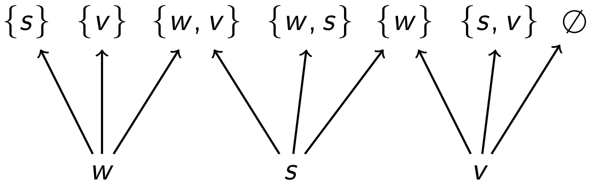
Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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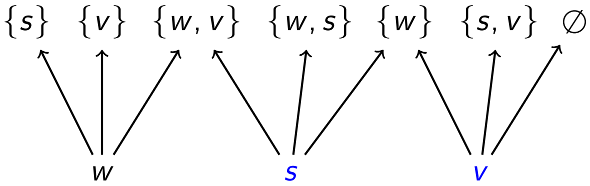

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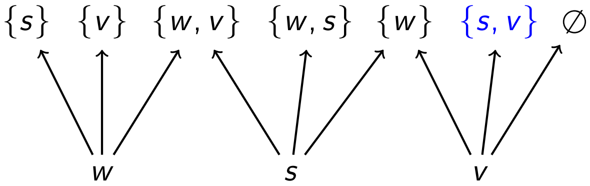

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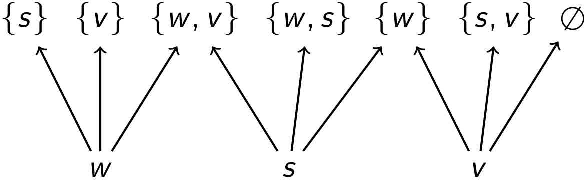

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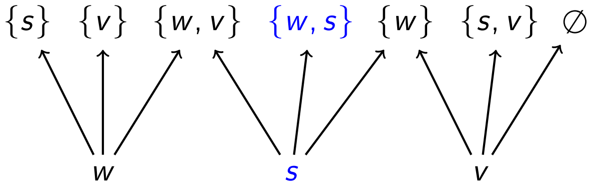

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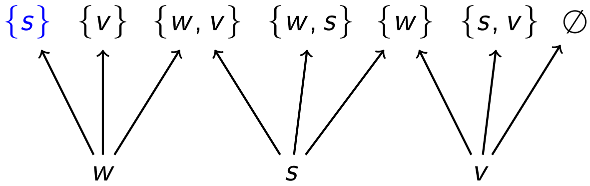

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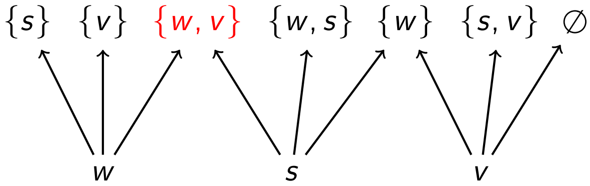

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Neighborhood Modalities

- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(w)$
- $\mathcal{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathcal{M}} \notin N(w)$

Other modal operators

- $\mathcal{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathcal{M}, v \models \varphi$

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Lemma

Let $\mathcal{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathcal{M}, w \models \Box\varphi$ then $\mathcal{M}, w \models \langle\ ]\varphi$*
2. *if $\mathcal{M}, w \models [\ \rangle\varphi$ then $\mathcal{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

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Lemma

1. *If $\varphi \to \psi$ is valid, then so is $\langle\ ]\varphi \to \langle\ ]\psi$.*
2. $\langle\ ](\varphi \wedge \psi) \to (\langle\ ]\varphi \wedge \langle\ ]\psi)$ *is valid in $\mathcal{M}$*

Investigate analogous results for the other modal operators defined above.

Instantial Neighborhood Logic

$\mathcal{M}, w \models \Box(\psi_1, \ldots, \psi_k; \varphi)$ iff there is an $X \in N(w)$ such that

- for all $x \in X$, $\mathcal{M}, x \models \varphi$ and
- for all $i \in \{1, \ldots, k\}$ there is a $x_i \in X$ such that $\mathcal{M}, x_i \models \psi_i$

Johan van Benthem, Nick Bezhanishvili, Sebastian Enqvist, and Junhua Yu (2017). *Instantial Neighbourhood Logic*. The Review of Symbolic Logic 10(1), pp. 116 - 144.

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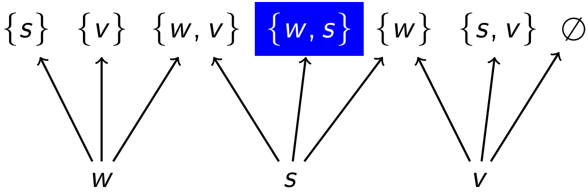

$$\mathcal{M}, s \models \Box(q, \neg q, p)$$

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Validity

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(Similar definitions for relational models/frames)

Examples

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- $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$is not valid on neighborhood frames
- $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$ is not valid on neighborhood frames
- $\Box\top$ is not valid on neighborhood frames

A **logic** is a set of formulas **L** satisfying certain closure conditions. We write $\vdash_{\mathbf{L}} \varphi$ iff $\varphi \in \mathbf{L}$.

Rule of inference: "From $\varphi_1, \ldots, \varphi_n$ infer φ", denoted $\dfrac{\varphi_1 \ \varphi_2 \ \cdots \ \varphi_n}{\varphi}$, where $n \geq 0$. A logic is closed under a rule of inference means that if $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subseteq \mathbf{L}$, then $\varphi \in \mathbf{L}$

Uniform Substitution (US)

$$\frac{\varphi}{\psi}$$

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Axiom Schemes vs. Axioms:

- The logic contains all instances of $\alpha \to (\beta \to \alpha)$
- The logic contains the axiom $p \to (q \to p)$ and is closed under uniform substitution

Normal Modal Logic

A **normal modal logic** is a logic that:

- contains all instances of propositional tautologies
- is closed under modus ponens: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

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4. Let **K** be the smallest normal modal logic: The smallest set of formulas containing all propositional tautologies, all instances of K, all instances of *Dual*, closed under Modus Ponens, and closed under Necessitation.

Normal Modal Logics

PC: All propositional tautologies

N: The rule of necessitation: $\dfrac{\varphi}{\Box\varphi}$

Some Axioms

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
D	$\Box\varphi \to \Diamond\varphi$
T	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
5	$\neg\Box\varphi \to \Box\neg\Box\varphi$
L	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$

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Some Normal Modal Logics

K	$PC + N + K$
T	$PC + N + K + T$
K4	$PC + N + K + 4$
S4	$PC + N + K + T + 4$
S5	$PC + N + K + T + 4 + 5$
KD45	$PC + N + K + D + 4 + 5$
GL	$PC + N + K + L$

Non-Normal Modal Logics

PC	Propositional Calculus + MP
E	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
M	$\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$
C	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$
N	$\Box\top$
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
RE	$\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$
Nec	$\dfrac{\varphi}{\Box\varphi}$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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E is the smallest classical modal logic.

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(RM)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

Non-Normal Modal Logics

PC Propositional Calculus + MP

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

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C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

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E is the smallest classical modal logic.

EM is the logic **E** + *RM*

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RM $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

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EM is the logic **E** + *RM*

EC is the logic **E** + *C*

Non-Normal Modal Logics

PC Propositional Calculus + MP

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EMC is the smallest regular modal logic

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A logic is normal if it contains all instances of *E*, *C* and is closed under *RM* and *Nec*

Non-Normal Modal Logics

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EC is the logic **E** + *C*

EMC is the smallest regular modal logic

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Non-Normal Modal Logics

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K = **EMCN**

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E is the smallest classical modal logic.

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EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

An equivalent definition of a normal modal logic: A **normal modal logic** is a logic that

- contains all instances of propositional tautologies
- is closed under modus ponens: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$
- contains all instances of
  - *Dual*: $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$,
  - *M*: $\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)$
  - *C*: $(\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$
  - *N*: $\Box\top$
- is closed under *RE*: $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Relationship Between Key Axioms

Each of K, M and C are logically independent:

- **EC** $\nvdash K$
- **EM** $\nvdash K$
- **EMC** $\vdash K$
- **EK** $\nvdash M$
- **EK** $\nvdash C$

Alternative Definition of Normal Modal Logics

(RE) $$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

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(RR) $$\frac{(\varphi_1 \wedge \varphi_2) \rightarrow \psi}{(\Box\varphi_1 \wedge \Box\varphi_2) \rightarrow \Box\psi}$$

(RK) $$\frac{(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi}{(\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n) \rightarrow \Box\psi} \qquad (n \geq 0)$$

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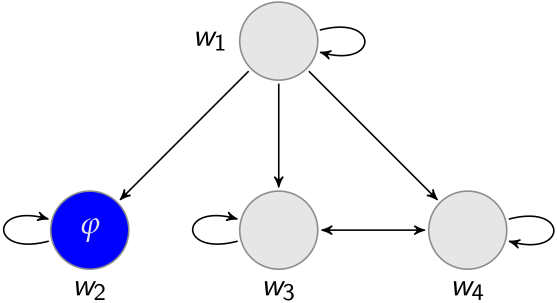
Are there non-normal extensions of **K**? Yes!

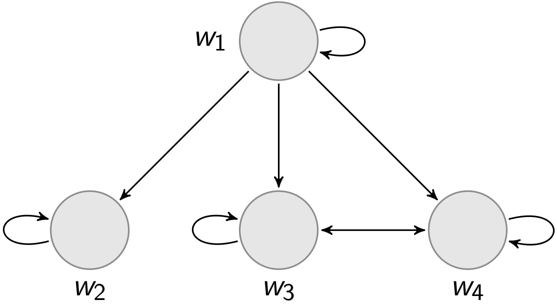
Let **L** be the smallest modal logic containing

- **S4** $(\mathbf{K} + \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi)$
- all instances of M: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4**.



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$$



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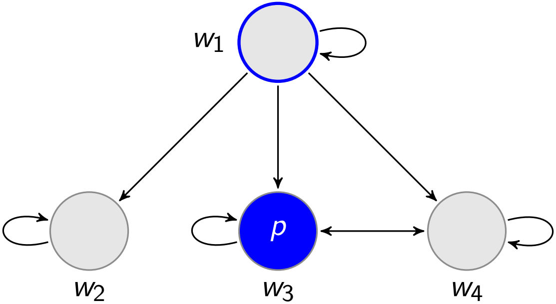

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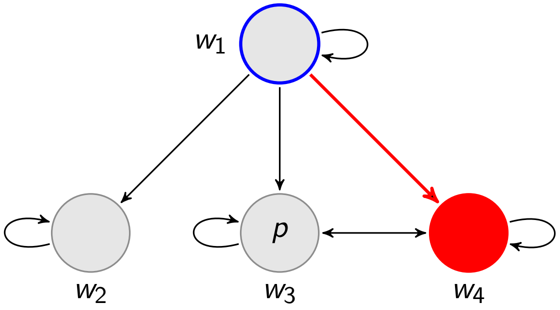

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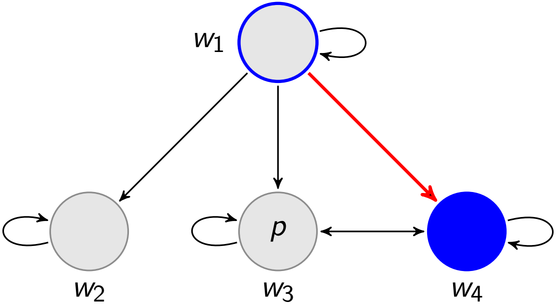

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Some Terminology: Subset Spaces

Let W be a set and $\mathcal{X} \subseteq \wp(W)$.

- $\mathcal{X}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{X}$, then $\cap_{i \in I} X_i \in \mathcal{X}$.
- $\mathcal{X}$ is closed under unions if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{X}$, then $\cup_{i \in I} X_i \in \mathcal{X}$.
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Lemma

$\mathcal{X}$ is supplemented iff if $X \cap Y \in \mathcal{X}$ then $X \in \mathcal{X}$ and $Y \in \mathcal{X}$.

A few more definitions

- $\mathcal{X}$ is a filter if $\mathcal{X}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{X}$ is a proper filter if in addition $\mathcal{X}$ does not contain the emptyset.
- $\mathcal{X}$ is an ultrafilter if $\mathcal{X}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{X}$ or $X^C \in \mathcal{X}$.
- $\mathcal{X}$ is a topology if $\mathcal{X}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.
- $\mathcal{X}$ is augmented if $\mathcal{X}$ contains its core and is supplemented.

Some Facts

Lemma

If $\mathcal{X}$ is augmented, then $\mathcal{X}$ is closed under arbitrary intersections. In fact, if $\mathcal{X}$ is augmented then $\mathcal{X}$ is a filter.

Fact

There are consistent filters that are not augmented.

Lemma

If $\mathcal{X}$ is closed under binary intersections (i.e., if $X, Y \in \mathcal{X}$ then $X \cap Y \in \mathcal{X}$), then $\mathcal{X}$ is closed under finite intersections.

Corollary

If W is finite and $\mathcal{X}$ is a filter over W, then $\mathcal{X}$ is augmented.

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Logical consequence

Suppose that Γ is a set of formulas and $\mathbb{F}$ is a set of frames. We write $\mathcal{M}, w \models \Gamma$ iff $\mathcal{M}, w \models \alpha$ for all $\alpha \in \Gamma$.

$\Gamma \models_{\mathbb{F}} \varphi$ iff for all frames $\mathcal{F} \in \mathbb{F}$, for all models $\mathcal{M}$ based on $\mathcal{F}$ and all states w in $\mathcal{M}$, $\mathcal{M}, w \models \Gamma$ implies $\mathcal{M}, w \models \varphi$.

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Over the class of relational frames:

- $\models (\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$
- $\{\Box p \to \Diamond p\} \models \Diamond\top$
- $\{\Box p \to p\} \models \Box p \to \Diamond p$

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Over the class of neighborhood frames:

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- $\{\Box p \rightarrow \Diamond p\} \not\models \Diamond\top$
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Soundness and Completeness

- A logic **L** is **sound** with respect to $\mathbb{F}$, provided $\vdash_{\mathbf{L}} \varphi$ implies $\models_{\mathbb{F}} \varphi$.

- A logic **L** is **weakly complete** with respect to a class of frames $\mathbb{F}$, if $\models_{\mathbb{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$.

- A logic **L** is **strongly complete** with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$.

A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas.

The **L-proof set** of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Let $\mathbf{L}$ be a logic and $\varphi, \psi \in \mathcal{L}$. Then

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \to \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. For any maximally $\mathbf{L}$-consistent set Γ, if $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$, then $\psi \in \Gamma$
7. For any maximally $\mathbf{L}$-consistent set Γ, If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$

Lindenbaum's Lemma. For any consistent set of formulas Γ, there exists a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

Canonical Model

Definition

A neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ is canonical for **L** provided

- $W = \{\Gamma \mid \Gamma$ is a maximally **L**-consistent set $\}$

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- for all $p \in \mathsf{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathcal{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,

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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathcal{M}_{\mathbf{L}}^{max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{max}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,

$$N_{\mathbf{L}}^{max}(\Gamma) = N_{\mathbf{L}}^{min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin P_{\mathbf{L}}\}$$