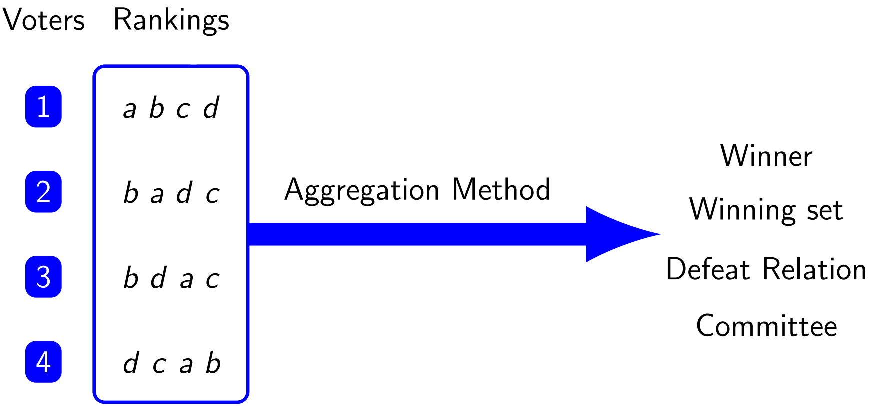
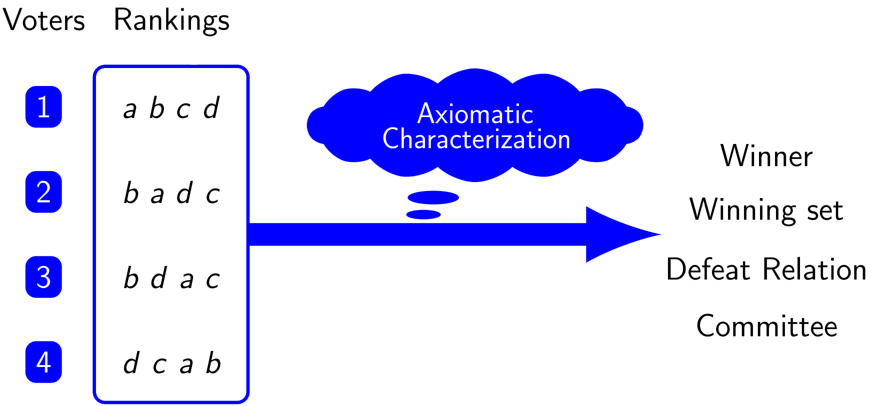
Probabilistic Methods in Social Choice

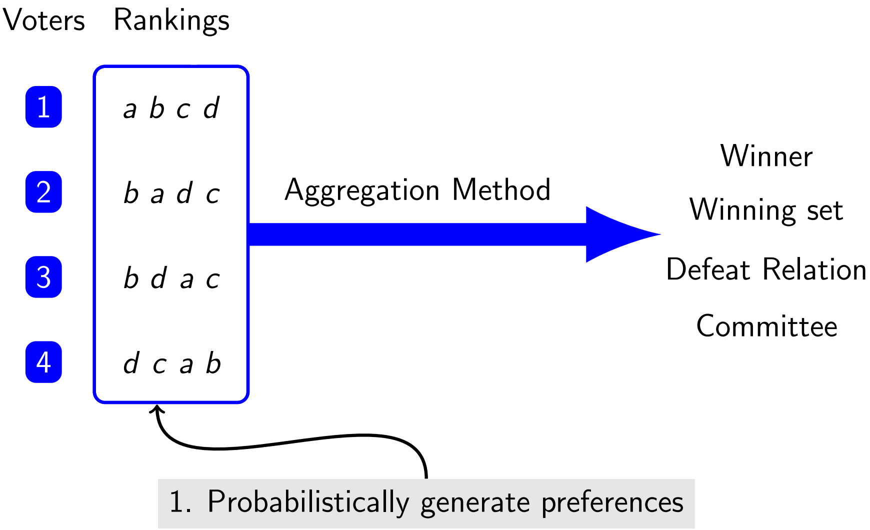
Eric Pacuit
University of Maryland

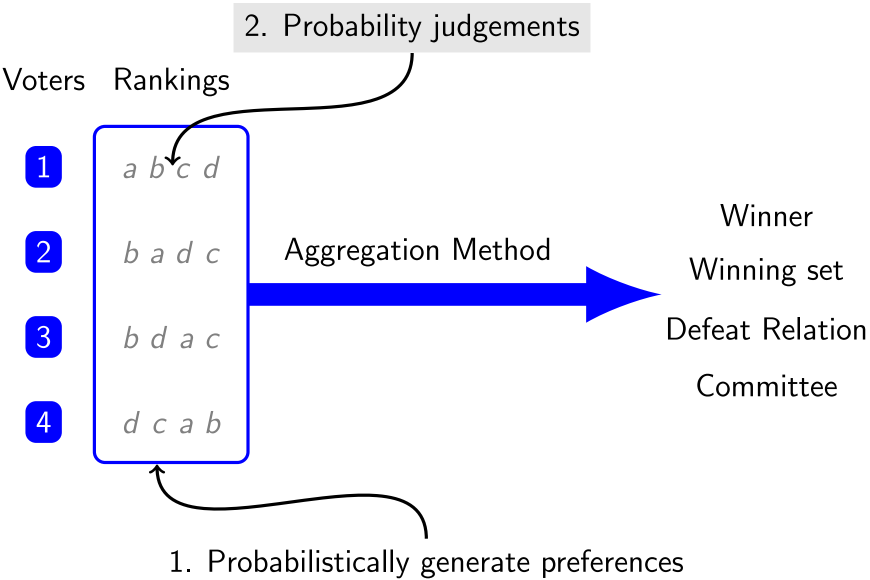
August 9, 2021

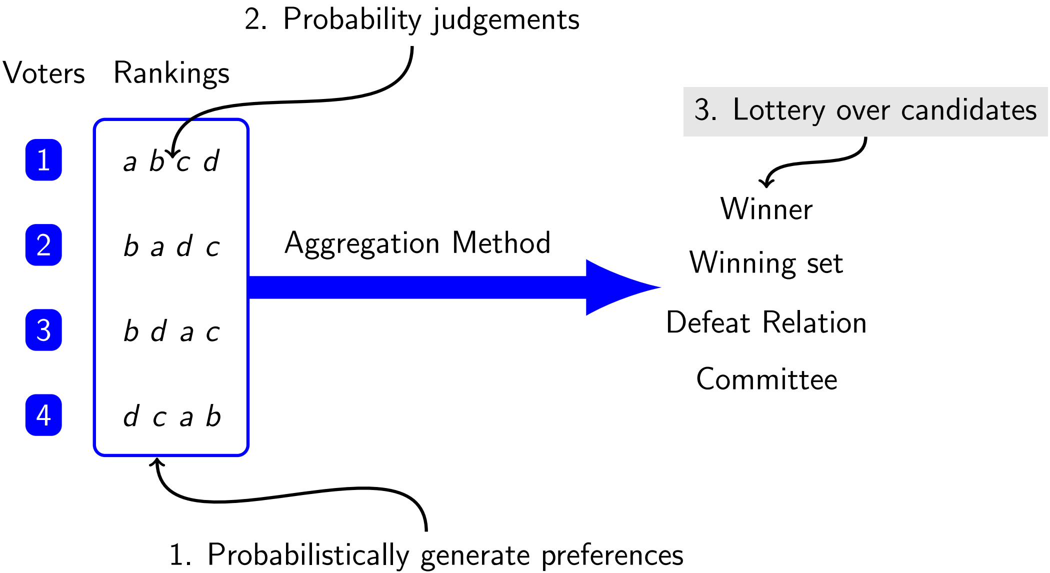


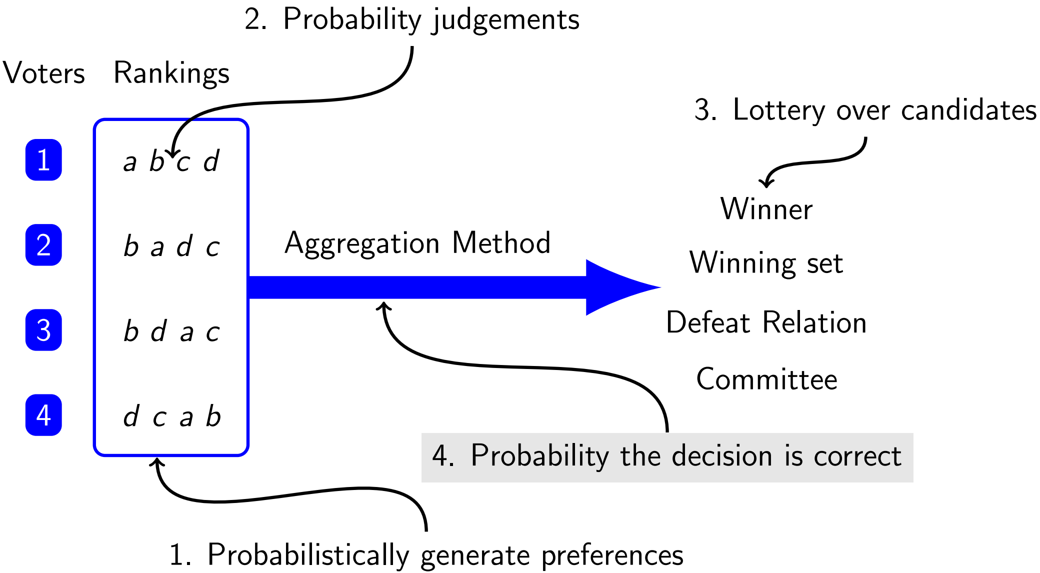
Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee


Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Axiomatic Characterization
Winner
Winning set
Defeat Relation
Committee


Voters
Rankings
1
a b c d
2
b a d c
3
b d a c
4
d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee
1. Probabilistically generate preferences


2. Probability judgements
Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee
1. Probabilistically generate preferences


2. Probability judgements
Voters
Rankings
3. Lottery over candidates
1
a b c d
2
b a d c
3
b d a c
4
d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee
1. Probabilistically generate preferences


2. Probability judgements
Voters
Rankings
3. Lottery over candidates
1
a b c d
2
b a d c
3
b d a c
4
d c a b
Aggregation Method
Winner
Winning set
Defeat Relation
Committee
4. Probability the decision is correct
1. Probabilistically generate preferences

Plan

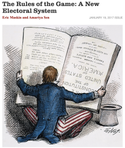
- Background on voting theory
- Generating preference profiles
- Quantitative analysis of voting methods
- Probabilistic voting methods
- Condorcet jury theorem and related results
- Aggregating probabilistic judgements

Voting

POLITICS • ELECTION

New York City Voters Just Adopted Ranked-Choice Voting in Elections. Here's How It Works

A poll workers explains the voting process to a voter at a public school polling location in New York on Nov. 5, 2019.



The Rules of the Game: A New Electoral System

Eric Maskin and Amartya Sen

JANUARY 19, 2017 ISSUE

n p r | WAMU 88.5

SIGN IN | NPR SHOP | DONATE

NEWS | ARTS & LIFE | MUSIC | SHOWS & PODCASTS | SEARCH

ELECTIONS

Ranked-Choice Voting Gets A Prime-Time Shot Under New York City's Bright Lights

Updated June 23, 2021 · 4:05 PM ET

DOMENICO MONTANARO

The New York Times

Opinion

A Better Electoral System in Maine

By Eric Maskin and Amartya Sen

The New York Times

Opinion

OPINION

How Majority Rule Might Have Stopped Donald Trump

By Eric Maskin and Amartya Sen

April 28, 2016

Cambridge, Mass. — DONALD TRUMP has won Republican primaries in 23 states to date — far more than any of his opponents. Yet in the first 17 states he won, several of the other major candidates might well have beaten him in a one-on-one contest.

There is no contradiction here. In the early contests, Mr. Trump attracted less than 50 percent of the vote (in Arkansas he got only 33 percent); a majority of voters rejected him. But he faced more than one opponent every time, so that the non-Trump vote was split. That implies he could have been defeated in most (given his extreme views on many subjects) had the opposition coalesced around one of his leading rivals. In such a scenario, he might have been out of contention long before he could ride his plurality victories toward his first outright majority win — in New York, last month.

American primaries are not the only recent elections to produce winners lacking the support of a majority of voters. In India, the Bharatiya Janata Party received only 31 percent of the vote in the last general election, but a majority of parliamentary seats. (Even including political allies, their vote share was no more than 39 percent.) The B.J.P., a right-wing party with a Hindu ideology

Subscribe | Latest Issues

SCIENTIFIC AMERICAN

Sign In | Newsletters

THE SCIENCES

Ranking Candidates Is More Accurate Than Voting

A ballot-counting system that allows voters to rank the candidates could provide more accurate results

By Partha Dasgupta and Eric Maskin on October 6, 2008

Editor's note: This story was originally posted in the March 2004 issue, and has been reposted to highlight the long history of Nobelists publishing in Scientific American.

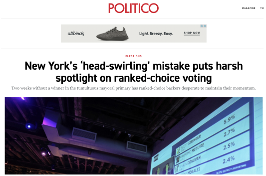
Most American and French citizens—indeed, those of democracies the world over—spend little time contemplating their voting systems. That preoccupation is usually left to political and electoral analysts. But in the past few years, a large segment of both these countries' populations have found themselves utterly perplexed. People in France wondered how a politician well outside the political mainstream made it to the final two-

N.Y.C. Mayoral Race

Why Some N.Y.C. Lawmakers Want to Rethink Ranked-Choice Voting

The new system was approved by voters in 2019, but critics, including at least one top mayoral candidate, fear that it may disenfranchise minority voters.

Under the new system being challenged in court, when New York City voters go to the polls for next year's mayoral primary, they would be allowed to choose as many as five candidates, ranked in order of preference.



POLITICO

ELECTIONS

New York's 'head-swirling' mistake puts harsh spotlight on ranked-choice voting

Two weeks without a winner in the tumultuous mayoral primary has ranked-choice backers desperate to maintain their momentum.

https://www.electology.org http://www.fairvote.org

Rankings

Let X be a set of candidates and V a set of voters.

A **ranking** of X is a strict linear order P on X: a relation $P \subseteq X \times X$ satisfying the following conditions for all $x, y, z \in X$:

> *asymmetry*: if $x \, P \, y$ then *not* $y \, P \, x$;
>
> *transitivity*: if $x \, P \, y$ and $y \, P \, z$, then $x \, P \, z$;
>
> *weak completeness*: if $x \neq y$, then $x \, P \, y$ or $y \, P \, x$.

Let $\mathcal{L}(X)$ be the set of all strict linear orders on X.

Rankings

CITY AND COUNTY 市縣						
MAYOR 市長	**1** 1st Choice 第一選擇	**2** 2nd Choice 第二選擇	**3** 3rd Choice 第三選擇	**4** 4th Choice 第四選擇	**5** 5th Choice 第五選擇	**6** 6th Choice 第六選擇
ELLEN LEE ZHOU / 李愛晨 Behavioral Health Clinician 行為健康臨床治療師	● 1	2	3	4	5	6
LONDON N. BREED / 倫敦·布里德 Mayor of San Francisco 三藩市市長	1	2	● 3	4	5	6
JOEL VENTRESCA / 喬爾·范崔斯卡 Retired Airport Analyst 退休機場分析師	1	2	3	4	● 5	6
WILMA PANG / 彭德慧 Retired Music Professor 退休音樂教授	1	2	3	● 4	5	6
ROBERT L. JORDAN, JR. / 小羅伯特·L·喬丹 Preacher 傳教士	1	2	3	4	5	● 6
PAUL YBARRA ROBERTSON / 保羅·伊巴拉·羅伯森 Small Business Owner 小企業業主	1	● 2	3	4	5	6
	1	2	3	4	5	6

Variable candidate/voter profiles

Fix infinite sets $\mathcal{V}$ and $\mathcal{X}$ of *voters* and *candidates*, respectively.

For $X \subseteq \mathcal{X}$, let $\mathcal{L}(X)$ be the set of all strict linear orders on X.

A *profile* is a function $\boldsymbol{P} : V(\boldsymbol{P}) \to \mathcal{L}(X(\boldsymbol{P}))$ for some nonempty finite $V(\boldsymbol{P}) \subseteq \mathcal{V}$ and nonempty finite $X(\boldsymbol{P}) \subseteq \mathcal{X}$.

We call $V(\boldsymbol{P})$ and $X(\boldsymbol{P})$ the sets of *voters in* $\boldsymbol{P}$ and *candidates in* $\boldsymbol{P}$, respectively.

We call $\boldsymbol{P}(i)$ voter i's *ranking*, and we write '$x\boldsymbol{P}_i y$' for $(x, y) \in \boldsymbol{P}(i)$. As usual, we take $x\boldsymbol{P}_i y$ to mean that voter i strictly prefers candidate x to candidate y.

Anonymous profiles

40	35	25
t	r	k
k	k	r
r	t	t

Margin

Let $\boldsymbol{P}$ be a profile and $a, b \in X(\boldsymbol{P})$. Then the **margin of** a **over** b is:

$$Margin_{\boldsymbol{P}}(a, b) = |\{i \in V(\boldsymbol{P}) \mid a\boldsymbol{P}_i b\}| - |\{i \in V(\boldsymbol{P}) \mid b\boldsymbol{P}_i a\}|.$$

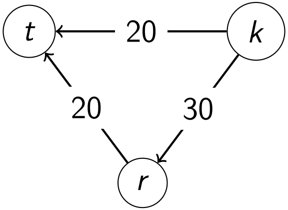
We say that a is **majority preferred** to b in $\boldsymbol{P}$ when $Margin_{\boldsymbol{P}}(a, b) > 0$.

Margin Graph

The **margin graph of** P, $\mathcal{M}(P)$, is the weighted directed graph whose set of nodes is $X(\mathbf{P})$ with an edge from a to b weighted by $Margin(a, b)$ when $Margin(a, b) > 0$. We write

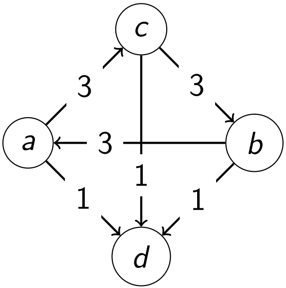
$$a \xrightarrow{\alpha}_{\mathbf{P}} b \text{ if } \alpha = Margin_{\mathbf{P}}(a, b) > 0.$$

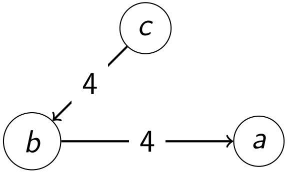
40	35	25
t	r	k
k	k	r
r	t	t



Margin Graph

A **margin graph** is a weighted directed graph $\mathcal{M}$ where all the weights have the same parity.





Theorem (Debord, 1987)

For any margin graph $\mathcal{M}$, there is a profile $\boldsymbol{P}$ such that $\mathcal{M}$ is the margin graph of $\boldsymbol{P}$.

A **voting method** is a function F on the domain of all profiles such that for any profile $\mathbf{P}$, $\varnothing \neq F(\mathbf{P}) \subseteq X(\mathbf{P})$ (also called a **variable social choice correspondence** VSCC).

A **variable-election collective choice rule** (VCCR) is a function f on the domain of all profiles such that for any profile $\mathbf{P}$, $f(\mathbf{P})$ is an asymmetric binary relation on $X(\mathbf{P})$, which we call *the defeat relation for* $\mathbf{P}$ *under* f.

For $x, y \in X(\mathbf{P})$, we say that x *defeats* y *in* $\mathbf{P}$ *according to* f when $(x, y) \in f(\mathbf{P})$.

Majority Defeat

Majority Defeat: if a candidate does not win in an election, they must have been *defeated* by some other candidate in the election,

Majority Defeat

Majority Defeat: if a candidate does not win in an election, they must have been *defeated* by some other candidate in the election, and a candidate should defeat another only if a *majority* of voters prefer the first candidate to the second.

Majority Defeat

> **Majority Defeat**: if a candidate does not win in an election, they must have been *defeated* by some other candidate in the election, and a candidate should defeat another only if a *majority* of voters prefer the first candidate to the second.

Widely used voting systems can violate the principle of Majority Defeat.

Majority Defeat

> **Majority Defeat**: if a candidate does not win in an election, they must have been *defeated* by some other candidate in the election, and a candidate should defeat another only if a *majority* of voters prefer the first candidate to the second.

Widely used voting systems can violate the principle of Majority Defeat.

Example

In the 2000 U.S. presidential election in Florida, George W. Bush defeated Al Gore and Ralph Nader according to Plurality voting, which only allows voters to vote for one candidate. Yet assuming that most Nader voters preferred Gore to Bush, it follows that a majority of all voters preferred Gore to Bush.

Majority Defeat

> **Majority Defeat**: if a candidate does not win in an election, they must have been *defeated* by some other candidate in the election, and a candidate should defeat another only if a *majority* of voters prefer the first candidate to the second.

Widely used voting systems can violate the principle of Majority Defeat.

Example

In the 2000 U.S. presidential election in Florida, George W. Bush defeated Al Gore and Ralph Nader according to Plurality voting, which only allows voters to vote for one candidate. Yet assuming that most Nader voters preferred Gore to Bush, it follows that a majority of all voters preferred Gore to Bush.

Under Plurality, Nader spoiled the election for Gore, handing victory to Bush.

Anonymity and Neutrality

Anonymity: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by swapping the ballots assigned to two voters, then x still defeats y in $\boldsymbol{P}'$.

Neutrality: if x defeats y in $\boldsymbol{P}$, and $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by swapping x and y on each voter's ballot, then y defeats x in $\boldsymbol{P}'$.

Availability: for all profiles $\boldsymbol{P}$, there is some undefeated candidate.

Monotonicity

Monotonicity (resp. Monotonicity for two-candidate profiles): if x defeats y in a profile (resp. two-candidate profile) $\boldsymbol{P}$, and $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by some voter i moving x above the candidate that i ranked immediately above x in $\boldsymbol{P}$, then x defeats y in $\boldsymbol{P}'$.

Lemma

If f satisfies Anonymity, Neutrality, and Monotonicity with respect to two-candidate profiles, then f satisfies Special Majority Defeat: for any two-candidate profile $\mathbf{P}$, x defeats y in $\mathbf{P}$ according to f only if x is majority preferred to y.

Other rules satisfying Anonymity, Neutrality and Monotonicity: The completely indecisive method; Unanimity; Quota rules (cf. Fishburn 1974, Section 1)

Neutral Reversal: if P' is obtained from P by adding two voters with reversed ballots, then x defeats y in P if and only if x defeats y in P'.

Neutral Reversal: if P' is obtained from P by adding two voters with reversed ballots, then x defeats y in P if and only if x defeats y in P'.

Pareto: if for all profiles P and $x, y \in X(P)$, if xP_iy for all $i \in V(P)$, then x defeats y in P.

Characterizing Majority Rule

Proposition

For any VCCR f, the following are equivalent:

1. *f coincides with majority rule on two-candidate profiles;*
2. *f satisfies the following axioms with respect to two-candidate profiles: Anonymity, Neutrality, Monotonicity, Pareto, and Upward Neutral Reversal.*

W. Holliday and EP. *Axioms for Defeat in Democratic Elections*. Forthcoming Journal of Theoretical Politics, `https://arxiv.org/pdf/2008.08451.pdf`, 2021.

Characterizing Majority Rule

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

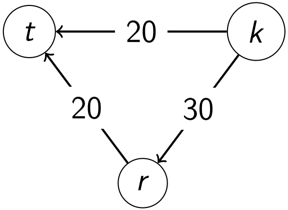
G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

More than 2 candidates...

40	35	25
t	r	k
k	k	r
r	t	t



Plurality winner	t
Instant Runoff winner	r
Borda winner	k
Condorcet winner	k
Condorcet loser	t

Positional scoring rules

A *scoring vector* is a vector $\langle s_1, \dots, s_n \rangle$ of numbers such that for each $m \in \{1, \dots, n-1\}$, $s_m \geq s_{m+1}$.

Given a profile $\mathbf{P}$ with $|X(\mathbf{P})| = n$, $x \in X(\mathbf{P})$, a scoring vector $\vec{s}$ of length n, and $i \in V(\mathbf{P})$, define $score_{\vec{s}}(x, \mathbf{P}_i) = s_r$ where $r = Rank(x, \mathbf{P}_i)$.

Let $score_{\vec{s}}(x, \mathbf{P}) = \sum_{i \in V(\mathbf{P})} score_{\vec{s}}(x, \mathbf{P}_i)$. A voting method F is a **positional scoring rule** if there is a map $\mathcal{S}$ assigning to each natural number n a scoring vector of length n such that for any profile $\mathbf{P}$ with $|X(\mathbf{P})| = n$,

$$F(\mathbf{P}) = \mathrm{argmax}_{x \in X(\mathbf{P})} score_{\mathcal{S}(n)}(x, \mathbf{P}).$$

Examples

Borda:	$\mathcal{S}(n) = \langle n-1, n-2, \ldots, 1, 0\rangle$
Plurality:	$\mathcal{S}(n) = \langle 1, 0, \ldots, 0\rangle$
Anti-Plurality:	$\mathcal{S}(n) = \langle 1, 1, \ldots, 1, 0\rangle$

1	3	2	4
a	*b*	*b*	*c*
c	*a*	*c*	*a*
b	*c*	*a*	*b*

Borda winner	*c*
Plurality winner	*b*
Anti-Plurality winner	*a*

Plurality vs. Borda

1	1
a	c
b	b
c	a

Plurality winners: a, b

Borda winners: a, b, c

Iterative procedures: Instant Runoff

- If some alternative is ranked first by an absolute majority of voters, then it is declared the winner.
- Otherwise, the alternative ranked first be the fewest voters (the plurality loser) is eliminated.
- Votes for eliminated alternatives get transferred: delete the removed alternatives from the ballots and "shift" the rankings (e.g., if 1st place alternative is removed, then your 2nd place alternative becomes 1st).

Also known as Ranked-Choice, STV, Hare

How should you deal with ties? (e.g., multiple alternatives are plurality losers)

Non-neutral tiebreaking: Fix a linear ordering of the candidates

Remove all: Remove all candidates tied for the smallest plurality score

Parallel universe tiebreaking: A candidate a wins if a wins according to some linear ordering of the candidates

1	3	2	1	1
c	c	b	a	a
a	b	a	c	b
b	a	c	b	c

Instant Runoff: $\{c\}$ Instant Runoff PUT: $\{a, c\}$

S. Obraztsova, E. Elkind and N. Hazon. *Ties Matter: Complexity of Voting Manipulation Revisited*. Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence.

J. Wang, S. Sikdar, T. Shepherd, Z. Zhao, C. Jiang and L. Xia. *Practical Algorithms for Multi-Stage Voting Rules with Parallel Universes Tiebreaking*. Proceedings of AAAI, 2019.

Iterative procedures

Variants:

- Plurality with runoff: remove all candidates except top two plurality score;
- Coombs: remove candidates with most last place votes;
- Baldwin: remove candidate with smallest Borda score;
- Nanson: remove candidates with below average Borda score

Example

1	1	1	1	1
c	b	a	b	d
a	d	b	c	a
d	a	c	d	b
b	c	d	a	c

Instant Runoff	$\{b\}$
Plurality with Runoff	$\{a, b\}$
Coombs	$\{d\}$
Baldwin	$\{a, b, d\}$
Strict Nanson	$\{a\}$