J Philos Logic
DOI 10.1007/s10992-015-9356-8

# On the use (and abuse) of Logic in Game Theory

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Received: 5 July 2013 / Accepted: 27 June 2014


**Keywords** Logic and games · Epistemic game theory · Epistemic logic

## 1 Introduction

A quick glance at the opening paragraphs in many of the classic logic textbooks reveals a common view: Logical methods highlight the reasoning patterns of a single (idealized) agent engaged in some form of *mathematical* thinking.[1] However, this traditional view of the "subject matter" of logic is expanding. There is a growing literature using phrases such as "rational interaction" or "information flow" to describe its subject matter while still employing traditional logical methods. The clearest example of this can be found in the work of Johan van Benthem and others on *logical dynamics* [38, 49, 59]; Rohit Parikh and others on *social software* [42, 60]; Samson Abramsky and others on *game semantics* for linear logic [1], and Mike Wooldridge, Valentin Goranko and others on *logics for multiagent systems* [25, 58, 62]. There are many issues driving this shift in thinking about what logic is *about* (see [50] for a discussion). One important reason for this shift is the close connection between logic and game theory.

For centuries, logicians have used game-theoretic notions in their analyses. (See [26, 31] for an overview of applications of game theory in logic.) In the past 20 years or so, the influence has started to run in the other direction. There is now a very active

[1] A (biased) sampling from my bookshelf: Shoenfield's *Mathematical Logic*: "Logic is the study of reasoning; and mathematical logic is the study of the type of reasoning done by mathematicians"; Enderton's *A Mathematical Introduction of Logic*: "Symbolic logic is a mathematical model of deductive thought"; and Chiswell and Hodges *Mathematical Logic*: "In this course we shall study some ways of proving statements."

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Published online: 18 March 2015

research area focused on adapting existing logical systems and developing new ones to reason about different aspects of a game situation. Full coverage of this fascinating research area would require much more space than I have in this short article. So, I will not attempt a comprehensive overview of the use of logic in game theory[2] and, instead, will focus on just a few key topics.

The first topic is the question of motivation: Why should game theorists learn some logic, and, vice versa, why should logicians learn some game theory? There is a very simple reason why logicians have become interested in game theory: The mathematical models that game theorists use to describe a game situation are natural models for many existing logical languages (e.g., epistemic, doxastic and preference logics, as well as conditional logics, temporal logics and logics of action). Thus, game theory provides a fertile testing ground for a variety of logical systems. The following quote nicely explains why game theorists have become interested in logic. The late economist Michael Bacharach wrote in 1997:

> Game theory is full of deep puzzles, and there is often disagreement about proposed solutions to them. The puzzlement and disagreement are neither empirical nor mathematical but, rather, concern the meanings of fundamental concepts ('solution', 'rational', 'complete information') and the soundness of certain arguments....Logic appears to be an appropriate tool for game theory both because these conceptual obscurities involve notions such as reasoning, knowledge and counter-factuality which are part of the stock-in-trade of logic, and because it is a prime function of logic to establish the validity or invalidity of disputed arguments [9].

Bacharach made some of the earliest contributions using logical methods to analyze games situations. His general idea is that various logical frameworks can be used to make precise the important notions (such as knowledge and counter-factuality) that are often left informal or implicit in game-theoretical analyses.

## 2 What is a Logical Analysis of a Game?

Open any book or journal on game theory, and you are immediately struck by the level of mathematical rigor. Precise mathematical results are in no short supply in game theory, so *prima facie* it seems odd to suggest that game theorists are in need of *more* formalism. Indeed, a question that one often hears when presenting a logical system that is intended to reason about game situations is: In what way does the logical system enhance or improve a standard game-theoretic analysis? One answer is that a logical analysis brings with it a precise syntax and the identification of key proof rules and axioms. This is important if one is interested in systematizing a collection of results and can uncover subtleties in the arguments. Of course, this is true of *any* logical analysis and does not explain the usefulness of logic in game theory *in particular*. In order to appreciate the insights that a logical analysis of a game can

[2] See [16, 54, 56, 57] for general overviews of logic in game theory.

deliver, it is important to distinguish the different ways that logical systems have been used to reason about game situations.

- Logics for reasoning about the *analysis* of games. The goal is to identify and study the patterns of reasoning that are used to analyze various game situations. For example, logical methods can be used to formally verify the proofs of key game-theoretic results (for instance, [61] uses the coq proof assistant to formally verify the proof of the existence of a Nash equilibrium in a restricted class of games). Other researchers have used various modal logics (with fixed-point operators) to highlight the logical structure of the *backwards induction* algorithm [55] and to formalize the standard analysis of the backwards induction algorithm found in many game theory textbooks [22].
- Logics for reasoning *about* games. As I remarked in the introduction, the mathematical models that game theorists use to describe game situations provide a natural semantics for many logical languages. The first question is: What is the "right" logical language for reasoning about these structures? The goal here is not simply to formally describe the various game-theoretic models. That could be done in a number of ways, often in a first-order language with an appropriate signature. Rather, the logician will aim for a well-behaved language, with a good balance between the level of formalization and other desirable properties, such as perspicuous axiomatization, low computational complexity of model checking and satisfiability, and the existence of an elegant meta-theory for the system. See [57] and [56] for broad surveys of the different logical systems for reasoning about games.
- Logics for reasoning *with* games. Games can be composed and iterated to form larger games. This compositional structure can be brought out in a logical system [cf. Parikh's *game logic* [41]). In addition, logical systems have been developed to reason about the compositional structure of *strategies* in a game. The key idea is that rather than treating a strategy as an unanalyzed recipe telling the players what to do at each move, a strategy is a complex object with an interesting logical structure of its own that can be analyzed in an appropriate logic [24, 44].
- Logics for reasoning *in* games. The goal is to analyze how people behave during strategic interactions, taking into account their knowledge and beliefs about the circumstances of the game, the other players' knowledge and beliefs, and the other players' rationality, as well as about the knowledge and beliefs about the knowledge and beliefs of the other players. I discuss the relevant logical frameworks in Section 3.

These categories are meant to be neither strict nor exhaustive. The same logical system can be used in a variety of ways. For example, standard epistemic logic has proven to be an invaluable tool for each type of analysis listed above. Furthermore, one often sees the same logical system being used in different ways within the same paper. For example, [35] develop a simple modal logic for reasoning about strategic games. Along the way, they present a formal proof of Harsanyi's famous observation that all uncertainty about the game situation can be reduced to uncertainty about the utilities (cf., also, [32] for a related proof). Still, it is useful to keep these categories in mind when studying different logical systems for reasoning about games.

## 3 Taking the Players' Perspective

Much of the standard game theory literature takes a "top-down" approach to analyzing games. The game situation is described in terms of the players' available actions and preferences over the *outcomes* of the game (where an outcome depends on the choices of *all* the players). Then, various *solution concepts* (e.g., the Nash equilibrium) are used to identify the outcomes that will arise under the assumption that all players behave "rationally". However, games can also be analyzed using a "bottom-up" approach. For this, we must extend the mathematical model of a game situation with a description of what the players know and believe about the game situation and each other. More broadly, the idea is to explicitly describe the players' "view" of the game situation:

> [C]onventional game theory confuses the world as seen by the theorist with the word as seen by the decision-making agent.....[I]t is essential to distinguish between the decision that appears in the theorist's model and the problem as the agent represents it to herself [8].

The logical analysis of games is very conducive to this first-person perspective on game theory. In this section, I discuss different ways to analyze games using the bottom-up approach sketched above. I focus on work that is influenced by or uses logical methods. This area lies at the intersection of logic and game theory. Indeed, I cannot clearly characterize the work discussed here as an application of logic in game theory or as a purely logical approach to analyzing games.

### 3.1 Describing the Informational Context of a Game

The *informational context* of a game describes the players' "knowledge" about the game situation *and* their opinions about the choices and beliefs of the other players. Researchers interested in the foundation of decision theory, epistemic and doxastic logic and formal epistemology have developed many different formal models that can describe the many varieties of informational attitudes important for assessing the choice of a *rational* agent in a game situation. Consult [33, 37, 53], and [27] for an overview of the different models and references to the relevant literature.

To make things more concrete, in this section, I focus on models of a *strategic game*. Recall that the definition of a strategic game for a set of players $N$ consists of (1) a set $A_i$ of actions for each $i \in N$, and (2) a utility function or preference ordering on the set of outcomes. For simplicity, one often identifies the outcomes with the set $S = \Pi_{i\in N} A_i$ of *strategy profiles*. Given a strategy profile $s \in S$ with $s = (a_1, \dots, a_n)$, $s_i$ is the $i$th projection (*i.e.*, $s_i = a_i$), and $s_{-i}$ lists the choices of all agents except agent $i$: $s_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.

Fix a strategic game $G$. A model of $G$ typically consists of three main components. The first is a set $W$ of states, or possible worlds. Each element of $W$ is associated with an outcome of the game $G$—i.e., a strategy profile in $G$. For $w \in W$, let $\sigma(w)$ denote the strategy profile associated with state $w$. Note that the possible worlds are *richer* than strategy profiles, as they also are associated with a description of the players' knowledge and beliefs. An *event* is any subset of $W$ and is intended to represent a property of the game situation.

The second component in a model of the informational context of a game is a description of the players' *knowledge* about the game situation. Following the standard approach of the epistemic logic literature, each player is associated with an equivalence relation $\sim_i$ on the set of states $W$. The intended interpretation of $w \sim_i v$ is that "the states $w$ and $v$ are *indistinguishable* for player $i$." In other words, player $i$'s information cell at $w$, defined as $[w]_i = \{v \mid w \sim_i v\}$, is the set of states that $i$ has not"ruled out" at state $w$. Then, player $i$ knows an event $E$ at state $w$, provided that $E$ contains all the states $i$ has not ruled out—i.e., $[w]_i \subseteq E$.

The third component in a model of the informational context of a game is a description of the players' *beliefs*. Broadly speaking, there are two ways to describe the players' beliefs in a game situation. The first is to assign to each player $i$, a reflexive and transitive ordering $\preceq_i$ on the states $W$. The intended interpretation of $w \preceq_i v$ is that "player $i$ considers $w$ at least as *plausible* as $v$" (let $w \prec_i v$ denote that $w$ is *strictly* more plausible than $v$ for player $i$—i.e., $w \preceq_i v$ and $v \not\preceq_i w$). Typically, there are two additional assumptions:

1. *plausibility implies possibility*: if $w \preceq_i v$, then $w \sim_i v$.
2. *locally-connected*: if $w \sim_i v$, then either $w \preceq_i v$ or $v \preceq_i w$.

In addition, the following is also standard: The *best worlds assumption* states that the set $best_i([w]_i) = \{v \mid \text{there is no v' } \in [w]_i \text{such that v' } \prec_i v\}$ is non-empty. Then, player $i$ believes an event $E$ at state $w$ provided $E$ contains all the best worlds at $w$, i.e., $best_i([w]_i) \subseteq E$. The second approach, which is more common in the game theory literature, is to use probabilities to represent the players' beliefs. As usual, the set of events is assumed to form a $\sigma$-algebra $\mathcal{F}$, and each player's probability measure $\pi_i : \mathcal{F} \to [0, 1]$ satisfies the usual Kolmogrov axioms. Further, it natural to assume that the players' information cells are measurable and assigned a non-zero probability. Then, player $i$'s degree of belief in an event $E$ is calculated using the conditional probability $\pi_i(E \mid [w]_i)$.

Much has been written about different approaches to modeling the knowledge and beliefs of the players in a game situation (see, for example, [15] and [40] for references to the relevant literature). I conclude this section with two issues that arise when the informational context of a game situation is made explicit. The first is a general point about the interpretation of the states in the models sketched above:

> The viewpoint is *descriptive*. Not 'why,' not 'should,' just *what*. Not that $i$ does $a$ *because* he believes $E$; simply that he does $a$ and believes $E$." The viewpoint adopted here ... is *descriptive*. Not *why* the players do what they do, not what *should* they do; just what *do* they do, what *do* they believe. Are they rational, are they irrational, are their actions—or beliefs—in equilibrium? Not "why," not "should," just *what*. Not that $i$ does $a$ *because* he believes $b$; simply that he does $a$, and believes $b$. [6, original italics, pgs. 1174, 1175]

The second point is that the perspective stressed in this section is that the game models describe the beliefs that the players acquire after a process of "rational" deliberation. This suggests additional constraints on a game model. One natural constraint is that the deliberation was successful in the sense that each player comes to know the choice that she has made:

**Knowledge of own action**: Players know their own actions provided that for all states $w$ and $v$, if $w \sim_i v$, then $(\sigma(w))_i = (\sigma(v))_i$.

Another natural constraint is that players cannot *rule out* the fact that their opponents will choose an optimal outcome[3] (in general, there will be more than one "choice-worthy" option for a player). This property was called "privacy of tie breaking" by Cubitt and Sugden [20, pg. 8] and "no extraneous restrictions on beliefs" by Asheim and Dufwenberg [4]. Rabinovich [43] takes this idea even further and argues that from the principle of indifference, players must assign equal probability to all choice-worthy options.

There are a number of ways to make the above idea precise. I need some notation. First, given any probability measure over the strategy profiles $S = \Pi_{i \in \mathcal{A}} S_i$, the *expected utility* of action $a \in S_i$ with respect to $\pi \in \Delta(S)$ is

$$E_\pi(a) = \sum_{s_{-i} \in S_{-i}} \pi(a, s_{-i}) \cdot u_i(a, s_{-i}).$$

Let $\mathcal{M} = \langle W, \{\sim_i\}_{i \in N}, \{\pi_i\}_{i \in N}, \sigma \rangle$ be an epistemic-model for a game $G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$. Each probability measure $\pi_i$, generates a probability measure over the strategy profiles as follows: for each $s \in S$, $\widehat{\pi_i}(s) = \pi_i(\sigma^{-1}(s))$. Using this machinery, I can define what it means for an action $a \in S_i$ to be **optimal for agent** $i$. Formally, for each $a \in S_i$, define the proposition $Opt_i(a)$ as follows:

$$\mathcal{M}, w \models Opt_i(a) \text{ iff for all} a' \in S_i, E_{\widehat{\pi_i}(\cdot \mid [w]_i)}(a) \geq E_{\widehat{\pi_i}(\cdot \mid [w]_i)}(a').$$

I can now formally state two constraints on epistemic probability models. These are natural assumptions to impose if we assume that the epistemic-probability models describe the players' knowledge and beliefs as the players deliberate about what to do in a game situation. The first is[4]:

**Regularity**: For all $i \in N$ and $w \in W$, for all $v \in [w]_i$, $\pi_i(v) > 0$.

The idea is that the first step of a reasoning process is that each player $i$ rules out all the states that are initially considered *impossible* (i.e., assigned probability 0 by $\pi_i$). The second is the "privacy of tie breaking" property discussed above:

**Privacy of tie breaking**: If $\mathcal{M}, w \models Opt_i(a)$ then for all $j \neq i$, there exists a $v \in [w]_j$ such that $\sigma_i(v) = a$.

The explanation of this property is given above. Cubitt and Sugden show that there is a conflict between the above two properties and the standard assumption that there

[3] That is, if, according to player $i$'s beliefs, strategy $s$ is optimal for player $j$, then $i$ cannot rule out *all* states where player $j$ follows strategy $s$.

[4] I am assuming that the set of states is finite, so that individual states can be assigned a non-zero probability.

is *common knowledge* that the players choose optimally[5]. Consider the following three-person game:

$in_3$:

| | $in_2$ | $out_2$ |
|---|---|---|
| $in_1$ | $1,1,1$ | $1,1,1$ |
| $out_1$ | $1,1,1$ | $0,1,1$ |

$out_3$:

| | $in_2$ | $out_2$ |
|---|---|---|
| $in_1$ | $1,1,1$ | $1,0,1$ |
| $out_1$ | $1,1,0$ | $0,0,0$ |

**Proposition 1** ([19, 20]) *There is no model of the above game satisfying privacy of tie breaking, regularity, knowledge of own choice where there is common knowledge that the players choose optimally.*

I give a high-level sketch of the proof (a more formal proof can be found in [20][6]). Suppose that player 1 considers $out_2$ possible. Since, $in_1$ weakly dominates $out_1$, then $in_1$ is the only rational choice for player 1.[7] Thus, both players 2 and 3 know that $out_1$ is impossible. Since player 3 thinks $out_1$ is impossible, both $in_3$ and $out_3$ are rational choices. Again, players 1 and 2 know this, and so by privacy of tie breaking must consider both outcomes possible. Since player 2 considers both $in_3$ and $out_3$ possible and $in_2$ weakly dominates $out_2$, $in_2$ must be the only rational choice for player 2. But, players 1 and 3 know this, and so 1 cannot consider $out_2$ possible.

Suppose that 1 does not consider $out_2$ possible. This means that both $in_1$ and $out_1$ are rational choices for player 1. By privacy of tie breaking, both players 2 and 3 consider these outcomes possible. Since $in_3$ weakly dominates $out_3$, $in_3$ is the only rational choice for player 3. Players 1 and 2 know this fact, and so do not consider $out_3$ possible. Since player 2 does not consider $out_3$ possible, both $in_2$ and $out_2$ are rational choices. By privacy of tie breaking, players 1 and 3 must consider these outcomes possible. In particular, player 1 considers $out_2$ possible, which is a contradiction.

## 3.2 Modeling Strategic Reasoning

The models introduced in the previous section describe the players' beliefs and choices in a game situation. However, this is only one step towards an analysis of

[5] I assume that the reader is familiar with the standard formulation of common knowledge: An event $E$ is commonly known provided everyone knows $E$, everyone knows that everyone knows $E$, everyone knows that everyone knows $E$, and so on *ad infinitum*.

[6] Note that the framework used in [20] differs in small but important ways from the epistemic-probability models introduced in this paper. These technical details are not important for the main point I am making here, and, indeed, this argument can be made more formal using epistemic probability models.

[7] This follows from the well-known fact that a strategy is weakly dominated iff it does not maximize expected utility with respect to any probability that assigns non-zero probability to all of the opponent's choices.

game situations from the players' perspective. What is not represented is the process of deliberation that leads the players to their beliefs and choices:

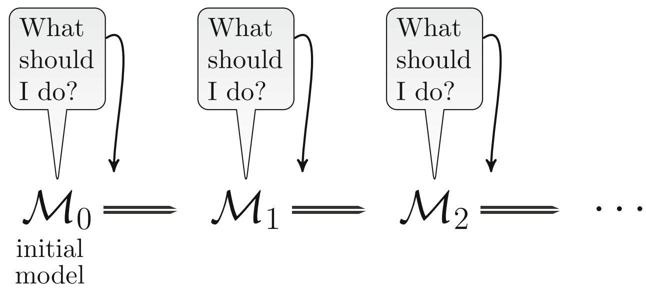
> Discussions of substantive rationality take place in an essentially *static* framework. Thus, equilibrium is discussed without explicit reference to any dynamic process by means of which the equilibrium is achieved. Similarly, prior beliefs are taken as given, without reference to how a rational agent acquires these beliefs. Indeed, all questions of the procedure by means of which rational behavior is achieved are swept aside by a methodology that treats this procedure as completed and reifies the supposed limiting entities by categorizing them axiomatically. [13, pg. 180]

Exactly how the players (should) incorporate the fact that they are interacting with other (actively reasoning) agents into their own decision-making process is the subject of much debate. A number of frameworks set out to model this "rational deliberation" process in game situations. Key examples include:

1. Harsanyi's *tracing procedure* [30]. The goal of the tracing procedure is to identify a unique Nash equilibrium in a strategic game. The idea is to analyze a continuum of games starting with the original game, in which the payoffs are replaced by expected payoffs. In fact, Harsanyi himself thought of this procedure as "being a mathematical formalization of the process by which rational players coordinate their choices of strategies."
2. Brian Skyrms' model of "dynamic deliberation": Players deliberate by calculating their expected utility and then use this new information to recalculate their probabilities about the states of the world and their expected utilities [46].
3. Binmore's analysis of "eductive reasoning": Players are represented as Turing machines that can *compute* their rational choice(s) [13].
4. Cubitt and Sugden's recent contribution developing a "reasoning-based expected utility" procedure for solving games (building on David Lewis's "common modes of reasoning") [20, 21].
5. van Benthem *et col.*'s analysis of solution concepts as fixed points of iterated "(virtual) rationality announcements" [2, 10, 51, 55, 56].

Although the details of these frameworks are quite different, they share a common thought: In contrast to conventional game theory, solution concepts are no longer the basic object of study. Instead, the "rational solutions" of a game are arrived at through a process of "rational deliberation".

Drawing on the extensive work on *dynamic* logics of knowledge and belief (see [38, 53] for a discussion and references to the relevant literature), here is a general way to think about this deliberational process: First, use some type of formal model to describe the players' beliefs about their own choices and their opponents' choices. These models are meant to be a "snapshot" of the players' beliefs during the deliberational process. At each stage of the deliberation, the players determine which of the options are "optimal" and which options the players ought to avoid (typically, the players are guided by some decision-theoretic choice rule). Using the information about the players' own choices and what they expect their opponents to do, the current model of the players' beliefs is *transformed*. The picture to keep in mind is:




Deliberation concludes when the players reach a fixed point in the above process. The frameworks mentioned above differ in how they represent the players' beliefs during the deliberation process:

1. Cubitt and Sugden categorize the players' actions as a pair of sets $(A_i^+, A_i^-)$ where elements of $A_i^+$ are the "admissible" actions, elements of $A_i^-$ are deemed "irrational" and actions such that $a \notin A_i^+ \cup A_i^-$ (if any) are not (yet) categorized.
2. Skyrms represents the players' beliefs as a state of indecision: For each player $i$, $i$'s state of indecision is a probability measure on $A_i$. This is the mixed strategy that the players would adopt if deliberation ended at this stage.
3. van Benthem and colleagues use the game models introduced in the previous section to describe what the players know and believe about their own choices, their opponents' choices and what they believe about their opponents' beliefs.

The second aspect of the model of the players' deliberation is the operations that are used to transform the model of the players' beliefs. Different types of transformations represent how confident the players are in their assessment of which of the available choices are "rational".

In this short section, I was able to only scratch the surface of the literature on deliberation in games. The goal is not to develop a formal account of the players' *practical reasoning* in game situations. Rather, it is to describe deliberation in terms of a sequence of belief changes about what the players are doing or what their opponents may be thinking. The central question is: What are the update mechanisms that match different game-theoretic analyses? The interested reader is invited to consult [46] and [39] for a more in-depth discussion.

### 3.3 Framing the Game Situation

I conclude this section with a very brief discussion of two topics that deserve more attention from researchers working at the intersection of logic and game theory. The logical analysis of games has focused mainly on many of the basic assumptions underlying any game-theoretic analysis. Still, two key assumptions have not been extensively analyzed:

1. The way that the players perceive their own decision problem matches the one specified by the game theorist in the model of the game situation.
2. The outcomes of the game are described only in terms of the actions chosen by the players (i.e., the strategy profiles).

On the one hand, for many game-theoretic analyses, these assumptions can be viewed as harmless idealizations. On the other hand, if we take seriously the idea that we want to represent the game from the players' perspective, then these assumptions are much more significant. Although I can point to a few papers that address these issues, much more analysis is needed. Already, it is clear that logical methods will be important tools for this endeavor.

Bacharach [7] developed a model that weakens the first assumption . In his *variable frame theory*, the players describe their decision problem differently. More formally, a *frame* is a set of possible descriptions of subsets of actions in a strategic game. A *framed game* consists of (1) a normal-form game in which each player chooses a description in her own frame; and (2) a probability assigned to each description specifying how likely it is that the player uses the particular description as she reasons about what she is going to do in the game.

It is also natural to consider models that relax the second assumption. This is useful to represent a player that prefers outcomes of the game in which her opponents have certain beliefs about the action that she is choosing. In this way, we can represent players that feel, for example, surprise, regret or trust in a game situation. Adam Brojndahl, Joe Halpern and Rafael Pass [28] have some initial work in this direction, in which the outcomes of a game are described using a modal language with belief operators. They argue that their model is a natural generalization of earlier work on *psychological games* [23] while being much easier to use.

There is much more to be said about the topics raised in this section. I leave that for another occasion and simply note that logical methods can be an invaluable tool for researchers who want to take into account the above observations in their analyses of game situations.

## 4 Concluding Remarks

Any discussion of *rationality*, *agency* and *social interaction* naturally touches on a variety of disciplines and often evokes strong opinions. This is especially true when we take into account experiments showing how "humans *really* behave" in game situations. However, the situation is not any simpler when we engage in "pure" conceptual modeling. Here, the use of formal models often generates a number of interesting technical issues. Furthermore (and often more interesting), there are questions that are not mathematical in nature but that involve the meaning of the fundamental concepts employed in the formal analyses. In other words, the debate concerns the very nature of *rationality* [29]; what constitutes an *agent* [18, 53]; underlying assumptions about *rational decision making* [14, 17, 36, 46, 47]; and the role that formal models play in philosophy and the social sciences [3, 5, 13, 34, 52].

Of course, there is no single approach that can address *all* of the complex phenomena that arise when rational and not-so-rational agents interact with one another and the environment. Thus, it is important to understand both the scope of a particular analysis and how different analyses from within and across the disciplines mentioned above can fit together.

The discussion in this paper has focused on analyzing *arbitrary* strategic (and extensive) games.[8] Focusing on *specific* game situations raises new and interesting questions. Consider one example, the so-called "hi-low" game:

| | | Bob | |
|---|---|---|---|
| | | $l$ | $r$ |
| Ann | $u$ | 3,3 | 0,0 |
| | $d$ | 0,0 | 1,1 |

Since there are no strictly or weakly dominated strategies, it is not hard to find epistemic (-plausibility/-probability) models in which there is common knowledge of rationality, and any of the outcomes of the game are realized. However, common sense and various empirical observations suggest that *rational* players somehow manage to reason to the *focal point* outcome $(u, l)$:

> There are these two broad empirical facts about Hi-Lo games, people almost always choose $A$ [Hi] and people with common knowledge of each other's rationality think it is obviously rational to choose $A$ [Hi]. [8, pg.42]

What are the underlying dynamics driving players' choices in hi-low games? See [12] for an interesting discussion that relates to many of the topics raised in this paper. Perhaps it involves some form of "group-think" or *team reasoning* [48], or perhaps it involves reasoning about which option is *salient* [11]. Each comes with interesting information dynamics that can be uncovered with the logical systems presented in this paper. Finding, a general logic of rational reasoning in such games may be too much to ask:

> The basic intellectual premise, or working hypothesis, for rational players in this game seems to be the premise that some rule must be used if success is to exceed coincidence, and that the best rule to be found, whatever its rationalization, is consequently a rational rule. [45, pg. 283, footnote 16]

[8] Well, I do restrict attention to games with a finite set of players and and finite sets of actions for each player.

Nonetheless, focusing on specific instances of strategic interaction raises many new and interesting issues for the logic analysis of games.

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