Neighborhood Semantics for Modal Logic
Lecture 5

Eric Pacuit, University of Maryland

August 9, 2024

Neighborhoods with nominals

$$p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid A\varphi$$

$p \in \mathsf{At}$ and $i \in \mathsf{Nom}$ (the set of nominals)

Neighborhood model with nominals $\langle W, N, V \rangle$, $V : \mathsf{At} \cup \mathsf{Nom} \to \wp(W)$, where for all $i \in Nom$, $|V(i)| = 1$.

- $\mathcal{M}, w \models i$ iff $V(w) = i$
- $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

$$\text{(BG)} \qquad \frac{\vdash E(i \wedge \Diamond j) \rightarrow E(j \wedge \varphi)}{\vdash E(i \wedge \Box \varphi)}$$

for $i \neq j$ and j not occurring in φ

$$\text{(BG)} \qquad \frac{\vdash E(i \wedge \Diamond j) \to E(j \wedge \varphi)}{\vdash E(i \wedge \Box \varphi)}$$

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A class of frames F **admits** a rule provided that every model that falsifies the consequent can be extended to a model that falsifies the premises.

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for $i \neq j$ and j not occurring in φ

A class of frames F **admits** a rule provided that every model that falsifies the consequent can be extended to a model that falsifies the premises.

Theorem. A neighborhood frame is augmented iff it *admits** the rule BG.

B. ten Cate and T. Litak (2007). *Topological Perspective on Hybrid Proof Rules*. Electronic Notes in Theoretical Computer Science, 174, pp. 79 - 94.

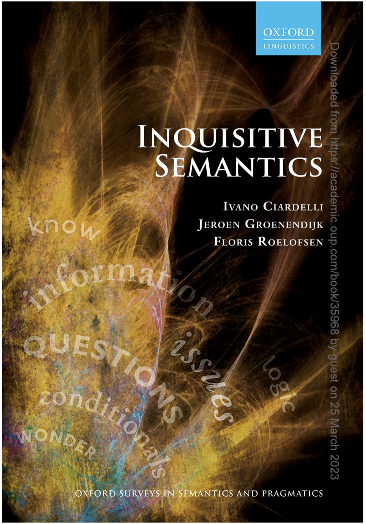
Course Plan

✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Neighborhood Structures, Motivating Weak Modal Logics/Neighborhood Semantics
(Monday, Tuesday)

✓ **Core Theory**: Non-Normal Modal Logic, Completeness, Decidability, Complexity, Incompleteness, Relationship with Other Semantics for Modal Logic, Model Theory
(Tuesday, Wednesday, Thursday)

3. **Extensions**: Inquisitive Logic on Neighborhood Models; First-Order Modal Logic, ~~Subset Spaces~~, ~~Common Knowledge/Belief~~, ~~Dynamics with Neighborhoods: Game Logic and Game Algebra~~, Dynamics on Neighborhoods **(Friday)**

I. Ciardelli. *Describing neighborhoods in inquisitive modal logic*. Proceedings of Advances in Modal Logic, 2022.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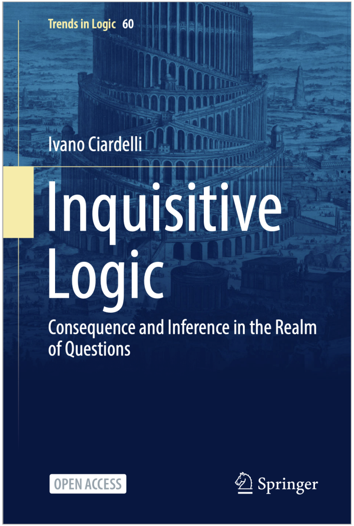
Trends in Logic 60
Ivano Ciardelli
Inquisitive
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Consequence and Inference in the Realm
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Support

Rather than taking semantics to specify in what circumstances a sentence is true, we may take it to specify what information it takes to *settle*, or *establish*, the sentence.

- Let W be a set of possible worlds. A *state* is an subset $s \subseteq W$.
- $s \models \varphi$ is read "s supports φ"

Entailment and the Conditional

Entailment: $\varphi \models \psi$ when for all models $\mathcal{M} = \langle W, V \rangle$ and $s \subseteq W$, $s \models \varphi$ implies $s \models \psi$

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Internalizing entailment: $s \models \varphi \to \psi$ when for all $t \subseteq s$, $t \models \varphi$ implies $t \models \psi$

Disjunction

Truth-support bridge: Let α be a statement and $\mathcal{M}$ a model. For any information state $s \subseteq W$ we should have:

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1. classic disjunction: $\forall w \in s,\ w \models \varphi$ or $w \models \psi$
2. inquisitive disjunction: either $\forall w \in s, w \models \varphi$ or $\forall w \in s, w \models \psi$

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   $\varphi \vee \psi$
2. inquisitive disjunction: either $\forall w \in s, w \models \varphi$ or $\forall w \in s, w \models \psi$
   $\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$

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   $\varphi \mathbin{\vee\!\!\!\vee} \psi$

For example:
$p \vee \neg p$ is a declarative statement that is a tautology
$p \mathbin{\vee\!\!\!\vee} \neg p$ is a question asking whether p (denoted $?p$)

$\mathcal{M} = \langle W, V \rangle$ where $W \neq \varnothing$ and $V : \mathsf{At} \to \wp(W)$

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \not\models \bot$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \to \psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}, w \models \psi$

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- $\mathcal{M}, s \models \varphi \to \psi$ iff for all $t \subseteq s$, $\mathcal{M}, t \models \varphi$ implies $\mathcal{M}, t \models \psi$

Neighborhood Semantics for Inquisitive Logic

IncCM

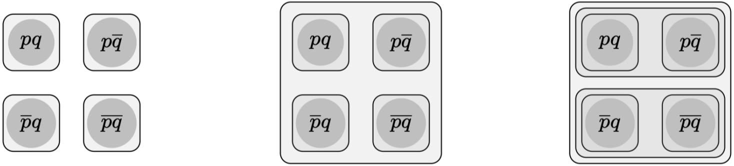
The language $\mathcal{L}$: $\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \vee\!\!\!\vee \varphi) \mid (\varphi \Rrightarrow \varphi)$

$\neg\varphi \ := \ \varphi \rightarrow \bot$, $\top \ := \ \neg\bot$, $\varphi \vee \psi \ := \ \neg(\neg\varphi \wedge \neg\psi)$, and $?\varphi \ := \ \varphi \vee\!\!\!\vee \neg\varphi$

Declaratives $\mathcal{L}_!$: $\alpha ::= p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \rightarrow \alpha) \mid (\varphi \Rrightarrow \varphi)$, where $\varphi \in \mathcal{L}$

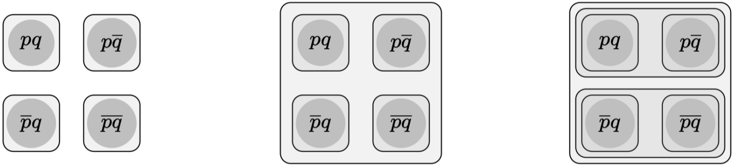
Models: $\langle W, \Sigma, V \rangle$ where

- $W \neq \varnothing$,
- $\Sigma : W \rightarrow \wp(\wp(W))$ such that for all $w \in W$, $\varnothing \notin \Sigma(w)$, and
- $V : \mathrm{At} \rightarrow \wp(W)$.



(a) $\Sigma(w_1)$ (b) $\Sigma(w_2)$ (c) $\Sigma(w_3)$

Fig. 1. Sets of neighborhoods associated with three worlds.



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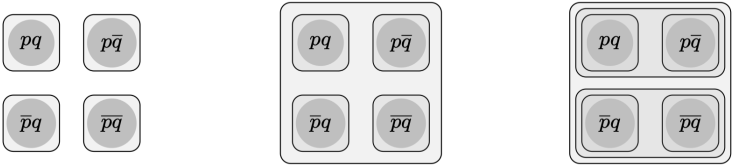
- every neighborhood of w_1 settles whether p (i.e., the truth value of p is constant within each neighborhood) while this is not the case for w_2 and w_3



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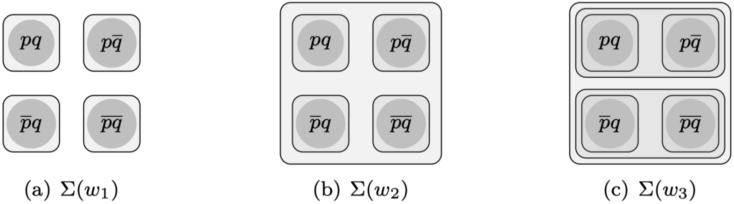
- every neighborhood of w_1 settles whether p (i.e., the truth value of p is constant within each neighborhood) while this is not the case for w_2 and w_3
- every neighborhood of w_2 that settles whether p also settles whether q, whereas this is not the case for w_3.



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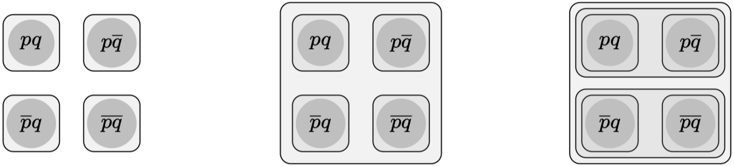
Suppose the above model represents what an agent can *force* to be true.



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Suppose the above model represents what an agent can *force* to be true.

$s \in \Sigma(w)$ means that the agent has an action that guarantees that s obtains.

From the perspective of logics of strategic ability, all situations represent an agent that is in effect a dictator who can force any of the outcomes, while other agents cannot prevent any outcome. Yet, there is a clear sense in which these situations are very different.

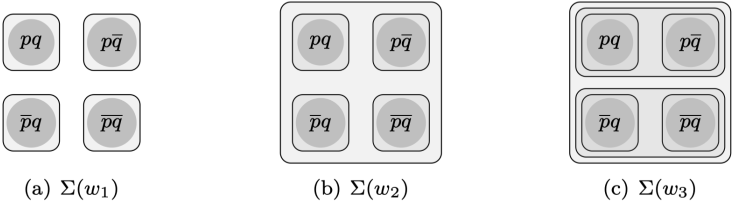
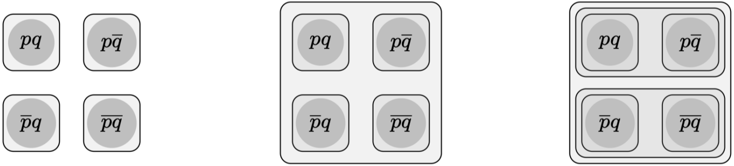


Fig. 1. Sets of neighborhoods associated with three worlds.

1. In w_1, not only is there an action the agent can perform that settles p, she *must* decide on p and q. $(\top \Rrightarrow ?p) \wedge (?p \Rrightarrow ?q)$



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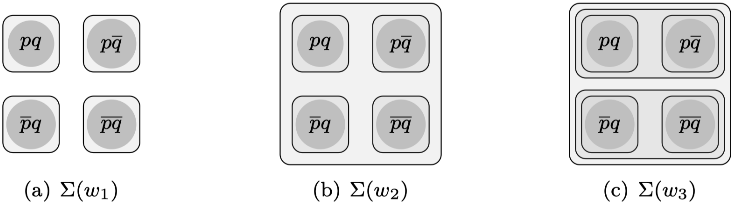

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2. In w_2, there is an action the agent can perform that settles p, but the agent must decide on q if she wants to decide on p. $?p \Rrightarrow ?q$
3. In w_3, there is an action the agent can perform that settles p, and the agent can delegate her decision on q

IncCM - Truth

- $\mathcal{M}, s \models p$ iff $s \subseteq V(p)$
- $\mathcal{M}, s \models \bot$ iff $s = \emptyset$
- $\mathcal{M}, s \models \varphi \wedge \psi$ iff $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, s \models \psi$
- $\mathcal{M}, s \models \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $\mathcal{M}, s \models \varphi$ or $\mathcal{M}, s \models \psi$
- $\mathcal{M}, s \models \varphi \rightarrow \psi$ iff for all $t \subseteq s$,
  $\mathcal{M}, t \models \varphi$ implies $\mathcal{M}, t \models \psi$
- $\mathcal{M}, s \models \varphi \Rrightarrow \psi$ iff for all $w \in s$, for all $t \in \Sigma(w)$,
  $\mathcal{M}, t \models \varphi$ implies $\mathcal{M}, t \models \psi$

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- $\mathcal{M}, s \models \varphi \Rrightarrow \psi$ iff for all $w \in s$, for all $t \in \Sigma(w)$,
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If $\alpha, \beta_1, \ldots, \beta_n$ are declaratives, then

$$w \models (\alpha \Rrightarrow (\mathbb{V}_{i \leq n} \beta_i)) \iff \forall s \in \Sigma(w) : \text{if } s \subseteq [\![\alpha]\!], \text{ then } s \subseteq [\![\beta_i]\!] \text{ for some } i$$

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Then $\neg(\alpha \Rrightarrow (\mathbb{V}_{i \leq n} \beta_i))$ expresses the existence of a neighborhood s such that α is true everywhere in s and for each $i \leq n$, β_i is true somewhere in s.

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This expresses the modality $\Box(\beta_1, \ldots, \beta_n; \alpha)$:

J. van Benthem, N. Bezhanishvili, S. Enqvist and J. Yu. *Instantial neighbourhood logic*. The Review of Symbolic Logic 10 (2017), pp. 116–144.

J. van Benthem, N. Bezhanishvili and S. Enqvist. *A new game equivalence, its logic and algebra*. Journal of Philosophical Logic 48 (2019), pp. 649–684.

J. van Benthem and EP. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica, 99(61), pp. 61 - 92, 2011.

Since $\varphi \Rrightarrow \psi$ is declarative, we have the following:

$$\mathcal{M}, w \models \varphi \Rrightarrow \psi \text{ iff for all } s \in \Sigma(w),\ \mathcal{M}, s \models \varphi \text{ implies } \mathcal{M}, s \models \psi$$

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- $p \Rrightarrow (q \to ?r)$: if we restrict to those neighborhoods that support p and then we restrict each of these neighborhoods to the q-worlds, all the resulting states settle whether r.

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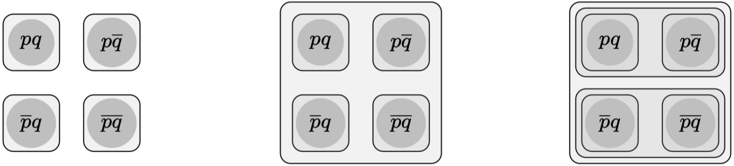
- $p \Rrightarrow (q \to ?r)$: if we restrict to those neighborhoods that support p and then we restrict each of these neighborhoods to the q-worlds, all the resulting states settle whether r.
- $?p \Rrightarrow (?q \to ?r)$: if we restrict to neighborhoods that settle whether p, and then look at the parts of such neighborhoods where the truth value of q is settled, each of these parts settles whether r.



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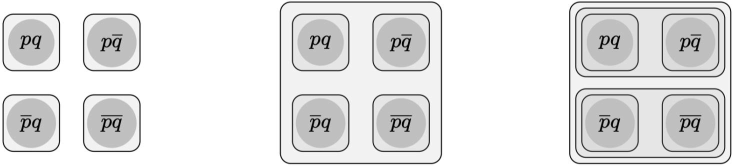
- $w_2 \models \neg(\top \Rrightarrow ?p) \wedge (?p \Rrightarrow ?q)$
- $w_3 \models \neg(\top \Rrightarrow ?p) \wedge \neg(?p \Rrightarrow ?q)$



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- $w_2 \models \neg(\top \Rrightarrow ?p) \land (?p \Rrightarrow ?q)$
- $w_3 \models \neg(\top \Rrightarrow ?p) \land \neg(?p \Rrightarrow ?q)$

Since the models in (a), (b), and (c) are monotonically bisimilar, these distinctions cannot be expressed in the basic modal language containing only the modality $\langle\ ]$. This means that monotonic bisimulation is not the appropriate notion of bisimulation for the language $\mathcal{L}$.

Monotonic Bisimulation

A bisimulation between $\mathcal{M} = \langle W, \Sigma, V \rangle$ and $\mathcal{M}' = \langle W', \Sigma', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $s \in \Sigma(w)$ then there is an $s' \subseteq W'$ such that

$s' \in \Sigma'(w')$ and $\forall w' \in s' \; \exists w \in s$ such that wZw'

Zag: If $s' \in \Sigma'(w')$ then there is an $s \subseteq W$ such that

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$$\forall w \in s \; \exists w' \in s' \text{ such that } wZw'$$

Bisimulation for $\mathcal{L}$

A bisimulation between $\mathcal{M} = \langle W, \Sigma, V \rangle$ and $\mathcal{M}' = \langle W', \Sigma', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

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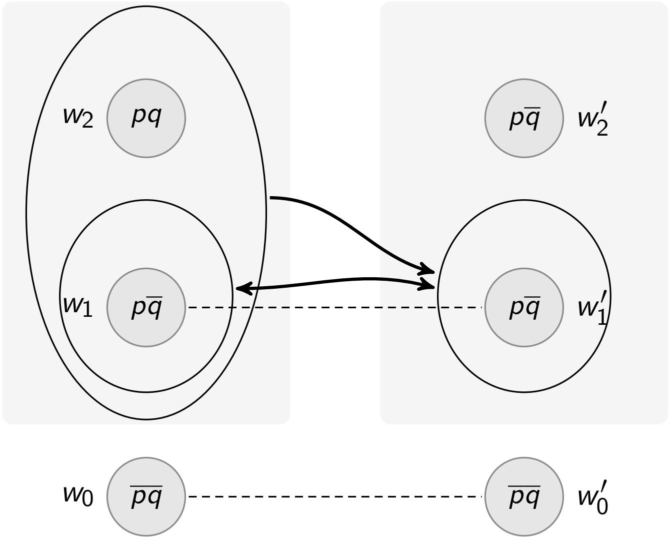
Zig: If $s \in \Sigma(w)$ then there is an $s' \in \Sigma'(w')$ such that

$\forall w' \in s' \; \exists w \in s$ such that wZw' and $\forall w \in s \; \exists w' \in s'$ such that wZw'

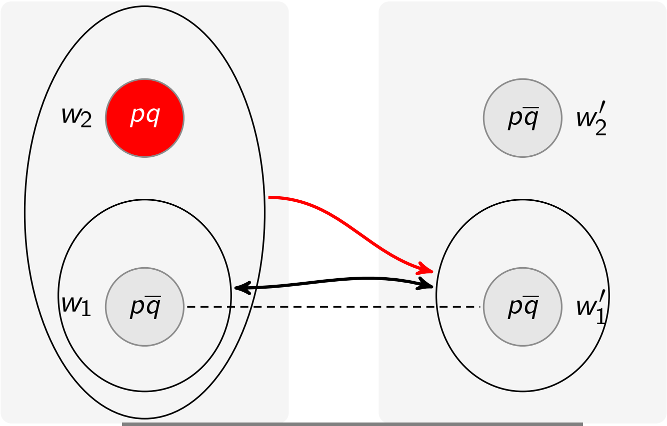
Zag: If $s' \in \Sigma'(w')$ then there is an $s \in \Sigma(w)$ such that

$\forall w \in s \; \exists w' \in s'$ such that wZw' and $\forall w' \in s' \; \exists w \in s$ such that wZw'

Monotonic Bisimulation



Not Bisimilar



$\forall w' \in \{w'_1\} \exists w \in \{w_1, w_2\} wZw'$

$w_0 \models \neg(p \Rightarrow \neg q)$ w_0 not $\forall w \in \{w_1, w_2\} \exists w' \in \{w'_1\} wZw'$ $w'_0 \models p \Rightarrow \neg q$

Proposition 3.3 For any worlds w, w', $w \underline{\leftrightarrow} w'$ implies $w \leftrightsquigarrow w'$; For any states s, s', $s \underline{\leftrightarrow} s'$ implies $s \leftrightsquigarrow s'$.

Theorem 3.4 If $\mathcal{M}$ and $\mathcal{M}'$ are image-finite, then for all worlds w, w', $w \leftrightsquigarrow w'$ implies $w \underline{\leftrightarrow} w'$; for all states s, s', $s \leftrightsquigarrow s'$ implies $s \underline{\leftrightarrow} s'$.

Axiomatization

- $\varphi \to (\psi \to \varphi)$
- $\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
- $\varphi \to (\psi \to (\varphi \wedge \psi))$
- $(\varphi \wedge \psi) \to \varphi, \quad (\varphi \wedge \psi) \to \psi$
- $\varphi \to (\varphi \vee \psi), \quad \psi \to (\varphi \vee \psi)$
- $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
- $\bot \to \varphi$

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- $(\varphi \wedge \psi) \to \varphi, \quad (\varphi \wedge \psi) \to \psi$
- $\varphi \to (\varphi \vee\!\!\!\vee \psi), \quad \psi \to (\varphi \vee\!\!\!\vee \psi)$
- $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee\!\!\!\vee \psi) \to \chi))$
- $\bot \to \varphi$
- $\neg\neg\alpha \to \alpha$
- $\alpha \to (\varphi \vee\!\!\!\vee \psi) \to ((\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi))$

Axiomatization

- $\varphi \to (\psi \to \varphi)$
- $\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
- $\varphi \to (\psi \to (\varphi \wedge \psi))$
- $(\varphi \wedge \psi) \to \varphi, \quad (\varphi \wedge \psi) \to \psi$
- $\varphi \to (\varphi \vee\!\!\!\vee \psi), \quad \psi \to (\varphi \vee\!\!\!\vee \psi)$
- $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee\!\!\!\vee \psi) \to \chi))$
- $\bot \to \varphi$
- $\neg\neg\alpha \to \alpha$
- $\alpha \to (\varphi \vee\!\!\!\vee \psi) \to ((\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi))$
- $((\varphi \Rrightarrow \psi) \wedge (\psi \Rrightarrow \chi)) \to (\varphi \Rrightarrow \chi)$
- $((\varphi \Rrightarrow \psi) \wedge (\varphi \Rrightarrow \chi)) \to (\varphi \Rrightarrow (\psi \wedge \chi))$
- $((\varphi \Rrightarrow \chi) \wedge (\varphi \Rrightarrow \chi)) \to ((\varphi \vee\!\!\!\vee \psi) \Rrightarrow \chi)$

Theorem The previous axiomatization is sound and complete with respect to neighborhood structures.

Concluding Remarks

- One could define a standard translation (into a two-sorted first-order logic) and aim for a van Benthem-style characterization of InqCM as the bisimulation-invariant fragment of first-order logic.

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Concluding Remarks

- One could define a standard translation (into a two-sorted first-order logic) and aim for a van Benthem-style characterization of InqCM as the bisimulation-invariant fragment of first-order logic.

- It would be interesting to develop modal correspondence theory for InqCM relating validity of InqCM-schemata over a neighborhood frame $\langle W, \Sigma \rangle$ to corresponding properties of the set of neighborhoods at each state.

- One can allow empty neighborhoods without substantive changes to the results of the paper.

Neighborhood Models for First-Order Modal Logic

H. Arlo Costa and E. Pacuit (2006). *First-Order Classical Modal Logic*. Studia Logica, 84, pp. 171 - 210.

Also, see:

G. Boella, D. Gabbay, V. Genovese, and L. van der Torre (2010). *Higher-Order Coalition Logic*. SeriesFrontiers in Artificial Intelligence and Applications, Volume 215: ECAI.

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

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$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \wedge A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)

State-of-the-art

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D.Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Draft available (2008).
http://lpcs.math.msu.su/~shehtman/QNCLfinal.pdf

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function I, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution σ' is said to be an x-**variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \dots, x_n)$ iff $\langle \sigma(x_1), \dots, \sigma(x_n)\rangle \in I(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M},\sigma}$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp(\wp(W))$.

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Suppose that σ is a substitution.

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2. $\mathcal{M}, w \models_\sigma \Box A$ iff $[\![\varphi]\!]_{\mathcal{M},\sigma} \in N(w)$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

Example

Suppose that F is a unary predicate symbol, $\mathcal{V} = \{x, y\}$, and $\langle W, N, D, I\rangle$ is a first order constant domain neighborhood model where

- $W = \{w, v, u\}$;
- $N(w) = \{\{w, v\}, \{u\}\}$, $N(v) = \{\{v\}\}$, $N(u) = \{\{w, v\}, \{v\}\}$;
- $D = \{a, b\}$; and
- $I(F, w) = \{a\}$, $I(F, v) = \{a, b\}$, and $I(F, u) = \emptyset$.

Example

$I(F, w) = \{a\}$, $I(F, v) = \{a, b\}$, and $I(F, u) = \emptyset$

There are four possible substitutions:

- $\sigma_1 : \mathcal{V} \to D$ where $\sigma_1(x) = a$, $\sigma_1(y) = b$;
- $\sigma_2 : \mathcal{V} \to D$ where $\sigma_2(x) = b$, $\sigma_2(y) = a$;
- $\sigma_3 : \mathcal{V} \to D$ where $\sigma_3(x) = \sigma_3(y) = a$; and
- $\sigma_4 : \mathcal{V} \to D$ where $\sigma_4(x) = \sigma_4(y) = b$

- $[\![F(x)]\!]_{\mathcal{M},\sigma_1} = \{w, v\}$;
- $[\![F(x)]\!]_{\mathcal{M},\sigma_2} = \{v\}$;
- $[\![F(x)]\!]_{\mathcal{M},\sigma_3} = \{w, v\}$; and
- $[\![F(x)]\!]_{\mathcal{M},\sigma_4} = \{v\}$.

Example

In general, every formula $\varphi \in \mathcal{L}_1$ is associated with a function

$$[\![\varphi]\!] : D^{\mathcal{V}} \to \wp(W)$$

Example

- $W = \{w, v, u\}$;
- $N(w) = \{\{w, v\}, \{u\}\}$, $N(v) = \{\{v\}\}$, $N(u) = \{\{w, v\}, \{v\}\}$;
- $D = \{a, b\}$; and
- $I(F, w) = \{a\}$, $I(F, v) = \{a, b\}$, and $I(F, u) = \emptyset$.

- $[\![\Box F(x)]\!]_{\mathcal{M},\sigma_1} = [\![\Box F(x)]\!]_{\mathcal{M},\sigma_3} = \{w, u\}$
  $[\![\Box F(x)]\!]_{\mathcal{M},\sigma_2} = [\![\Box F(x)]\!]_{\mathcal{M},\sigma_4} = \{v, u\}$;
- $[\![\Box \forall x F(x)]\!]_{\mathcal{M},\sigma_1} = \{v\}$; and
- $[\![\forall x \Box F(x)]\!]_{\mathcal{M},\sigma_1} = \{v, u\}$.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \to \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \to \forall x \Box A(x)$

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Observation 1: *CBF* is provable in $\mathbf{FOL} + \mathbf{EM}$

Observation 2: *BF* and *CBF* both valid on relational frames with constant domains

Observation 3: *BF* is valid in a *varying* domain relational frame iff the frame is anti-monotonic; *CBF* is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

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Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the '**lottery paradox**':

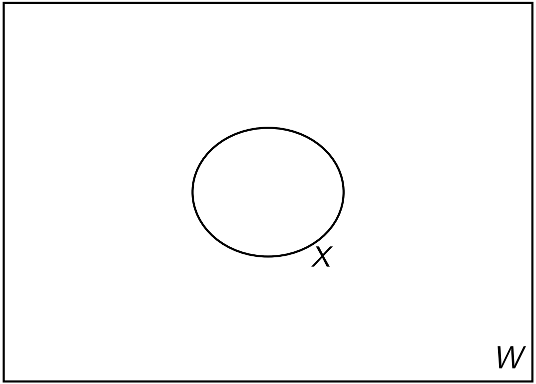
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

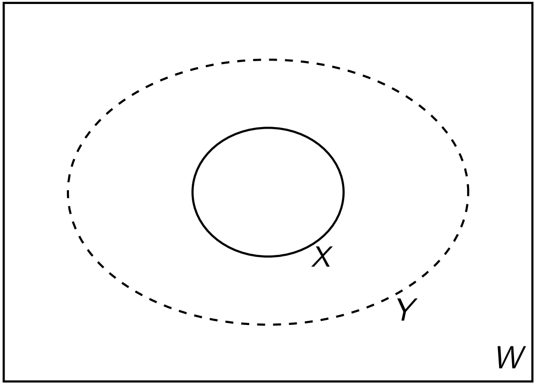
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

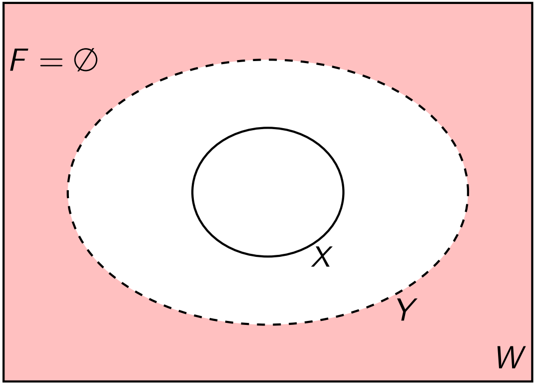
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



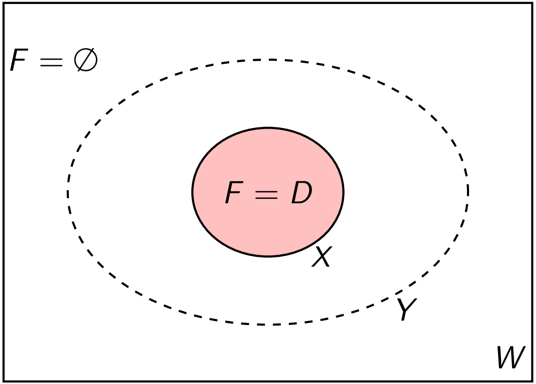
$$X \in N(w)$$



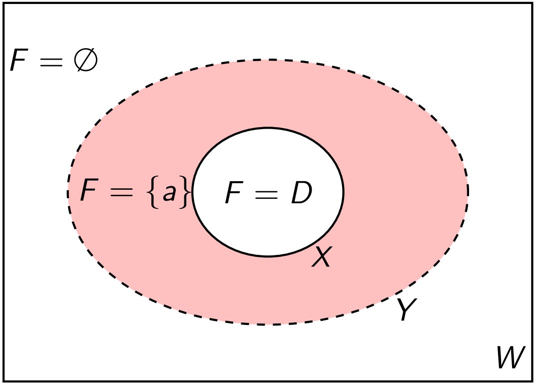
$$Y \notin N(w)$$



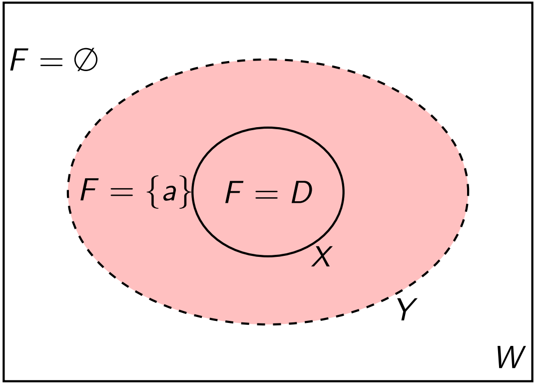
$$\forall v \notin Y, \quad I(F, v) = \emptyset$$



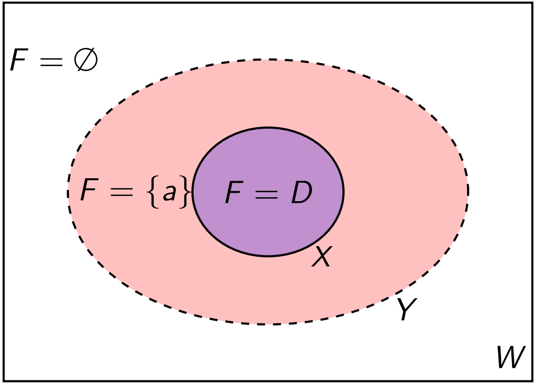
$$\forall v \in X, \quad I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \quad I(F, v) = D = \{a\}$$



$$(F[a])^{\mathcal{M}} = Y \notin N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

$$\text{hence} \quad w \models \Box \forall x F(x)$$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Suppose that **L** is a propositional modal logic. Let $\mathbf{FOL} + \mathbf{L}$ denote the set of formulas closed under the following rules and axiom schemes

L All axiom schemes and rules from **L**.

All $\forall x \varphi(x) \to \varphi[y/x]$ is an axiom scheme, where y is free for x in φ.

Gen $\dfrac{\varphi \to \psi}{\varphi \to \forall x \psi}$, where x is not free in φ.

Theorem $\mathbf{FOL} + \mathbf{E}$ is sound and strongly complete with respect to the class of **all** constant domain neighborhood frames.

CBF

$$\vdash_{\mathbf{FOL+EM}} \Box\forall x\varphi(x) \to \forall x\Box\varphi(x)$$

$$\nvdash_{\mathbf{FOL+E}+\mathrm{CBF}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$$

Completeness Theorems

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Theorem $\mathbf{FOL} + \mathbf{E} + CBF$ is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

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Observation The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$. In fact, the closure under infinite intersection of the minimal canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$.

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Lemma The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K} + BF$ is a canonical for $\mathbf{FOL} + \mathbf{K} + BF$.

Theorem $\mathbf{FOL} + \mathbf{K} + BF$ is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

Dynamics on Neighborhoods

J. van Benthem and EP. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica, 99(61), pp. 61 - 92, 2011.

Minghui Ma, Katsuhiko Sano (2018). *How to update neighbourhood models*. Journal of Logic and Computation, 28:8, pp. 1781 - 1804.

Richer Languages

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \langle\,]\varphi$$

- $\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(W)$
- $\mathcal{M}, w \models \langle\,]\varphi$ iff there is a $X \in N(W)$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$

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- $\mathcal{M}, w \models [B]\varphi$ iff for all max-f.i.p. $\mathcal{X} \subseteq N(w)$, $\bigcap \mathcal{X} \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [B]^{\psi}\varphi$ iff for all maximal ψ-f.i.p. $\mathcal{X}^{\psi} \subseteq N(w)$, $\bigcap \mathcal{X} \cap [\![\psi]\!]_{\mathcal{M}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$

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Accept evidence from an infallible source.

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Accept evidence from an infallible source.

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model and $\varphi \in \mathcal{L}$ a formula. The model $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, E^{!\varphi}, V^{!\varphi} \rangle$ is defined as follows: $W^{!\varphi} = [\![\varphi]\!]_{\mathcal{M}}$, for each $p \in \mathrm{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$ and for all $w \in W$,

$$E^{!\varphi}(w) = \{X \mid \emptyset \neq X = Y \cap [\![\varphi]\!]_{\mathcal{M}} \text{ for some } Y \in E(w)\}.$$

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$$E^{!\varphi}(w) = \{X \mid \emptyset \neq X = Y \cap [\![\varphi]\!]_{\mathcal{M}} \text{ for some } Y \in E(w)\}.$$

$[!\varphi]\psi$: “ψ is true after the public announcement of φ”

$\mathcal{M}, w \models [!\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}^{!\varphi}, w \models \psi$

Public Announcements: Recursion Axioms

$$
\begin{array}{lcl}
[!\varphi]p & \leftrightarrow & (\varphi \to p) \qquad (p \in \mathsf{At}) \\
[!\varphi](\psi \wedge \chi) & \leftrightarrow & ([!\varphi]\psi \wedge [!\varphi]\chi) \\
[!\varphi]\neg\psi & \leftrightarrow & (\varphi \to \neg[!\varphi]\psi) \\
[!\varphi]\Box\psi & \leftrightarrow & (\varphi \to \Box^{\varphi}[!\varphi]\psi) \\
[!\varphi]B\psi & \leftrightarrow & (\varphi \to B^{\varphi}[!\varphi]\psi) \\
[!\varphi]\Box^{\alpha}\psi & \leftrightarrow & (\varphi \to \Box^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi) \\
[!\varphi]B^{\alpha}\psi & \leftrightarrow & (\varphi \to B^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi) \\
[!\varphi]A\psi & \leftrightarrow & (\varphi \to A[!\varphi]\psi)
\end{array}
$$

Public Announcements: Recursion Axioms

$$[!\varphi]p \leftrightarrow (\varphi \to p) \quad (p \in \mathsf{At})$$

$$[!\varphi](\psi \wedge \chi) \leftrightarrow ([!\varphi]\psi \wedge [!\varphi]\chi)$$

$$[!\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[!\varphi]\psi)$$

$$[!\varphi]\Box\psi \leftrightarrow (\varphi \to \Box^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]B\psi \leftrightarrow (\varphi \to B^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]\Box^{\alpha}\psi \leftrightarrow (\varphi \to \Box^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi)$$

$$[!\varphi]B^{\alpha}\psi \leftrightarrow (\varphi \to B^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi)$$

$$[!\varphi]A\psi \leftrightarrow (\varphi \to A[!\varphi]\psi)$$

1. Other definition of public announcement
2. Dissecting the public announcement operation

Public Announcement

Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a monotonic neighborhood modeland $\emptyset \neq X \subseteq W$.

Intersection submodel
$N^{\Cap X}(w) = \{Y \mid \emptyset \neq Y = X \cap Z \text{ for some } Z \in N(w)\}$

Strong intersection submodel:
$N^{\cap X}(w) = \{Y \mid Y = Z \cap X \text{ for some } Z \in N(w)\}$.

Subset submodel: $N^{\subseteq X}(w) = \{Y \mid Y \subseteq X \text{ and } Y \in N(w)\}$.

- $[\varphi]^{\cap}\Box\psi \leftrightarrow (\varphi \to \Box[\varphi]^{\cap}\psi)$ is **valid** on monotonic frames.

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- $[\varphi]^{\Cap}\Box^{\alpha}\psi \leftrightarrow (\varphi \to \Box^{\varphi\wedge[\varphi]^{\Cap}\alpha}[\varphi]^{\Cap}\psi)$ is **valid** on monotonic frames.

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Evidence Addition

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and φ a formula in $\mathcal{L}$. The model $\mathcal{M}^{+\varphi} = \langle W^{+\varphi}, E^{+\varphi}, V^{+\varphi} \rangle$ has $W^{+\varphi} = W$, $V^{+\varphi} = V$ and for all $w \in W$,

$$E^{+\varphi}(w) = E(w) \cup \{[\![\varphi]\!]_{\mathcal{M}}\}$$

$[+\varphi]\psi$: "ψ is true after φ is accepted as an admissible piece of evidence"

$\mathcal{M}, w \models [+\varphi]\psi$ iff $\mathcal{M}, w \models E\varphi$ implies $\mathcal{M}^{+\varphi}, w \models \psi$

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Evidence Addition: Recursion Axioms

$$
\begin{array}{lcll}
[+\varphi]p & \leftrightarrow & (E\varphi \to p) & (p \in \mathsf{At}) \\
[+\varphi](\psi \wedge \chi) & \leftrightarrow & ([+\varphi]\psi \wedge [+\varphi]\chi) & \\
[+\varphi]\neg\psi & \leftrightarrow & (E\varphi \to \neg[+\varphi]\psi) & \\
[+\varphi]A\psi & \leftrightarrow & (E\varphi \to A[+\varphi]\psi) &
\end{array}
$$

Evidence Addition: Recursion Axioms

$$
\begin{array}{lcl}
[+\varphi]\Box\psi & \leftrightarrow & (E\varphi \to (\Box[+\varphi]\psi \vee A(\varphi \to [+\varphi]\psi))) \\
[+\varphi]\Box^{\alpha}\psi & \leftrightarrow & (E\varphi \to (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge \\
 & & A((\varphi \wedge [+\varphi]\alpha) \to [+\varphi]\psi))))
\end{array}
$$

Evidence Addition: Recursion Axioms

$$
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$$

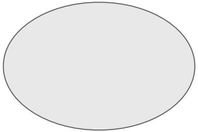
Evidence Addition: Recursion Axioms

$$[+\varphi]B\psi \quad \leftrightarrow \quad ????$$

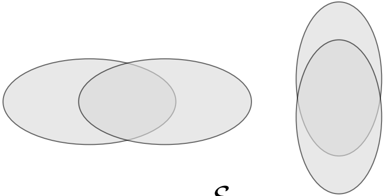
$$[+\varphi]B^{\alpha}\psi \quad \leftrightarrow \quad ????$$

Adding φ

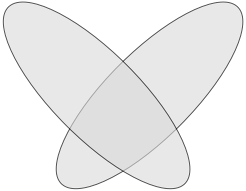
Adding φ



$\mathcal{E}_1$

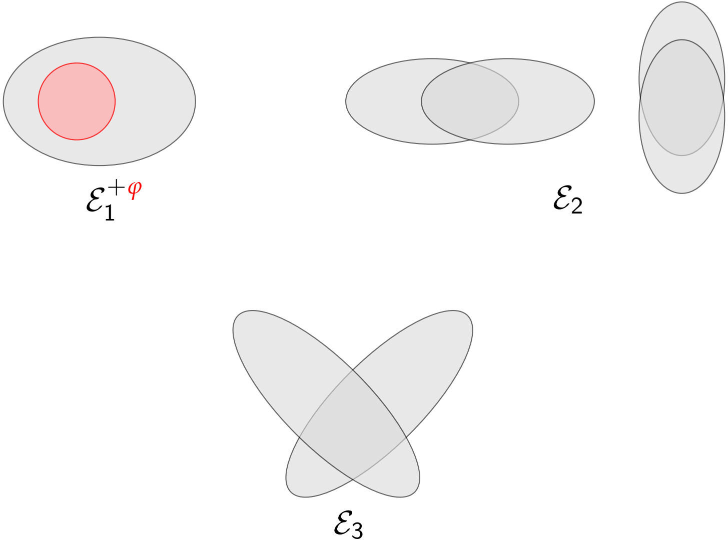


$\mathcal{E}_2$

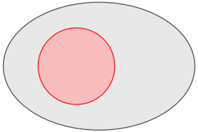


$\mathcal{E}_3$

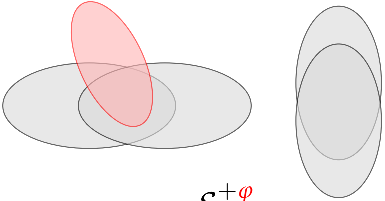
Adding φ



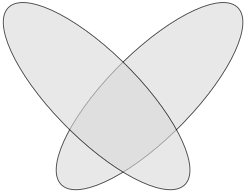
Adding φ



$\mathcal{E}_1^{+\varphi}$

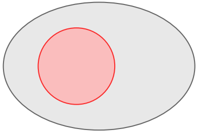


$\mathcal{E}_2^{+\varphi}$

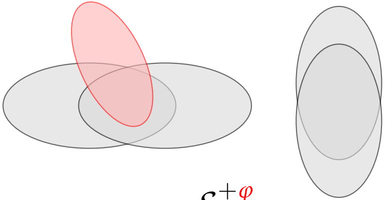


$\mathcal{E}_3$

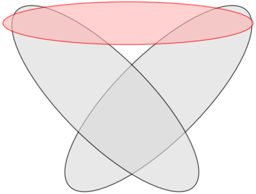
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2^{+\varphi}$



$\mathcal{E}_3^{+\varphi}$

Compatibile vs. Incompatible

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally **φ-compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and

Compatibile vs. Incompatible

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Conditional belief: $B^{+\varphi}\psi$ iff for each maximally φ-compatible $\mathcal{X} \subseteq E(w)$, $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$

Conditional Beliefs (Incompatibility Version): $\mathcal{M}, w \models B^{-\varphi}\psi$ iff for all maximal f.i.p., if $\mathcal{X}$ is incompatible with φ then $\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

$B^{+\neg\varphi}$ vs. $B^{-\varphi}$

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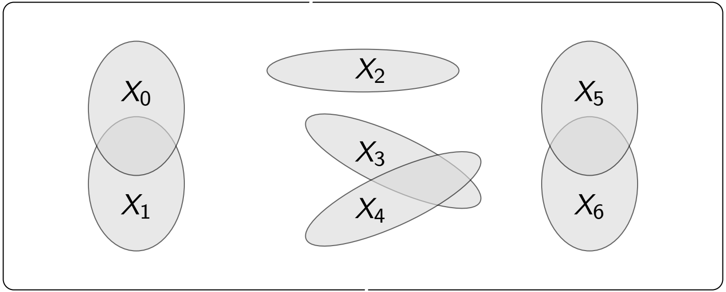

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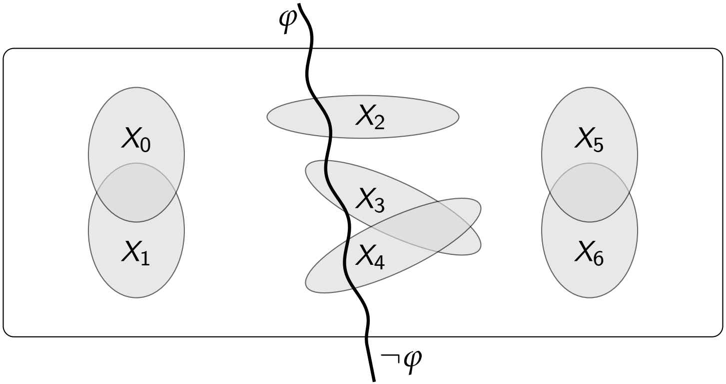

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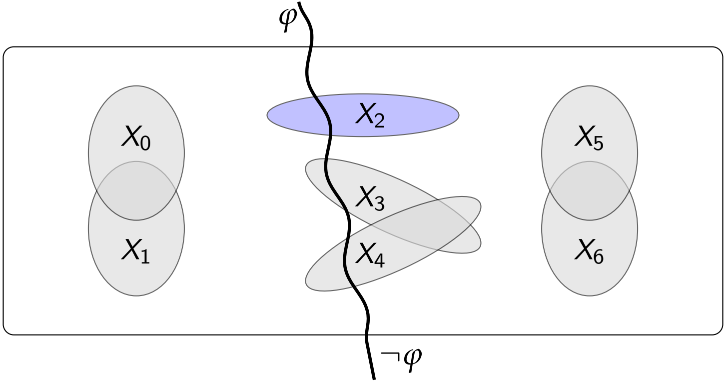

$\{X_2\}$ is (max.) compatible with $\neg\varphi$ but not maximally φ incompatible

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \to (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

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Language Extension: $\mathcal{M}, w \models B^{\varphi,\psi}\chi$ iff for all maximally φ-compatible sets $\mathcal{X} \subseteq E(w)$, if $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$, then $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\chi]\!]_{\mathcal{M}}$.

$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

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$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

Fact. The following is valid:

$$[+\varphi]B^{\psi,\alpha}\chi \leftrightarrow (E\varphi \rightarrow (B^{\varphi\wedge[+\varphi]\psi,[+\varphi]\alpha}[+\varphi]\chi \wedge B^{[+\varphi]\psi,\neg\varphi\wedge[+\varphi]\alpha}[+\varphi]\chi))$$

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Evidence Removal

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and $\varphi \in \mathcal{L}$. The model $\mathcal{M}^{-\varphi} = \langle W^{-\varphi}, E^{-\varphi}, V^{-\varphi} \rangle$ has $W^{-\varphi} = W$, $V^{-\varphi} = V$ and for all $w \in W$,

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$[-\varphi]\psi$: "after removing the evidence that φ, ψ is true"

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$

Fact. Evidence removal *extends* the language.

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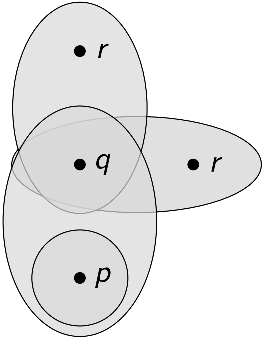
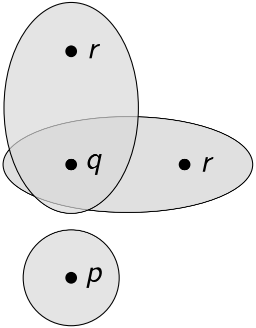


$\mathcal{E}_1$



$\mathcal{E}_2$

$[-p]\Box(p \vee q)$ is true in $\mathcal{M}_1$ but not in $\mathcal{M}_2$.

Compatible Evidence

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Recursion axiom: $[-\varphi]\Box\psi \leftrightarrow (\neg A\varphi \rightarrow \Box_{\neg\varphi}[-\varphi]\psi)$

Evidence Removal: Recursion Axioms

Langage $\mathcal{L}'$: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\alpha}_{\varphi}\psi \mid \Box^{\alpha}_{\varphi}\psi \mid A\varphi$

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- $\mathcal{M}, w \models B^{\alpha}_{\varphi}\psi$ iff for each maximal α-f.i.p. $\mathcal{X}$ compatible with $\overline{\varphi}$, $\bigcap \mathcal{X}^{\alpha} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

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$$
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\end{array}
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Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
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$B^{\varphi}\psi$:	"the agent believe χ conditional on φ"
$B^{\varphi}_{\overline{\gamma}}\psi$:	"the agent believe χ conditional on φ assuming compatibility with each of the γ_i"

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Complete logical analysis?

$$B^{\varphi}\psi \to B(\varphi \to \psi) \quad \text{and} \quad B(\varphi \to \psi) \to B^{\top,\varphi}\psi$$

Summary: Evidence Operations

$$
\begin{array}{lrcl}
\text{Public announcement:} & [!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \\
\text{Evidence addition:} & [+\varphi]B\psi & \leftrightarrow & (E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi)) \\
\text{Evidence removal:} & [-\varphi]B\psi & \leftrightarrow & (\neg A\varphi \rightarrow B_{\neg\varphi}[-\varphi]\psi)
\end{array}
$$

Thank you!!

https://pacuit.org/esslli2024/neighborhood-semantics/

https://pacuit.org/modal/neighborhoods/