Logical and Probabilistic Models of Belief Change

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Plan

- **Day 1** Introduction to belief revision, AGM, possible worlds models, Bayesian models (time permitted)
- **Day 2** Bayesian models (continued), Justifying Bayesian models (Dutch books, Accuracy-based arguments), Updating probabilities
- **Day 3** The value of learning, Lottery Paradox, Preface Paradox, Review Paradox, Iterated belief revision, Context shifts, Becoming aware
- **Day 4** The value of learning, Lottery Paradox, Preface Paradox, Review Paradox, Iterated belief revision, Context shifts, Becoming aware (continued)
- **Day 5** Interactive epistemology (Agreement Theorems, Belief Revision in Games)

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Plan for today

- Quick recap (AGM, possible worlds models)
- Bayesian models
-
- Updating probabilities

$$K_0 \qquad \Longrightarrow \qquad K_t$$

$$K_0 \quad \overset{\substack{\text{Learn that } \varphi \\ \text{Suppose that } \varphi}}{\Longrightarrow} \quad K_t = K_0 * \varphi$$

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$$p_0 \Longrightarrow p_t$$

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$$p_0 \overset{\text{Learning experience}}{\Longrightarrow} p_t = ???$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
K_0 \arrow[r, Rightarrow, "\substack{\text{Learn that } \varphi \\ \text{Suppose that } \varphi}"] \arrow[d, leftrightarrow] & K_t = K_0 * \varphi \arrow[d, leftrightarrow] \\
p_0 \arrow[r, Rightarrow, "\text{Learning experience}"] & p_t = ???
\end{tikzcd}$$

Conditioning

$$p_0 \quad \stackrel{\text{Learn that } E}{\Longrightarrow} \quad p_t(\cdot) = p_0(\cdot \mid E)$$

Conditional Probability

The probability of E *given* F, dented $p(E|F)$, is defined to be

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

provided $P(F) > 0$.

Setting $p_t(\cdot) = p_0(\cdot \mid E)$ is demonstrably the correct thing to do just in case, for all propositions $H \in \Sigma$, both:

1. Certainty: $p_t(E) = 1$
2. Rigidity: $p_t(H \mid E) = p_0(H \mid E)$

People are often not aware of all that they have learnt or they fail to adequately represent it, and it is only the failure of the Rigidity condition that alerts us to this.

Three Prisoner's Problem

Three prisoners A, B and C have been tried for murder and their verdicts will told to them tomorrow morning. They know only that one of them will be declared guilty and will be executed while the others will be set free. The identity of the condemned prisoner is revealed to the very reliable prison guard, but not to the prisoners themselves. Prisoner A asks the guard "Please give this letter to one of my friends — to the one who is to be released. We both know that at least one of them will be released".

Three Prisoner's Problem

An hour later, A asks the guard "Can you tell me which of my friends you gave the letter to? It should give me no clue regarding my own status because, regardless of my fate, each of my friends had an equal chance of receiving my letter." The guard told him that B received his letter.

Prisoner A then concluded that the probability that he will be released is $1/2$ (since the only people without a verdict are A and C).

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Explain what is wrong with A's reasoning.

A's reasoning

Consider the following events:

G_A: "Prisoner A will be declared guilty" (we have $p(G_A) = 1/3$)

I_B: "Prisoner B will be declared innocent" (we have $p(I_B) = 2/3$)

A's reasoning

Consider the following events:

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We have $p(I_B \mid G_A) = 1$: "If A is declared guilty then B will be declared innocent."

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Bayes Theorem:

$$p(G_A \mid I_B) = p(I_B \mid G_A)\frac{p(G_A)}{p(I_B)} =$$

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Bayes Theorem:

$$p(G_A \mid I_B) = p(I_B \mid G_A)\frac{p(G_A)}{p(I_B)} = 1 \cdot \frac{1/3}{2/3} = 1/2$$

A's reasoning, corrected

But, A did not receive the information that B will be declared innocent, but rather that "the guard said that B will be declared innocent." So, A should have conditioned on the event:

I'_B: "The guard said that B will be declared innocent"

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Given that $p(I'_B \mid G_A)$ is $1/2$ (given that A is guilty, there is a 50-50 chance that the guard could have given the letter to B or C).

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I'_B: "The guard said that B will be declared innocent"

Given that $p(I'_B \mid G_A)$ is $1/2$ (given that A is guilty, there is a 50-50 chance that the guard could have given the letter to B or C). This gives us the following correct calculation:

$$p(G_A \mid I'_B) = p(I'_B \mid G_A)\frac{p(G_A)}{p(I'_B)} = 1/2 \cdot \frac{1/3}{1/2} = 1/3$$

Setting $p_t(\cdot) = p_0(\cdot \mid E)$ is demonstrably the correct thing to do just in case, for all propositions $H \in \Sigma$, both:

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Observation by candlelight

An agent inspects a piece of cloth by candlelight, and gets the impression that it is green (G), although he concedes that it might be blue (B) or even (but very improbably) violet (V).

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$$p_0(G) = p_0(B) = 0.3,\ \ p_0(V) = 0.4$$

$$\Downarrow$$

$$p_t(G) = 0.7,\ \ p_t(B) = 0.25,\ \ p_t(V) = .05$$

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Is there a proposition E such that $p_t(\cdot) = p_0(\cdot \mid E)$?

Jeffrey Conditionalization

When an observation bears directly on the probabilities over a partition $\{E_i\}$, changing them from $p(E_i)$ to $q(E_i)$, the new probability for any proposition H should be

$$q(H) = \sum_i p(H \mid E_i)q(E_i)$$

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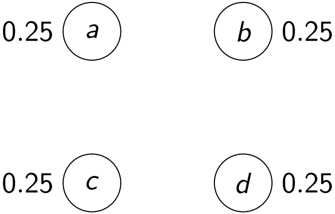
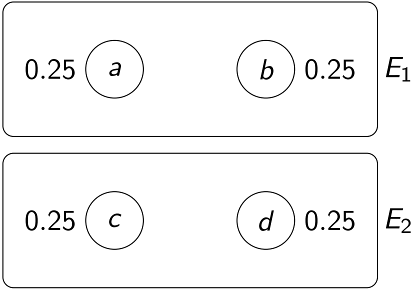
Fact: If q is obtained from p by Jeffrey Conditioning on the partition $\{E, \overline{E}\}$ with $q(E) = 1$, then $q(\cdot) = p(\cdot \mid E)$.

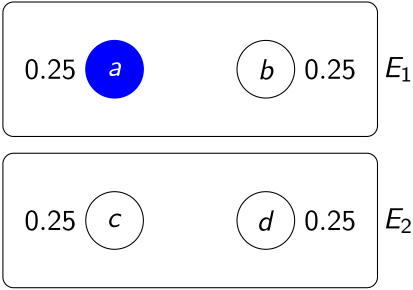

0.25 a
b 0.25
0.25 c
d 0.25



The probability that the guilty party is left-handed is 0.8

$E_1 = \{a, b\}$, $E_2 = \{c, d\}$

$$p(E_1) = 0.8 \qquad p(E_2) = 0.2$$



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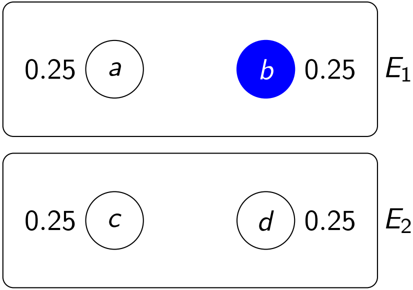
$$p(a) = p_0(\{a\} \mid E_1) * p(E_1) + p_0(\{a\} \mid E_2) * p(E_2) = 0.5 * 0.8 + 0 = 0.4$$



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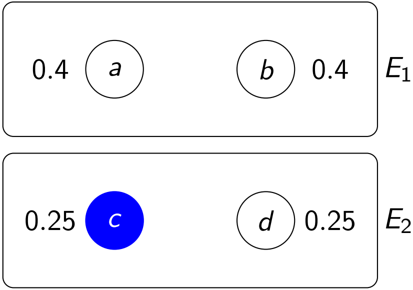

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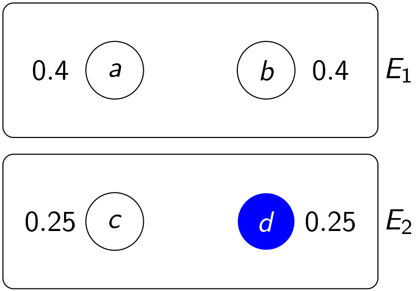
$p(c) = p_0(\{c\} \mid E_1) * p(E_1) + p_0(\{c\} \mid E_2) * p(E_2) = 0 + 0.5 * 0.2 = 0.1$



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$$E_1 = \{a, b\},\ E_2 = \{c, d\}$$

$$p(E_1) = 0.8 \qquad p(E_2) = 0.2$$

$$p(d) = p_0(\{d\} \mid E_1) * p(E_1) + p_0(\{d\} \mid E_2) * p(E_2) = 0 + 0.5 * 0.2 = 0.1$$

P. Diaconis and S. Zabell. *Updating Subjective Probability*. Journal of the American Statistical Association, Vol. 77, No. 380., pp. 822-830 (1982).

Fact. Jeffrey conditioning is not commutative.

Commutativity on Experiences Any rule for updating degrees of belief on experiences should be such that the result of updating credences on one experience and then another should be the same as the result of updating on the same two experiences in reverse order.

Holism For any experience and any proposition, there is a "defeater" proposition, such that your degree of belief in the first proposition, upon having the experience, should depend on your degree of belief in the defeater proposition.

J. Weisberg. *Commutativity or Holism? A Dilemma for Conditionalizers*. British Journal of the Philosophy of Science, 60(4), pp. 793-812, 2009.

M. Lange. *Is Jeffrey Conditionalization Defective in Virtue of Being NonCommutative? Remarks on the Sameness of Sensory Experience*. Synthese 123: 393-403, 2000.

C. Wagner. *Probability kinematics and commutativity*. Philosophy of Science 69, 266-278, 2002.

Conditioning

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Conditioning

- If q comes from p by conditioning on E, then $q(E) = 1$
- If $p(E_1) = 1$ then $p(E_1 \mid E_2) = 1$ for any E_2 consistent with E_1
- If $p(E_1) = 1$ then $p(A \mid E_2)$ is undefined whenever E_2 is inconsistent with E_1, since $p(E_2) = 0$

Updating probabilities

Orthodox Bayesian Policy

- accept as admissible input only propositions;
- as response to such an input the only admissible change is conditioning the prior on the proposition in question.

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Departing from a (orthodox) Bayesian policy:

1. accept as admissible a wider variety of inputs (e.g. expected values);
2. an admissible response to such an input can be a change in the prior that is not the result of conditioning;
3. an admissible response to such an input may be non-unique, that is, the posterior may not be uniquely determined by the prior + input.

p_0

$$\downarrow$$

$$\boxed{\begin{array}{c} E \\ p_0(E) > 0 \end{array}}$$

$$\downarrow$$

$$p(\cdot) = p_0(\cdot \mid E)$$

p_0

$$(E_1 : q_1, \ldots, E_k : q_k)$$

$\{E_i\}$ is a partition, $\sum_i q_i = 1$

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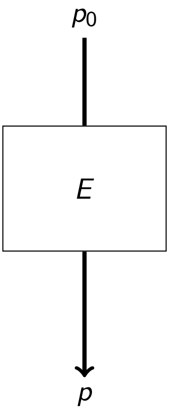
p_0
E
p

$$p_0$$

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$$\downarrow$$

$$p$$

Conditional Probabilities

What is the probability that a Republican will be the next president?

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A. Hájek. *What conditional probability could not be*. Synthese, 137, pp. 273 - 323, 2003.

"When conditional probability is defined by the ratio rule, it has limited expressive capacity. We would like to allow propositions that have been accorded zero probability to serve as conditions for the probability of other propositions. This is impossible when $p(x \mid a)$ is put as $p(a \wedge x)/p(a)$, for it is undefined when $p(a) = 0$."

D. Makinson. *Conditional Probability in the Light of Qualitative Belief Change*. Journal of Philosophical Logic.

Problem: The condition a is consistent but of zero probability (the **critical zone**).

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Solutions:

- Carnap: If $p(x) = 0$, then x is inconsistent.

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  Define $p_a(\cdot)$ as $p(\cdot \mid a)$. By the left projection, $p_a(x) = p(x \mid a)$, then $p_a(\neg a) = p(\neg a \mid a) = 0$ since $p(a)$. Thus, $p_a(\neg a) = 0$ even though $\neg a$ is inconsistent.

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- $p(x \mid a) = 1$ for every value x when $p(a) = 0$.

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- $p(x \mid a)$ is the limit of the values of $p(x \mid a')$ for suitable infinite sequence of non-critical approximations a' to a. *Only defined on special domains.*

CPS (Popper Space)

A **conditional probability space** (CPS) over $(W, \mathfrak{A})$ is a tuple $(W, \mathfrak{A}, \mathfrak{B}, \mu)$ such that $\mathfrak{A}$ is an algebra over W, $\mathfrak{B}$ is a set of subsets of W (not necessarily an algebra) that does not contain $\emptyset$ and $\mu : \mathfrak{A} \times \mathfrak{B} \to [0, 1]$ satisfying the following conditions:

1. $\mu(U \mid U) = 1$ if $U \in \mathfrak{B}$
2. $\mu(E_1 \cup E_1 \mid U) = \mu(E_1 \mid U) + \mu(E_2 \mid U)$ if $E_1 \cap E_2 = \emptyset$, $U \in \mathfrak{B}$ and $E_1, E_2 \in \mathfrak{A}$
3. $\mu(E \mid U) = \mu(E \mid X) * \mu(X \mid U)$ if $E \subseteq X \subseteq U$, $U, X \in \mathfrak{B}$ and $E \in \mathfrak{A}$.

$$p : \mathcal{L} \times \mathcal{L} \to [0, 1]$$

van Fraassen Axioms:

- $vF1$ $p(x, a) = p(x, a')$ whenever $a \equiv a'$
- $vF2$ p_a is a one-place Kolmogorov probability function with $p_a(a) = 1$
- $vF3$ $p(x \wedge y, a) = p(x, a) * p(y, a \wedge x)$ for all a, x, y

"for 'most' values of the right argument of the two-place function, the left projections should be proper one-place Kolmogorov functions, while in the remaining cases it should be the unit function."

(Positive): when $p(a, \top) > 0$ then p_a is a proper Kolmogorov function.

(Carnap) When a is consistent then $p(a, \top) > 0$.

(Unit) When a is consistent but $p(a, \top) = 0$, then p_a is the unit function.

(HL) When a is consistent but $p(a, \top) = 0$, then p_a is a proper Kolmogorov probability function.

What does 'most propositions' mean?

- The van Fraassen system: an unspecified subset (possibly empty) of the consistent propositions,
- The Popper system: all propositions that are above the critical zone or in an unspecified subset (possibly empty) of it,
- The Unit system: for all propositions above the critical zone but no others,
- The Hosiasson-Lindenbaum system: for all propositions above or in the critical zone,
- Carnaps system: we can say any of the last three, since the critical zone is declared empty.

LPS (Lexicographic Probability Space)

A **lexicographic probability space** (LPS) (of length α) is a tuple $(W, \Sigma, \vec{\mu})$ where W is a set of possible worlds, Σ is an algebra over W and $\vec{\mu}$ is a sequence of (finitely/countable additive) probability measures on (W, Σ) indexed by ordinals $< \alpha$.

Fix an *LPS* $\vec{\mu} = (\mu_0, \ldots, \mu_n)$

- E is certain: $\mu_0(E) = 1$

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- E is certain: $\mu_0(E) = 1$
- E is absolutely certain: $\mu_i(E) = 1$ for all $i = 1, \ldots, n$
- E is *assumed*: there exists k such that $\mu_i(E) = 1$ for all $i \leq k$ and $\mu_i(E) = 0$ for all $k < i < n$.

NPS (non-standard probability measures)

$\mathbb{R}^*$ is a *non-Archimedean* field that includes the real numbers as a subfield but also has *infinitesimals*.

For all $b \in \mathbb{R}^*$ such that $-r < b < r$ for some $r \in \mathbb{R}$, there is a unique closest real number a such that $|a - b|$ is an infinitesimal. Let $st(b)$ denote the closest standard real to b.

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A **nonstandard probability space** (NPS) is a tuple (W, Σ, μ) where W is a set of possible worlds, Σ is an algebra over W and μ assigns to elements of Σ, nonnegative elements of $\mathbb{R}^*$ such that $\mu(W) = 1$, $\mu(E \cup F) = \mu(E) + \mu(F)$ if E and F are disjoint.

J. Halpern. *Lexicographic probability, conditional probability, and nonstandard probability*. Games and Economic Behavior, 68:1, pgs. 155 - 179, 2010.

$$p_0$$

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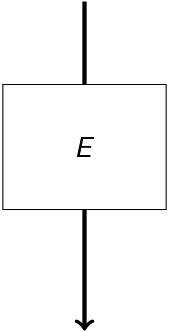
$$p_0(\cdot, \top)$$



$$p(\cdot) = p_0(\cdot, E)$$

p_0

$\mathcal{C}$: set of constraints

p satisfies $\mathcal{C}$

MAXENT

Let us start with the simplest case, where our outcome space, X, contains only a finite number of points, $x_1, x_2, \ldots, x_n$. Then the **entropy** of a probability, p, on this space is:

$$-\sum_i p(x_i) \log p(x_i)$$

and the **information** is the negative of the entropy.

The minimum information or maximum entropy probability is the one which makes the states equiprobable: $p(x_i) = \frac{1}{n}$.

Consider three die x_1, x_2, x_3 and a random variable f such that $f(x_i) = i$.

$$\mathbb{E}[f] = p(x_1)f(x_1) + p(x_2)f(x_2) + p(x_3)f(x_3)$$

What probabilities maximize entropy under the constraint that $\mathbb{E}[f]$ have different values?

MAXENT

$\mathbb{E}[f]$	$p(x_1)$	$p(x_2)$	$p(x_3)$
1	1	0	0
0.1	0.907833	0.084333	0.007834
0.2	0.826297	0.147407	0.026297
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.8	0.438371	0.323257	0.238271
0.9	0.384586	0.330829	0.284586
2.0	0.333333	0.333333	0.333333
2.1	0.284586	0.330829	0.384586
2.2	0.238372	0.323257	0.438370
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2.8	0.026297	0.147407	0.826296
2.9	0.007834	0.084332	0,907834
3.0	0	0	1

Kullback-Leibler

Suppose that we start with a prior probability, p_0, and move to a posterior p_1 which satisfies certain constraints. The Kullback-Leibler "distance" is:

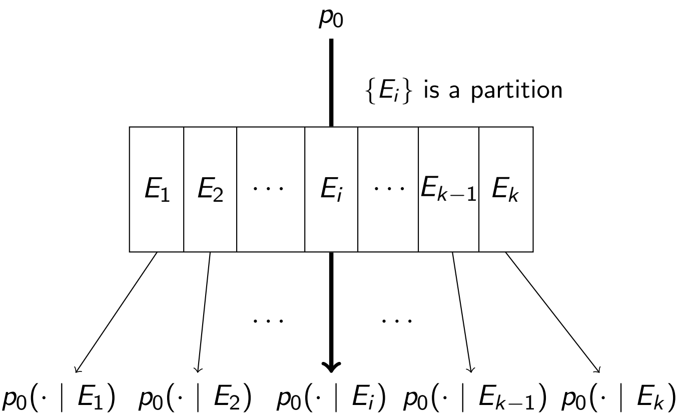
$$I(p_1, p_0) = \sum_i p_1(x_i) \log \frac{p_1(x_i)}{p_0(x_i)}$$

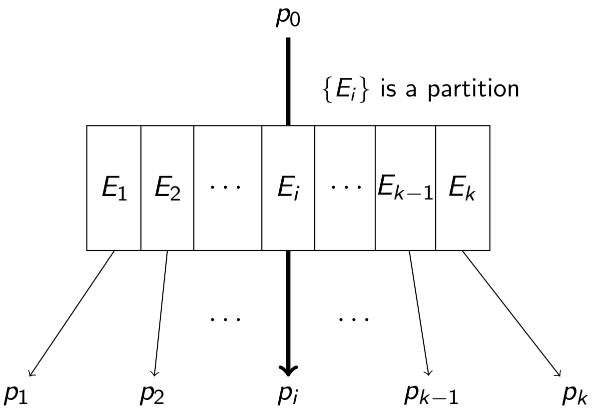
p_0

$\mathcal{C}$: set of constraints

p satisfies $\mathcal{C}$

p minimizes $I(p, p_0)$


p_0
$\{E_i\}$ is a partition
E_1
E_2
$\cdots$
E_i
$\cdots$
E_{k-1}
E_k
$\cdots$
$\cdots$
$p_0(\cdot \mid E_1)$
$p_0(\cdot \mid E_2)$
$p_0(\cdot \mid E_i)$
$p_0(\cdot \mid E_{k-1})$
$p_0(\cdot \mid E_k)$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$\{E_i\}$ is a partition
E_1
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E_i
$\cdots$
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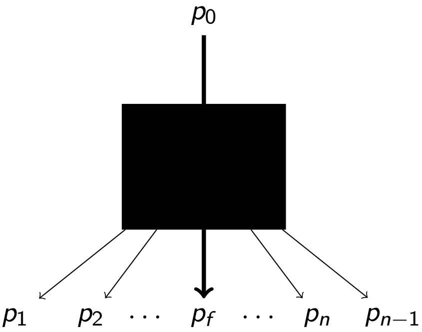
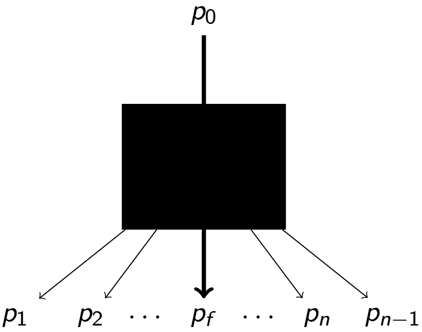
E_k
$\cdots$
$\cdots$
p_1
p_2
p_i
p_{k-1}
p_k

Suppose that you are in a learning situation even more amorphous than the kind which motivates Jeffrey's idea. There is no nontrivial partition that you expect with probability one to be sufficient for your belief change....Perhaps you are in a novel situation where you expect the unexpected observational input....You are going to just think about some subject matter and update as a result of your thoughts...I will consider the learning situation a kind of black box and attempt no analysis of its internal structure.

(Skyrms, pg. 96, 97)


p_0
p_1 p_2 $\cdots$ p_f $\cdots$ p_n p_{n-1}



(Martingale Property) $\quad p_0(A \mid p_f) = p_f(A)$

It was suggested by Skyrms (1990) that this principle provides a plausible way to distinguish learning situations from situations where one expects probabilities to change for other reasons, such as getting drunk, having a brain lesion or having a dangerously low blood sugar level.

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Huttegger develops an account in which the reflection principle is a necessary condition for a black-box probability update to count as a *genuine learning experience*.

Simon Huttegger. *Learning Experiences and the Value of Knowledge*. Philosophical Studies, 2013.

The Value of Knowledge

Why is it better to make a “more informed” decision?
Suppose that you can either choose know, or perform a costless experiment and make the decision later. What should you do?

I. J. Good. *On the principle of total evidence*. British Journal for the Philosophy of Science, 17, pgs. 319 - 321, 1967.

“Never decide today what you might postpone until tomorrow in order to learn something new”

Choose between n acts $A_1, \ldots, A_n$ (with states K_i) or perform a cost-free experiment with possible results $\{e_k\}$, then decide.

$$EU(A) = \sum_i p(K_i) U(A \ \& \ K_i)$$

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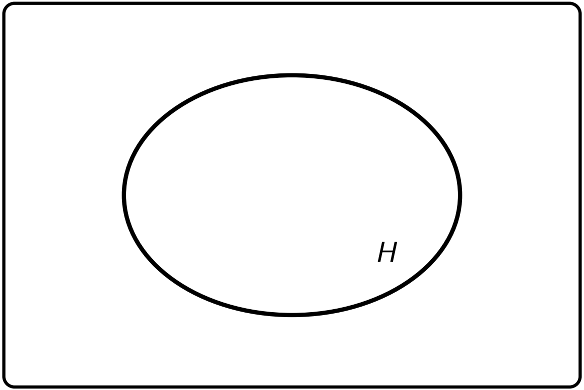
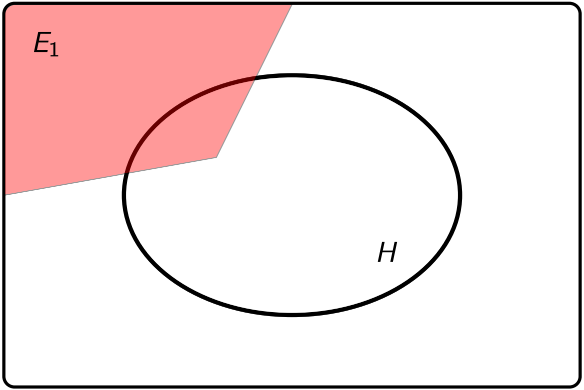
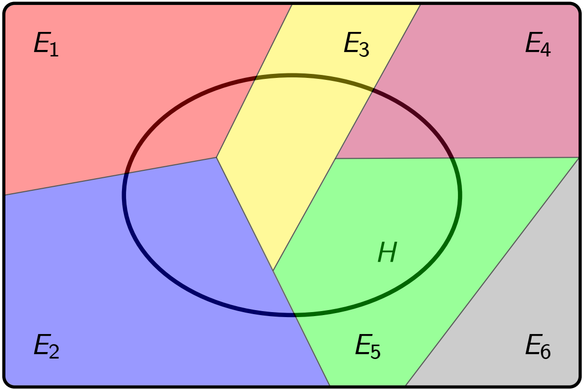
A basic result about probabilities.

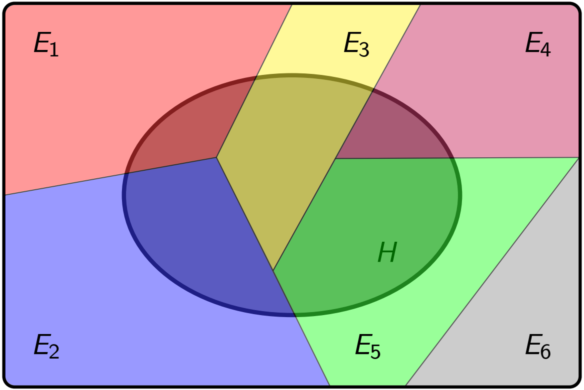
For any finite partition $\{E_i\}$ of the state space and any event H,

$$p(H) = \sum_i p(H \mid E_i)$$

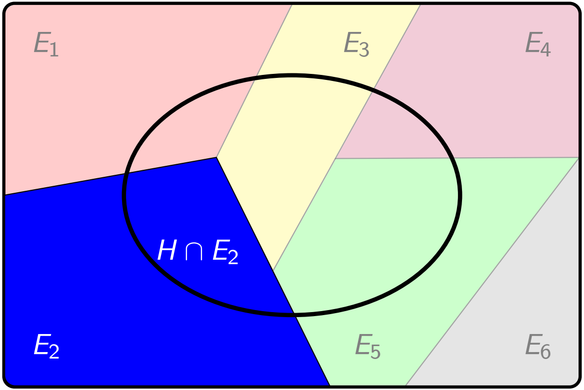

H


E_1
H

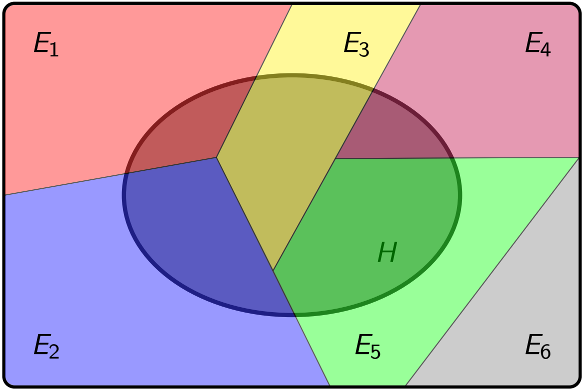

E_1
E_3
E_4
H
E_2
E_5
E_6



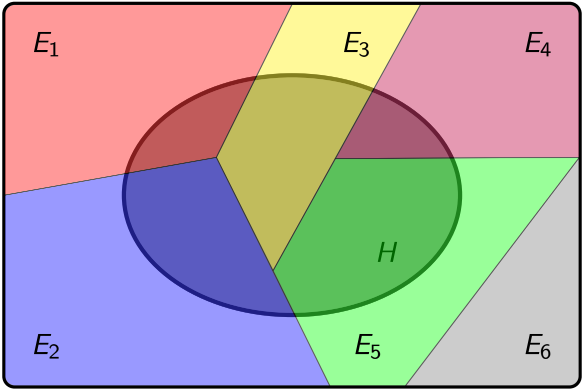
$$p(H) \quad = \quad p(H \cap E_1) + \cdots + p(H \cap E_6)$$



$$p(H) \quad = \quad p(H \cap E_1) + p(H \cap E_2) + \cdots + p(H \cap E_6)$$



$$
\begin{aligned}
p(H) &= p(H \cap E_1) + \cdots + p(H \cap E_6) \\
&= \frac{p(E_1)}{p(E_1)} p(H \cap E_1) + \cdots + \frac{p(E_6)}{p(E_6)} p(H \cap E_6)
\end{aligned}
$$



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&= \textstyle\sum_i p(E_i) P(H \mid E_i)
\end{aligned}
$$

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Bayes Theorem. $p(K_i|e_j) = p(e_j|K_i)\frac{p(K_i)}{p(e_j)}$

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The value of an informed decision conditional on e:

$$\max_j \sum_i p(K_i \mid e)U(A_j \,\&\, K_i)$$

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$$\begin{aligned} U(\text{Learn, Choose }) &= \textstyle\sum_k p(e_k)\max_j \sum_i p(K_i \mid e_k)U(A_j \;\&\; K_i) \\ &= \textstyle\sum_k p(e_k)\max_j \sum_i (\frac{p(e_k \mid K_i)p(K_i)}{p(e_k)})U(A_j \;\&\; K_i) \end{aligned}$$

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Compare $\max_j \sum_k \sum_i p(K_i) p(e_k \mid K_i) U(A_j \;\&\; K_i)$ and $\sum_k \max_j \sum_i p(e_k \mid K_i) p(K_i) U(A_j \;\&\; K_i)$

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$$\begin{array}{rcl} U(\text{Learn, Choose }) & = & \sum_k p(e_k) \max_j \sum_i p(K_i \mid e_k) U(A_j \,\&\, K_i) \\ & = & \sum_k p(e_k) \max_j \sum_i (\frac{p(e_k \mid K_i) p(K_i)}{p(e_k)}) U(A_j \,\&\, K_i) \\ & = & \sum_k \max_j \sum_i p(e_k \mid K_i) p(K_i) U(A_j \,\&\, K_i) \end{array}$$

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$\sum_k \max_j g(k,j)$ is greater than or equal to $\max_j \sum_k g(k,j)$, so the second is greater than or equal to the first.

(M) implies value of knowledge

Suppose that (M) holds.

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Suppose that (M) holds.
Then (assuming that each $p(p_f)$ is positive) your value for choosing an act now is

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$$= \max_j \sum_f \sum_i p_f(K_i)p(p_f)u(A_j \& K_i)$$

The value of choosing after the learning experience is:

$$\sum_f p(p_f) \max_j \sum_i p_f(K_i)u(A_j \& K_i)$$

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The value of choosing after the learning experience is:

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The latter term cannot be less than the former term on general mathematical grounds.

- The experiment is assumed to be essentially costless;

- You know that you are an expected utility maximizer and that you will be one after learning the true member of the partition.

- In the classical theorem you know that you will update by conditioning; in Skyrms' extension, you know that you will honor the martingale principle.

- By working within Savages decision theory, the states and acts are probabilistically independent (choosing an act does not give any information about the state).

- The states, acts and utilities are the same before and after the learning experience.

- Having the learning experience does not by itself alter your probabilities for states of the world (although the outcomes of the experience usually do); the learning experience and the states of the world are probabilistically independent.

...the martingale principle should not be applied to belief changes in epistemologically defective situations. In situations of memory loss, of being brainwashed or being under the influence of drugs, (M) should obviously not hold. If you believe that in an hour you will think you can fly because you're about to consume some funny looking pills, then you should not already now have that belief.

So, the martingale principle is claimed to apply if you learn something in the black-box, but not if you learn nothing or other things happen besides learning.

A genuine learning situation is partially characterized in the following way:

Postulate. If a belief change from p to $\{pf\}$ constitutes a genuine learning situation, then

$$\sum_f p(p_f) \max_j \sum_i p_f(K_i)u(A_j\&K_i) \geq \max \sum_i p(K_i)u(A_j\&K_i)$$

for all utility values $u(A_j\&K_i)$ with strict inequality unless the same act maximizes expected utility irrespective of which of the p_f occurs.

If a belief change leads you to foreseeably make worse choices than you could already make now in some decision situations, then it cannot be a pure learning experience. Perhaps you are bolder after having taken those funny looking pills, for example. From your current perspective, this might help you in some decision problems, but it will be harmful in others.