Aggregating Judgements: Logical and Probabilistic Approaches

Lecture 1

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Plan

Monday  Representing judgements; Introduction to judgement aggregation; Aggregation paradoxes

Tuesday  Axiomatic characterizations of aggregation methods I

Wednesday  Axiomatic characterizations of aggregation methods II, Distance-based characterizations

Thursday  Opinion pooling; Merging of probabilistic opinions (Blackwell-Dubins Theorem); Aumann’s agreeing to disagree theorem and related results

Friday  Belief polarization; Diversity trumps ability theorem (The Hong-Page Theorem)
Judgement aggregation model

- Group of experts
- Agenda
- Judgement
- Aggregation method
Group of experts

- Evidence: shared or independent
- Communication: Allow communication/sharing of opinions
- Opinionated
- Coherent: logically and/or probabilistically
Agenda

- Single issue/proposition
- Set of independent issues/propositions
- Set of logically connected issues/propositions

Is $P$ true?

Do you accept $P_1$?
Do you accept $P_2$?
\vdots
Do you accept $P_n$?

Do you accept $P$?
Do you accept $P \rightarrow Q$?
Do you accept $Q$?
Agenda

- Single issue/proposition
- Set of independent issues/propositions
- Set of logically connected issues/propositions
- Value from some range (quantity/chance)
- Causal relationships between variables

What is the chance that $E$ will happen?
What is the value of $x$?

Which intervention will be most effective?
Judgements

- Expressions of judgement vs. expressions of preference
- Qualitative: Accept/Reject; Orderings; Grades

Accept $P$

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$P_2$</th>
<th>...</th>
<th>$P_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>$N$</td>
<td>...</td>
<td>$Y$</td>
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</table>

Reject $P$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$P \rightarrow Q$</th>
<th>$Q$</th>
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</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>$N$</td>
<td>$N$</td>
</tr>
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</table>

$P \geq Q \geq R \geq \cdots$

$P$ is very likely

$Q$ is very likely

$R$ is very unlikely

\[ \vdots \]
Judgements

- Expressions of judgement vs. expressions of preference
- Qualitative: Accept/Reject; Orderings; Grades
- Quantitative: Probabilities; Imprecise probabilities
- Causal models
- Do the experts provide their reasons/arguments/confidence?

\[ Pr(P) = p \]

\[ Pr(P) = [l, h] \]

![Causal model diagram](image)
Tutorial: What is the chance that humans will land on Mars by 2024?

This question is in round 1, and round 2 will open on 31 Dec, 2022 at 06:59 (EDT).
Aggregation method

- Functions from *profiles* of judgements to judgements.
- Is the group judgement the same type as the individual judgements?
- Hides *disagreement* among the experts.
Aggregation method

- Epistemic considerations: How likely is it that the group judgement is correct?
- Procedural/fairness considerations: Does the group judgement reflect the individual judgements?

\[ J_1 \quad J_2 \quad \ldots \quad J_n \quad \rightarrow \quad F \quad \rightarrow \quad J \text{ (Group judgement)} \]
Wisdom of the Crowds
Collective Intelligence


Group of experts
  Assume that there are an odd number of experts

Agenda:
  A single proposition $P$

Judgements:
  Accept $P$/Judge that $P$ is true
  Reject $P$/Judge that $P$ is false
  Suspend judgement about $P$

Aggregation method
  Majority rule: Accept $P$ if more people accept $P$ than reject $P$; Reject $P$ if more people reject $P$ than accept $P$
Group of voters
   Assume that there are an odd number of experts

Candidates:
   Two candidates $A$ and $B$

Preferences:
   Rank $A$ above $B$
   Rank $B$ above $A$
   Indifferent between $A$ and $B$

Aggregation method
   Majority rule: $A$ wins if more voters rank $A$ above $B$ than $B$ above $A$; $B$ wins if more voters rank $B$ above $A$ than $A$ above $B$;
Characterizing Majority Rule

May’s Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of $n$ experts/voters.

**Aggregation function:** $F : \{1, 0, -1\}^n \to \{1, 0, -1\}$, where

- 1 means Accept $P$ or rank $A$ above $B$
- $-1$ means Reject $P$ or rank $B$ above $A$
- 0 means Suspend judgement about $P$ or $A$ and $B$ are tied
May’s Theorem: Details

Aggregation function: \( F : \{1, 0, -1\}^n \rightarrow \{1, 0, -1\} \)

For \( v \in \{1, 0, -1\}^n \) and \( x \in \{1, 0, -1\} \), let \( N_v(x) = \{ i \in N \mid v_i = x \} \)

\[
F_{Maj}(v) = \begin{cases} 
1 & \text{if } |N_v(1)| > |N_v(-1)| \\
0 & \text{if } |N_v(1)| = |N_v(-1)| \\
-1 & \text{if } |N_v(-1)| > |N_v(1)|
\end{cases}
\]
Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there?
Suppose that there are two voters and two candidates. How many social choice functions are there? 19,683

- There are three possible rankings for 2 candidates.
- When there are two voters there are $3^2 = 9$ possible profiles:
  $$\{(1, 1), (1, 0), (1, -1), (0, 1), (0, 0), (0, -1), (-1, 1), (-1, 0), (-1, -1)\}$$

- Since there are 9 profiles and 3 rankings, there are $3^9 = 19,683$ possible preference aggregation functions.
May’s Theorem: Details

- **Anonymity**: all voters should be treated equally.

- **Neutrality**: all candidates should be treated equally.
May’s Theorem: Details

- **Anonymity**: all voters should be treated equally.

\[ F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}) \] where \( v_i \in \{1, 0, -1\} \) and \( \pi \) is a permutation of the voters.

- **Neutrality**: all candidates should be treated equally.
May’s Theorem: Details

- **Anonymity**: all voters should be treated equally.

  \[ F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}) \]  
  where \( v_i \in \{1, 0, -1\} \) and \( \pi \) is a permutation of the voters.

- **Neutrality**: all candidates should be treated equally.

  \[ F(-v) = -F(v) \]  
  where \( -v = (-v_1, \ldots, -v_n) \).
May’s Theorem: Details

- **Positive Responsiveness** (Monotonicity): unidirectional shift in the voters’ opinions should help the alternative toward which this shift occurs.

For any $v, v' \in \{1, 0, -1\}$ with $v'_i \geq v_i$ for all $i \in N$ and $v'_j > v_j$ for some $j \in N$ we have $F(v) \in \{0, 1\}$ implies that $F(v') = 1$.

Similarly, for any $v, v' \in \{1, 0, -1\}$ with $v'_i \leq v_i$ for all $i \in N$ and $v'_j < v_j$ for some $j \in N$ we have $F(v) \in \{0, -1\}$ implies that $F(v') = -1$. 
Warm-up Exercise

Suppose that there are two experts/voters. How many aggregation functions satisfy anonymity?

**Anonymity**: all voters should be treated equally.

\[ F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}) \] where \( \pi \) is a permutation of the voters.

\[ F(1, 0) = F(0, 1) \]
\[ F(1, -1) = F(-1, 1) \]
\[ F(-1, 0) = F(0, -1) \]

This means that there are essentially 6 elements of the domain. So, there are 3^6 = 729 preference aggregation functions.
Warm-up Exercise

Suppose that there are two experts/voters. How many aggregation functions satisfy anonymity? 729

**Anonymity**: all voters should be treated equally.

\[ F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}) \] where \( \pi \) is a permutation of the voters.

- Imposing anonymity reduces the number of preference aggregation functions.
- If \( F \) satisfies anonymity, then \( F(1, 0) = F(0, 1), F(1, -1) = F(-1, 1) \) and \( F(-1, 0) = F(0, -1) \).
- This means that there are essentially 6 elements of the domain. So, there are \( 3^6 = 729 \) preference aggregation functions.
Anonymity

\[ F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}) \] where \( \pi \) is a permutation of the voters.

Alternative definition of anonymity:

For all \( v \), \( F(v) = sgn(\sum_{i \in N} v_i) \) where,

\[
sgn(r) = \begin{cases} 
1 & r > 0 \\
0 & r = 0 \\
-1 & r < 0 
\end{cases}
\]
May’s Theorem (1952) A social decision method $F$ satisfies neutrality, anonymity and positive responsiveness iff $F$ is majority rule.
Proof Idea

For any $v \in \{1, 0, -1\}$, if $|N_v(1)| = |N_v(-1)|$, then $F(v) = 0$. 

Anonymity implies $(−1, 0, 1)$ is assigned $1$ or $−1$.

Neutrality implies $(1, 0, −1)$ is assigned $−1$ or $1$.

Contradiction.
Proof Idea

For any $v \in \{1, 0, -1\}$, if $|N_v(1)| = |N_v(-1)|$, then $F(v) = 0$.

If $(1, 0, -1)$ is assigned 1 or $-1$ then
Proof Idea

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If $(1, 0, -1)$ is assigned 1 or $-1$ then

✓ Anonymity implies $(-1, 0, 1)$ is assigned 1 or $-1$
Proof Idea

For any $v \in \{1, 0, -1\}$, if $|N_v(1)| = |N_v(-1)|$, then $F(v) = 0$.

If $(1, 0, -1)$ is assigned 1 or $-1$ then

- Anonymity implies $(-1, 0, 1)$ is assigned 1 or $-1$

- Neutrality implies $(1, 0, -1)$ is assigned $-1$ or 1

Contradiction.
Proof Idea

For any $v \in \{1, 0, -1\}$, if $|N_v(1)| > |N_v(-1)|$, then $F(v) = 1$. 
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✓ Neutrality implies \((-1, -1, 1)\) is assigned 0 or 1
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Proof Idea

For any $v \in \{1, 0, -1\}$, if $|N_v(1)| > |N_v(-1)|$, then $F(v) = 1$.

If $(1, 1, -1)$ is assigned 0 or $-1$ then

✓ Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1

✓ Anonymity implies $(1, -1, -1)$ is assigned 0 or 1

✓ Positive Responsiveness implies $(1, 0, -1)$ is assigned 1
Proof Idea

For any \( v \in \{1, 0, -1\} \), if \(|N_v(1)| > |N_v(-1)|\), then \( F(v) = 1 \).

If \((1, 1, -1)\) is assigned 0 or \(-1\) then

✓ Neutrality implies \((-1, -1, 1)\) is assigned 0 or 1

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Contradiction.
Other characterizations


Positive Responsiveness

For any $v, v' \in \{1, 0, -1\}$ with $v'_i \geq v_i$ for all $i \in N$ and $v'_j > v_j$ for some $j \in N$ we have $F(v) \in \{0, 1\}$ implies that $F(v') = 1$.

Similarly, for any $v, v' \in \{1, 0, -1\}$ with $v'_i \leq v_i$ for all $i \in N$ and $v'_j < v_j$ for some $j \in N$ we have $F(v) \in \{0, -1\}$ implies that $F(v') = -1$. 
Positive Responsiveness

For any \( v, v' \in \{1, 0, -1\} \) with \( v'_i \geq v_i \) for all \( i \in N \) and \( v'_j > v_j \) for some \( j \in N \) we have \( F(v) \in \{0, 1\} \) implies that \( F(v') = 1 \).

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**Pareto Optimality**: For any \( v \), if \( v_i \geq 0 \) for all \( i \in N \) and \( v_j = 1 \) for some \( j \in N \), then \( F(v) = 1 \). Similarly, for any \( v \), if \( v_i \leq 0 \) for all \( i \in N \) and \( v_j = -1 \) for some \( j \in N \), then \( F(v) = -1 \).

**Monotonicity**: For all \( v, v' \), if \( v \leq v' \) (i.e., \( v_i \leq v'_i \) for all \( i \in N \)), then \( F(v) \leq F(v') \)
Aggregation function with variable domain: $F : \bigcup_{n \in \mathbb{N}} \{-1, 0, 1\}^n \to \{-1, 0, 1\}$. 

Path Independence: For all $v \in \{-1, 0, 1\}^n$ and $v' \in \{-1, 0, 1\}^{n'}$, we have $F(v \oplus v') = F(F(v) \oplus F(v'))$.

Weak Path Independence: For all $v \in \{-1, 0, 1\}^n$ and $v' \in \{-1, 0, 1\}^{n'}$ with $|F(v) - F(v')|$, we have $F(v \oplus v') = F(F(v) \oplus F(v'))$. 

19 / 49
Aggregation function with variable domain: \( F : \bigcup_{n \in \mathbb{N}} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\} \).

For \( n, n' \in \mathbb{N} \), let \( N = \{1, \ldots, n\} \) and \( N' = \{n + 1, \ldots, n + n'\} \) be disjoint populations. If \( \mathbf{v} \in \{-1, 0, 1\}^n \) and \( \mathbf{v}' \in \{-1, 0, 1\}^{n'} \), then let:

\[
\mathbf{v} \oplus \mathbf{v}' = (v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+n'})
\]
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Weak Path Independence: For all $v \in \{-1, 0, 1\}^n$ and $v' \in \{-1, 0, 1\}^{n'}$ with $|F(v) - F(v')| \neq 2$, we have $F(v \oplus v') = F(F(v) \oplus F(v'))$. 
Theorem (Asan and Sanver). An aggregation function

\[ F : \bigcup_{n \in \mathbb{N}} \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\} \]

satisfies Anonymity, Neutrality, Pareto Optimality and Weak Path Independence if and only if it is the majority rule.
Proof Sketch

If $F$ satisfies Anonymity and Neutrality, then for any $v \in \{1, 0, -1\}$, if $|N_v(1)| = |N_v(-1)|$, then $F(v) = 0$. 
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If $F$ satisfies Anonymity and Neutrality, then for any $v \in \{1, 0, -1\}$, if $|N_v(1)| = |N_v(-1)|$, then $F(v) = 0$.

Fix a $v$ with $|N_v(1)| > |N_v(-1)|$. Let $k = |N_v(1)| - |N_v(-1)|$. 

Proof Sketch

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Define a coalition $K \subseteq \{i \mid v_i = 1\}$ with $|K| = k$. 

By Weak Path Independence, $F(v) = F(v' \oplus v'') = F(F(v') \oplus F(v'')) = F(1, 0)$. 

By Pareto Optimality, $F(1, 0) = 1$. 


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Let $v' \in \{-1, 0, 1\}^k$ be a profile with $v'_i = v_i$ for all $i \in K$ and $v'' \in \{-1, 0, 1\}^{n-k}$ be a profile with $v''_i = v_i$ for all $i \in N - K$. Then, $v = v' \oplus v''$, $F(v') = 1$ and $F(v'') = 0$.

By Weak Path Independence, $F(v) = F(v' \oplus v'') = F(F(v') \oplus F(v'')) = F(1, 0) = 1$.

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21 / 49
Proof Sketch

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Then, $v = v' \oplus v''$, $F(v') = 1$ and $F(v'') = 0$. 
Proof Sketch

If \( F \) satisfies Anonymity and Neutrality, then for any \( \mathbf{v} \in \{1, 0, -1\} \), if \( |N_\mathbf{v}(1)| = |N_\mathbf{v}(-1)| \), then \( F(\mathbf{v}) = 0 \).

Fix a \( \mathbf{v} \) with \( |N_\mathbf{v}(1)| > |N_\mathbf{v}(-1)| \). Let \( k = |N_\mathbf{v}(1)| - |N_\mathbf{v}(-1)| \).

Define a coalition \( K \subseteq \{i \mid v_i = 1\} \) with \( |K| = k \).

Let \( \mathbf{v}' \in \{-1, 0, 1\}^k \) be a profile with \( v'_i = v_i \) for all \( i \in K \) and \( \mathbf{v}'' \in \{-1, 0, 1\}^{n-k} \) be a profile with \( v''_i = v_i \) for all \( i \in N - K \).

Then, \( \mathbf{v} = \mathbf{v}' \oplus \mathbf{v}'' \), \( F(\mathbf{v}') = 1 \) and \( F(\mathbf{v}'') = 0 \).

By Weak Path Independence, \( F(\mathbf{v}) = F(\mathbf{v}' \oplus \mathbf{v}'') = F(F(\mathbf{v}') \oplus F(\mathbf{v}'')) = F(1, 0) \).
Proof Sketch

If $F$ satisfies Anonymity and Neutrality, then for any $v \in \{1, 0, -1\}$, if 
\[|N_v(1)| = |N_v(-1)|,\] 
then $F(v) = 0$.

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By Weak Path Independence, $F(v) = F(v' \oplus v'') = F(F(v') \oplus F(v'')) = F(1, 0)$.

By Pareto Optimality, $F(1, 0) = 1$. 
Preference Aggregation vs. Judgement Aggregation

Preference Aggregation vs. Judgement Aggregation

Preferring one alternative to another is not the same as judging it to be better. Judgments of betterness, and in general value judgments, often accompany preferences and the latter might often be based on the former. But it is possible to prefer $a$ to $b$ even though one lacks a clear view about their relative value. Indeed, it is even possible to judge $b$ to be better than $a$ and still prefer $a$ to $b$; perhaps because one thinks that $a$ is better for oneself, even though one considers $b$ to be better overall; or perhaps because one is simply irrational. Consequently, aggregation of preferences is not reducible to aggregation of value judgments.

(Rabinowicz, pg. 11)
Preference Aggregation vs. Judgement Aggregation

Pareto: If every individual ranks $a$ at least as highly as $b$ and some individuals rank $a$ higher than $b$, then $a$ is ranked higher than $b$ by the collective.

“This condition is intuitively plausible for preference aggregation, if we think of collective preferences as primarily guides to choice and if we in addition take it to be important that the collective in its choices endeavors to satisfy individual preferences. ... By opting for $a$ rather than $b$, it will satisfy the preferences of some and frustrate the preferences of no one.”
Preference Aggregation vs. Judgement Aggregation

**Pareto**: If every individual ranks \( a \) at least as highly as \( b \) and some individuals rank \( a \) higher than \( b \), then \( a \) is ranked higher than \( b \) by the collective.

"When it comes to the aggregation of value rankings, things are different. In this aggregation process it is important to require that the collective judgment as far as possible approximates the judgments of the individuals. ... if some individuals believe \( a \) to be better than \( b \), but the overwhelming majority believes \( a \) and \( b \) to be equally good, then — it would seem — the collective value judgment should follow the majority view: \( a \) and \( b \) should be considered by the collective to be of equal value. "

Indifference: The collective ranking of the alternatives doesn’t change if voters who rank all the alternatives equally are removed from consideration (as long as some voters still remain to be considered).
In many group decision making problems, one of the alternatives is the *correct* one. Which aggregation method is best for finding the “correct” alternative?
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Group decision problems often exhibit a *combinatorial structure*. For example, selecting a committee from a set of candidates or voting on a number of yes/no issues in a referendum. Furthermore, the propositions in the agenda may be *interconnected*. 
Proceduralist Justifications

“identifies a set of ideals with which any collective decision-making procedure ought to comply. [A] process of collective decision making would be more or less justifiable depending on the extent to which it satisfies them...
Proceduralist Justifications

“identifies a set of ideals with which any collective decision-making procedure ought to comply. [A] process of collective decision making would be more or less justifiable depending on the extent to which it satisfies them...What justifies a [collective] decision-making procedure is strictly a necessary property of the procedure—one entailed by the definition of the procedure alone.”

Epistemic Justifications

“An epistemic interpretation of voting has three main elements: (1) an independent standard of correct decisions — that is, an account of justice or of the common good that is independent of current consensus and the outcome of votes;
Epistemic Justifications

“An epistemic interpretation of voting has three main elements: (1) an independent standard of correct decisions — that is, an account of justice or of the common good that is independent of current consensus and the outcome of votes; (2) a cognitive account of voting — that is, the view that voting expresses beliefs about what the correct policies are according to the independent standard, not personal preferences for policies;
“An epistemic interpretation of voting has three main elements: (1) an independent standard of correct decisions — that is, an account of justice or of the common good that is independent of current consensus and the outcome of votes; (2) a cognitive account of voting — that is, the view that voting expresses beliefs about what the correct policies are according to the independent standard, not personal preferences for policies; and (3) an account of decision making as a process of the adjustment of beliefs, adjustments that are undertaken in part in light of the evidence about the correct answer that is provided by the beliefs of others. (p. 34)”

The Condorcet Jury Theorem
Suppose that $P$ takes on two values: $P = 1$ (i.e., $P$ is true) or $P = 0$ (i.e., $P$ is false).
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Each voter $i$ can report two values: $V_i = 1$ ("$i$ says that $P$ is true") and $V_i = 0$ ("$i$ says that $P$ is false").
Suppose that $P$ takes on two values: $P = 1$ (i.e., $P$ is true) or $P = 0$ (i.e., $P$ is false).

Each voter $i$ can report two values: $V_i = 1$ ("$i$ says that $P$ is true") and $V_i = 0$ ("$i$ says that $P$ is false").

$i$’s competence $p_i \in [0, 1]$ is the probability of reporting correctly:

$$Pr(V_i = 1 \mid P = 1) = Pr(V_i = 0 \mid P = 0) = p_i.$$
Suppose that $P$ takes on two values: $P = 1$ (i.e., $P$ is true) or $P = 0$ (i.e., $P$ is false).

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$i$’s competence $p_i \in [0, 1]$ is the probability of reporting correctly:

$Pr(V_i = 1 | P = 1) = Pr(V_i = 0 | P = 0) = p_i$.

Given a profile $\mathbf{p}$ of competences and an aggregation method $F$, let $\pi(F, \mathbf{p})$ be the probability that $F$ identifies the correct answer.
Expert rule

Suppose that all competences are the same $p = (p_1, \ldots, p_n)$ with $p_i = p_j$ for all $i, j$. Then,

$$\pi(F^e, p) = p$$
Expert rule

Suppose that all competences are the same $p = (p_1, \ldots, p_n)$ with $p_i = p_j$ for all $i, j$. Then,

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In general, for a profile $p = (p_1, \ldots, p_n)$, we have that

$$\pi(F^e, p) = \sum_{i}^{n} Pr(\text{"choosing } i\text{"})p_i$$

In particular, if each expert is equally likely to be chosen:

$$\pi(F^e, p) = \sum_{i}^{n} \frac{1}{n}p_i$$
Majority Rule

\[ \pi(F^m, p) \]
Majority Rule

$$\pi(F^m, p) = p^3$$

The probability everyone is correct is $p^3$
Majority Rule

\[ \pi(F^m, p) = p^3 + 3p^2(1 - p) \]

The probability everyone is correct is \( p^3 \)

The probability that 1 and 2 are correct: \( p^2(1 - p) \)

The probability that 2 and 3 are correct: \( p^2(1 - p) \)

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Condorcet Jury Theorem tutorial.
Condorcet Jury Theorem

Suppose that the $P$ takes values 0 and 1

$R_i$ is the event that voter $i$ reports correctly.
Condorcet Jury Theorem

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$R_i$ is the event that voter $i$ reports correctly.

**Independence** The reports of the voters are independent conditional on the state of the world: $R_1, R_2, \ldots$ are independent conditional on $P$

**Competence**: For each voter, the probability that the reports correctly is greater than 1/2: for each $x \in \{0, 1\}$, $p(R_i \mid P = x) > \frac{1}{2}$ and
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**Condorcet Jury Theorem**. Suppose Independence and Competence. As the group size increases, the probability that majority opinion is correct (i) increases and (ii) converges to one.


In many group decision making problems, one of the alternatives is the *correct* one. Which aggregation method is best for finding the “correct” alternative?

Group decision problems often exhibit a *combinatorial structure*. For example, selecting a committee from a set of candidates or voting on a number of yes/no issues in a referendum. Furthermore, the propositions in the agenda may be *interconnected*. 
Multiple Elections Paradox

Voters are asked to give their opinion on three yes/no issues:

<table>
<thead>
<tr>
<th>YYY</th>
<th>YYN</th>
<th>YNY</th>
<th>YNN</th>
<th>NYY</th>
<th>NYN</th>
<th>NNY</th>
<th>NNN</th>
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</thead>
<tbody>
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<td>1</td>
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<td>1</td>
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<td>0</td>
</tr>
</tbody>
</table>

Outcome by majority vote:

- Proposition 1: \( N \) (7 - 6)
- Proposition 2: \( N \) (7 - 6)
- Proposition 3: \( N \) (7 - 6)

But there is no support for \( NNN \)
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Outcome by majority vote

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Multiple Elections Paradox

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Outcome by majority vote

**Proposition 1**: $N (7 - 6)$
**Proposition 2**: $N (7 - 6)$
**Proposition 3**: $N (7 - 6)$

But there is no support for $NNN$!
### Complete Reversal

<table>
<thead>
<tr>
<th>YYYN</th>
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<th>YNYY</th>
<th>NYYY</th>
<th>NNNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Outcome by majority vote**

**Proposition 1**: $Y$ (6 - 5)
**Proposition 2**: $Y$ (6 - 5)
**Proposition 3**: $Y$ (6 - 5)
**Proposition 4**: $Y$ (6 - 5)

$YYYY$ wins proposition-wise voting, but the “opposite” outcome $NNN$ has the *most* overall support!


A decision has to be made about whether or not to build a new swimming pool ($S$ or $\bar{S}$) and a new tennis court ($T$ or $\bar{T}$). Consider 5 voters with rankings over \{${ST, \bar{S}T, S\bar{T}, \bar{S}\bar{T}}$\}:

<table>
<thead>
<tr>
<th>rank</th>
<th>2 voters</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$S\bar{T}$</td>
<td>$\bar{S}T$</td>
<td>$ST$</td>
</tr>
<tr>
<td>2</td>
<td>$\bar{S}T$</td>
<td>$S\bar{T}$</td>
<td>$ST$</td>
</tr>
<tr>
<td>3</td>
<td>$\bar{S}T$</td>
<td>$\bar{S}\bar{T}$</td>
<td>$\bar{S}T$</td>
</tr>
<tr>
<td>4</td>
<td>$ST$</td>
<td>$ST$</td>
<td>$\bar{S}\bar{T}$</td>
</tr>
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A decision has to be made about whether or not to build a new swimming pool (\(S\) or \(\overline{S}\)) and a new tennis court (\(T\) or \(\overline{T}\)). Consider 5 voters with rankings over \(\{S \ T, \overline{S} \ T, S \ \overline{T}, \overline{S} \ \overline{T}\}\):

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(S \ \overline{T})</td>
<td>(\overline{S} \ T)</td>
<td>(S \ T)</td>
</tr>
<tr>
<td>2</td>
<td>(\overline{S} \ T)</td>
<td>(S \ \overline{T})</td>
<td>(S \ \overline{T})</td>
</tr>
<tr>
<td>3</td>
<td>(\overline{S} \ \overline{T})</td>
<td>(\overline{S} \ \overline{T})</td>
<td>(\overline{S} \ T)</td>
</tr>
<tr>
<td>4</td>
<td>(S \ T)</td>
<td>(S \ T)</td>
<td>(\overline{S} \ \overline{T})</td>
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</table>

The preferences of voters 1-4 are not separable. So, they will have a hard time voting on \(S\) vs. \(\overline{S}\) and \(T\) vs. \(\overline{T}\).
A decision has to be made about whether or not to build a new swimming pool (S or $\overline{S}$) and a new tennis court (T or $\overline{T}$). Consider 5 voters with rankings over \{ST, $\overline{S}$T, S$\overline{T}$, $\overline{S}$ $\overline{T}$\}:

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Assume that the voters are optimistic: They vote for the options that are top on their list.
A decision has to be made about whether or not to build a new swimming pool ($S$ or $\bar{S}$) and a new tennis court ($T$ or $\bar{T}$). Consider 5 voters with rankings over $\{S \ T, \bar{S} \ T, S \ \bar{T}, \bar{S} \ \bar{T}\}$:

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When voting on the individual issues, $S$ wins (3-2) and $T$ wins (3-2), but the outcome $S \ T$ is a Condorcet loser.
"Is a conflict between the proposition and combination winners necessarily bad?"
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“Is a conflict between the proposition and combination winners necessarily bad? ... The paradox does not just highlight problems of aggregation and packaging, however, but strikes at the core of social choice—both what it means and how to uncover it. In our view, the paradox shows there may be a clash between two different meanings of social choice, leaving unsettled the best way to uncover what this elusive quantity is.”  