Probabilistic Methods in Social Choice

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Plan

- ✓ Background on voting theory
- ✓ Generating preference profiles
- ✓ Quantitative analysis of voting methods
- ⇒ Probabilistic voting methods
- ▶ Condorcet jury theorem and related results
- ▶ Aggregating probabilistic judgements

Even-Chance Tiebreaking

All anonymous and neutral voting methods F may select more than one winner in a profile $\mathbf{P}$.

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When $F(\mathbf{P})$ is not a singleton set, one option is to use an even-chance lottery on $F(\mathbf{P})$ to break the tie and select a unique ultimate winner.

Definition

For a voting method F, let F^{eve} be the probabilistic voting method such that for any profile $\mathbf{P}$ and $x \in X(\mathbf{P})$, $F^{\text{eve}}(\mathbf{P})(x) = 1/|F(\mathbf{P})|$ if $x \in F(\mathbf{P})$ and $F^{\text{eve}}(\mathbf{P})(x) = 0$ otherwise.

F. Brandt. *Rolling the Dice: Recent Results in Probabilistic Social Choice*. Handbook of Computational Social Choice, 2016.

Let V be a set of voters, X a set of m alternatives.

The set of all lotteries over X is:

$$\Delta(X) = \{p \in \mathbb{R}^X \mid p(x) \geq 0 \text{ for all } x \in X \text{ and } \sum_{x \in X} p(x) = 1\}$$

For $p \in \Delta(X)$, $supp(p) = \{x \mid p(x) > 0\}$

p is **degenerate** when $|supp(p)| = 1$.

We write lotteries as convex combinations of alternatives, e.g., the uniform lottery on $\{a, b\}$ where $p(a) = p(b) = 1/2$ is denoted as $1/2a + 1/2b$.

Probabilistic Voting Methods

A **probabilistic social choice function (PSCF)** is a map $F: O(X)^V \rightarrow \wp(\Delta(A)) \setminus \emptyset$ such that for all $\boldsymbol{P}$, $F(\boldsymbol{P})$ is a convex set of lotteries.

Anonymity and neutrality can be defined as usual.

Random (Serial) Dictator

Random dictatorship: A voter is picked uniformly at random and this voter's most-preferred alternative is selected. Thus, the probabilities assigned by RD are directly proportional to the number of agents who top-rank a given alternative (or, in other words, the alternative's plurality score).

Random serial dictatorship (RSD): RSD selects a permutation of the agents uniformly at random and then sequentially allows agents in the order of the permutation to narrow down the set of alternatives to their most preferred of the remaining ones.

Proportional Borda

Proportional Borda: Assign probabilities to the alternatives that are proportional to their Borda scores.

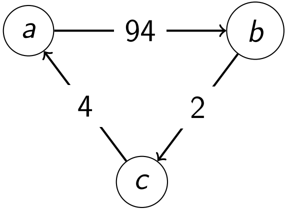
3	2
a	b
b	c
c	a

- $Plurality^{eve}$ lottery is a
- RD lottery is $\frac{3}{5}a + \frac{2}{5}b$
- $Borda^{eve}$ lottery is b
- $Borda_{pro}$ lottery is $\frac{6}{15}a + \frac{7}{15}b + \frac{2}{15}c$

Margin Matrix/Graph

49	48	3
c	a	b
a	b	c
b	c	a

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 94 & -4 \\ -94 & 0 & 2 \\ 4 & -2 & 0 \end{pmatrix} \end{array}$$



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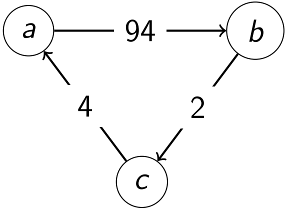

If the output of a neutral PSCF F only depends on the margin matrix/graph M, F is called **pairwise**. An advantage of pairwise PSCFs is that they are applicable even when individual preferences are incomplete or intransitive.

Maximal Lotteries

G. Kreweras. *Aggregation of preference orderings*. In Mathematics and Social Sciences I: Proceedings of the seminars of Menthon-Saint-Bernard, France (1–27 July 1960) and of Gösing, Austria (3 - 27 July 1962), pages 73 - 79, 1965.

P. C. Fishburn. *Probabilistic social choice based on simple voting comparisons*. Review of Economic Studies, 51(4):683-692, 1984.

Maximal Lotteries

H. Aziz, F. Brandl, F. Brandt, and M. Brill. *On the tradeoff between efficiency and strategyproofness*. Games and Economic Behavior, 110:1 - 18, 2018.

F.. Brandl, F. Brandt, and H. G. Seedig. *Consistent probabilistic social choice*. Econometrica, 84(5):1839 - 1880, 2016.

F. Brandl, F. Brandt, M. Eberl, and C. Geist. *Proving the incompatibility of efficiency and strategyproofness via SMT solving*. Journal of the ACM, 65(2):1 - 28, 2018.

F. Brandl, F. Brandt, and C. Stricker. *An Analytical and Experimental Comparison of Maximal Lottery Schemes* . forthcoming Social Choice and Welfare, 2021.

Maximal Lotteries

M admits a (weak) Condorcet winner if M contains a nonnegative row, i.e., there is a standard unit vector v such that

$$v^T M \geq 0$$

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1	1	1
a	b	c
b	a	a
c	c	b

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$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \begin{pmatrix}0 & 1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 1\end{pmatrix} \geq 0$$

Maximal Lotteries

A lottery p is maximal if $p^T M \geq 0$:

- randomized Condorcet winner
- p is "at least as good" as any other lottery:

  the expected number of agents who prefer the alternative returned by p to that returned by q is at least as large as the expected number of agents who prefer the outcome returned by q to that returned by p

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$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\end{pmatrix} < 0$$

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Maximal Lotteries

A lottery p is maximal if $p^T M \geq 0$:

1	1	1
a	b	c
b	c	a
c	a	b

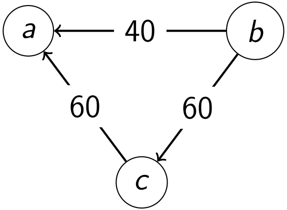
$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}^T \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \geq 0$$

Maximal Lotteries

- always exist due to the Von Neumann Minimax Theorem.
- almost always unique
- set of profiles with multiple maximal lotteries has measure zero
- always unique for odd number of voters with strict preferences
- does not require asymmetry, completeness, or even transitivity of individual preferences

20	20	60
c	a	b
a	b	c
b	c	a

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & -40 & -60 \\ 40 & 0 & 60 \\ 60 & -60 & 0 \end{pmatrix} \end{array}$$



$$\begin{array}{c|ccc|c} & a & b & c & \\ \hline a & 0, 0 & -40, 40 & -60, 60 & -60 \\ b & 40, -40 & 0, 0 & 60, -60 & 40 \\ c & 60, -60 & -60, 60 & 0, 0 & -60 \\ \hline & -60 & 40 & -60 & \end{array}$$

Nash equilibrium: (b, b)

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c	a	b
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$$
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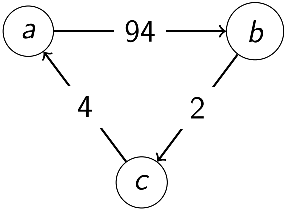

$$
\begin{array}{c c}
 & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0,0 & 94,-94 & -4,4 \\ -94,94 & 0,0 & 2,-2 \\ 4,-4 & -2,2 & 0,0 \end{pmatrix}
\end{array}
$$

There is no *pure strategy Nash equilibrium*

There is a mixed Nash equilibrium: $2/100a + 4/100b + 94/100c$.

Maximal Lotteries can be efficiently computed via linear programming

`https://voting.ml`

Maximal Lottery Schemes

Every **maximal lottery scheme** is based on an odd and monotone function $\tau : \mathbb{Z} \to \mathbb{R}$ with $\tau(1) = 1$.

$$ML^{\tau}(\boldsymbol{R}) = \{p \in \Delta(X) \mid \sum_{x,y \in X} p(x)q(y)\tau(m_{xy}) \geq 0 \text{ for all } q \in \Delta(X)\}$$

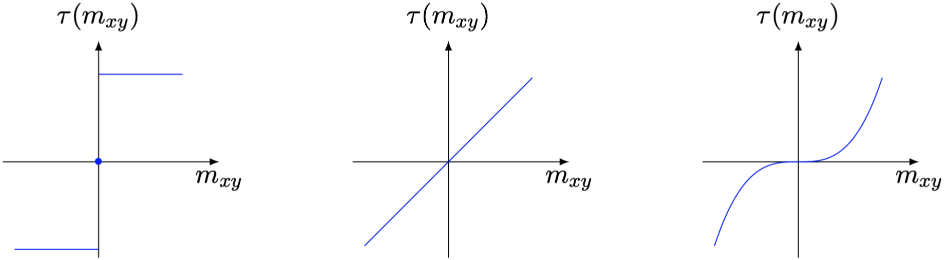

Figure 2: Different examples for τ (extended to $\mathbb{R}$): sign function, identity function, and cubic function.

$C1$-ML: ML schemes based on the sign function.

$C2$-ML: ML schemes based on the identity function.

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$C2$-ML: ML schemes based on the identity function.

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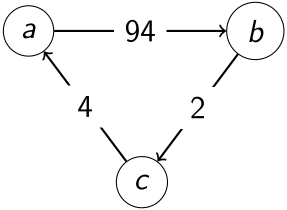

$$C1\text{-}ML\text{: } 1/3a + 1/3b + 1/3c$$

$$C2\text{-}ML\text{: } 2/100a + 4/100b + 94/100c$$

Theorem (Brandl, Brandt and Stricker) For any pair of ML schemes ML^{τ} and ML^{σ}, there is a preference profile $\boldsymbol{R}$ such that $ML^{\tau}(\boldsymbol{R}) = \{p\}$ and $ML^{\sigma}(\boldsymbol{R}) = \{q\}$ and $supp(p) \cap supp(q) = \varnothing$.

(cf. B. Dutta and J.-F. Laslier. Comparison functions and choice correspondences. Social Choice and Welfare, 16(4):513-532, 1999)

An SDS is **homogenous** when replacing every voter with a fixed number of identical clones (i.e., voters with the same preferences) does not change the outcome.

Theorem (Brandl, Brandt and Stricker) ML^{τ} is homogenous if and only if τ is based on τ' where there is a $t \geq 0$ such that $\tau'(k) = k^t$ for all $k \in \mathbb{N}$.

Efficiency

Pareto Efficiency: no voter can be made better off without making another voter worse off.

To define this, we need assumptions about how voters rank lotteries.

Suppose that $\succeq_i$ is voter i's weak preference relation.

$p \succeq_i^{DD'} q$ if and only if $x \succ_i y$ for all $x \in supp(p)$ and $y \in supp(q)$

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$p \succeq_i^{ST} q$ if and only if
$(supp(p) \setminus supp(q)) \succ_i (supp(p) \cap supp(q)) \succ_i (supp(q) \setminus supp(p))$
and $p(x) = q(x)$ for all $x \in supp(p) \cap supp(q)$

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Suppose that $a \succ_i b \succ_i c$. Then,
$2/3a + 1/3b \succ_i^{DD'} c$
$2/3a + 1/3b \succ_i^{DD} 1/2b + 1/2c$
$1/2a + 1/2b \succ_i^{ST} 1/2b + 1/2c$

Bilinear Dominance

$p \succeq_i^{BD} q$ if and only if $p(x)q(y) \geq p(y)q(x)$ for all $x, y \in X$ with $x \succ_i y$

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Suppose that $a \succ_i b \succ_i c$. Then,
$1/2a + 1/2b \succ_i^{BD} 1/3a + 1/3b + 1/3c$

(Fishburn 1984): p bilinearly dominates q iff p is preferable to q for every skew-symmetric bilinear (SSB) utility function consistent with $\succeq_i$

Stochastic Dominance

$p \succeq_i^{SD} q$ if and only if $\sum_{y \succeq_i x} p(y) \geq \sum_{y \succeq_i x} q(y)$ for all $x \in X$

Stochastic Dominance

$p \succeq_i^{SD} q$ if and only if $\sum_{y \succeq_i x} p(y) \geq \sum_{y \succeq_i x} q(y)$ for all $x \in X$

Suppose that $a \succ_i b \succ_i c$. Then,
$1/2a + 1/2c \succ_i^{SD} 1/2b + 1/2c$

p stochastically dominates q iff p is preferable to q for every von Neumann-Morgenstern utility function consistent with $\succeq_i$

$$\succeq^{DD'} \subseteq \succeq^{DD} \subseteq \succeq^{BD} \qquad \succeq^{DD'} \subseteq \succeq^{ST} \subseteq \succeq^{BD} \qquad \succeq^{BD} \subseteq \succeq^{SD} \subseteq \succeq^{PC}$$

A lottery p is *SD*-**efficient** for a preference profile $\boldsymbol{R}$ if there is no lottery $q \in \Delta(X)$ such that $q \succeq_i^{SD} p$ for all i and $q \succ_j^{SD} p$ for some j. (Similar definitions for other preferences over lotteries).

Theorem (Fishburn, 1984) Every ML^{τ} is *ex post efficient*: whenever there are alternatives x and y such that $x \succeq_i y$ for all i and $x \succ_j y$ for some j, then y should receive probability 0.

Theorem (Brandl, Brandt and Stricker). Every $C2$-ML schemes is SD-efficient. No other ML scheme is SD-efficient for all numbers of voters and candidates.

Theorem (Brandl, Brandt and Stricker). Suppose that m is the number of candidates and n is the number of voters. Every *majoritarian* and neutral $SPSC$ violates SD-efficiency for $m \geq 9$ (and $n = 5$, $n - 7$ or $n \geq 9$), even when preferences are strict.

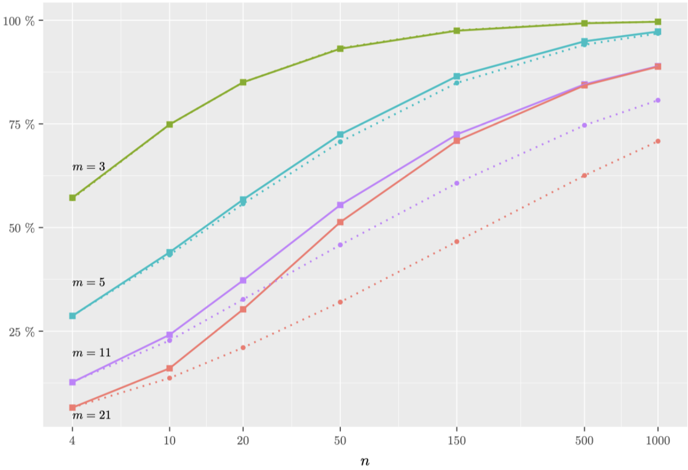

Figure 7: Frequencies of profiles R for which $|C2\text{-}ML(R)| = 1$ (solid lines) and $|C1\text{-}ML(R)| = 1$ (dotted lines), respectively. Each point is based on 100 000 samples according to the IAC model.

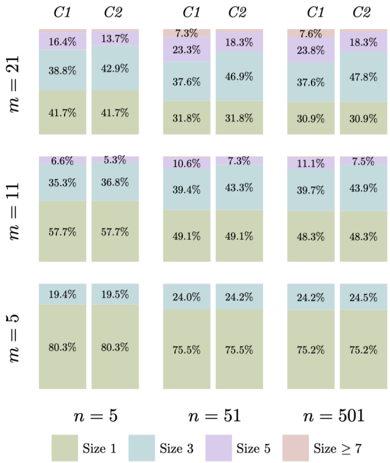

Figure 8: Distributions of the support sizes of *C1-ML* (left) and *C2-ML* (right) based on 100 000 samples according to the IAC model for every combination of parameters. The bars represent the frequency of a support with size $1, 3, 5$, and 7 or more stacked from bottom to top. Frequencies lower than 4% are not labeled.

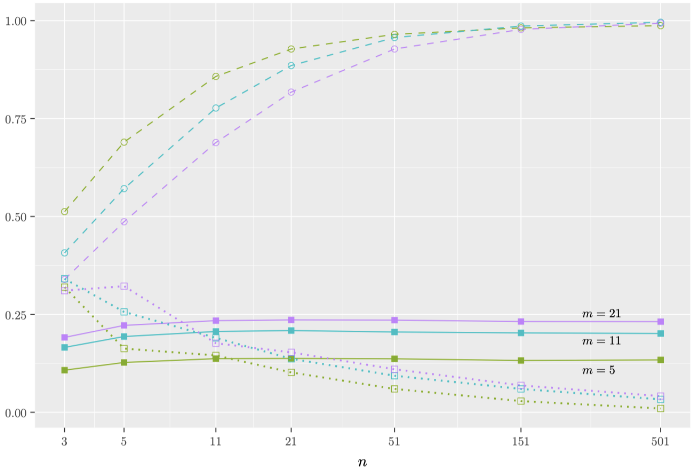

Figure 10: Average normalized Shannon entropies of *C2-ML* (solid line) compared to random dictatorship (dashed line) and the plurality rule (dotted line). Each point is based on 100 000 samples according to the IAC model.

F. Brandl and F. Brandt. *A Natural Adaptive Process for Collective Decision-Making*. manuscript, 2021.

Consider an urn filled with balls, each labeled with one of several possible collective decisions. Now, draw two balls from the urn, let a random voter pick her more preferred as the alternative, relabel the losing ball with the collective decision, put both balls back into the urn, and repeat. In order to prevent the permanent disappearance of some types of balls, once in a while, a randomly drawn ball is labeled with a random alternative.

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Brandl and Brandt prove that this process will almost surely converge to the $C2$-ML.

300	300	300
*A*1	*A*1	*A*2
*A*2	*A*3	*A*3
*A*3	*A*2	*A*1

300	300	300
*A*1	*A*2	*A*3
*A*2	*A*3	*A*1
*A*3	*A*1	*A*2