Puzzles, ESSLLI Riga

Hans van Ditmarsch & Eric Pacuit

Overview

- public information change: consecutive numbers, muddy children, sum and product
- non-public information change: gossip, one hundred prisoners
- probability and knowledge: surprise examination, lottery paradox, Ellsberg paradox
- probability and knowledge: Monty Hall, reviewer paradox, Judy Benjamin
- common knowledge and common belief: Byzantine generals, Brandenburg-Keisler paradox

Consecutive numbers

- Consecutive numbers

Consecutive numbers

Anne and Bill are each going to be told a natural number. Their numbers will be one apart. The numbers are now being whispered in their respective ears. They are aware of this scenario. Suppose Anne is told 2 and Bill is told 3. The following truthful conversation between Anne and Bill now takes place:

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number."
- Bill: "I know your number."

Explain why is this possible.

Consecutive numbers — representing uncertainties

(2,3)

Consecutive numbers — representing uncertainties

(2,1) — a — <u>(2,3)</u>

Consecutive numbers — representing uncertainties

$$(2,1) - a - \underline{(2,3)} - b - (4,3)$$

Consecutive numbers — representing uncertainties

$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3)$

Consecutive numbers — representing uncertainties

$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$

Consecutive numbers — representing uncertainties

(1,0) — a — (1,2) — b — (3,2) — a — (3,4) ——— ...

(0,1) — b — (2,1) — a — <u>(2,3)</u> — b — (4,3) ——— ...

Consecutive numbers — representing uncertainties

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne knows that her number is 2.
- Bill knows that Anne's number is 2 or 4.
- Anne and Bill commonly know that Bill's number is odd.
- ...

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: “I do not know your number.” ??

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number." eliminated states

Consecutive numbers — successive announcements

$(1{,}0) - a - (1{,}2) - b - (3{,}2) - a - (3{,}4) \text{———} \cdots$

$(2{,}1) - a - \underline{(2{,}3)} - b - (4{,}3) \text{———} \cdots$

- Anne: "I do not know your number."

Consecutive numbers — successive announcements

(1,0) — *a* — (1,2) — *b* — (3,2) — *a* — (3,4) ———— ⋯

(2,1) — *a* — <u>(2,3)</u> — *b* — (4,3) ———— ⋯

- Anne: "I do not know your number."
- Bill: "I do not know your number." ??

Consecutive numbers — successive announcements

$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$

$(2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$

- Anne: "I do not know your number."
- Bill: "I do not know your number." eliminated states

Consecutive numbers — successive announcements

$$(1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$\underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: “I do not know your number.”
- Bill: “I do not know your number.”

Consecutive numbers — successive announcements

$(1{,}2) - b - (3{,}2) - a - (3{,}4) \text{———} \cdots$

$\underline{(2{,}3)} - b - (4{,}3) \text{———} \cdots$

- Anne: “I do not know your number.”
- Bill: “I do not know your number.”
- Anne: “I know your number.” ??

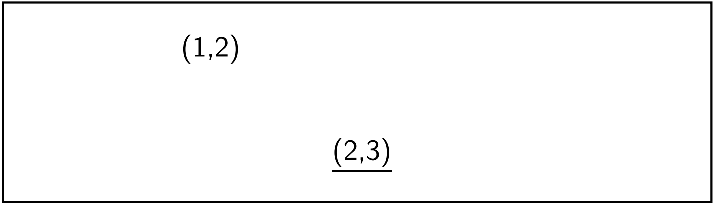
Consecutive numbers — successive announcements

(1,2) — b — (3,2) — a — (3,4) ——— · · ·

<u>(2,3)</u> — b — (4,3) ——— · · ·

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number." eliminated states

Consecutive numbers — successive announcements



- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number."

Consecutive numbers — successive announcements

(1,2)

(2,3)

- Anne: "I do not know your number."
- Bill: "I do not know your number."
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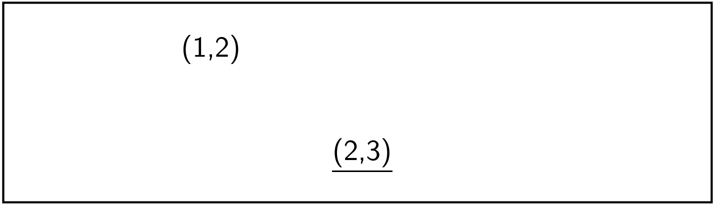
- Bill: "I know your number." ??

Consecutive numbers — successive announcements



- Anne: “I do not know your number.”
- Bill: “I do not know your number.”
- Anne: “I know your number.”
- Bill: “I know your number.” already common knowledge

Consecutive numbers — successive announcements

(1,2)

(2,3)

- Anne: “I do not know your number.”
- Bill: “I do not know your number.”
- Anne: “I know your number.”
- Bill: “I know your number.”

An alternative representation

Anne and Bill are each going to be told a natural number. Their numbers will be one apart. The numbers are now being whispered in their respective ears. They are aware of this scenario. Suppose Anne is told 2 and Bill is told 3.

Anne and Bill each have a natural number on their forehead. Their numbers are one apart. They only can see the number on the other's forehead. They are aware of this scenario. Suppose Anne has the number 3 and Bill has the number 2.

Muddy Children

- Muddy Children

A dynamic epistemic logic classic: Muddy Children

A group of children has been playing outside and are called back into the house by their father. The children gather round him. As one may imagine, some of them have become dirty from the play and in particular: they may have mud on their forehead. Children can only see whether other children are muddy, and not if there is any mud on their own forehead. All this is commonly known, and the children are, obviously, perfect logicians. Father now says: "At least one of you has mud on his or her forehead." And then: "Will those who know whether they are muddy step forward." If nobody steps forward, father keeps repeating the request. What happens?

Muddy Children

> Father says: "At least one of you has mud on his or her forehead." And then: "Will those who know whether they are muddy step forward." If nobody steps forward, father keeps repeating the request. What happens?

Let there be two children, Anne and Bill.

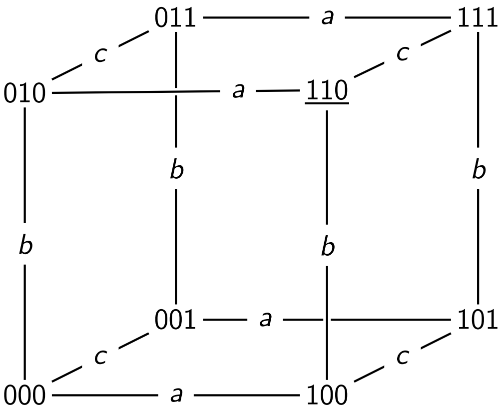
- Suppose neither Anne nor Bill is muddy. What happens?
- Suppose Anne is muddy. What happens?
- Suppose Bill is muddy. What happens?
- Suppose Anne and Bill are muddy. What happens?

Muddy Children

Let there be three children, Anne, Bill, and Cath.

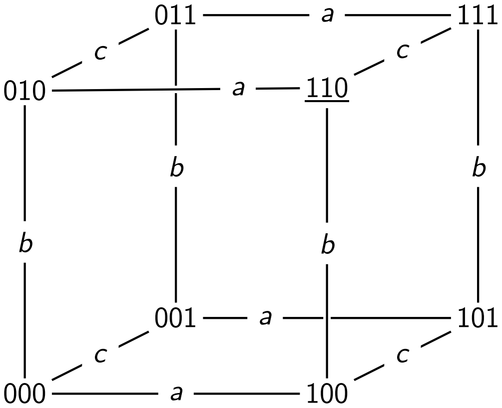
- Suppose one child is muddy. What happens?
- Suppose two children are muddy. What happens?
- Suppose all three children are muddy. What happens?

Muddy Children



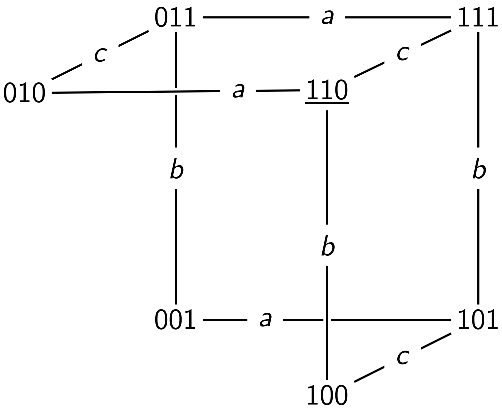
– The children can see each other.

Muddy Children



– The children can see each other.
– At least one of you has mud on his or her forehead.

Muddy Children



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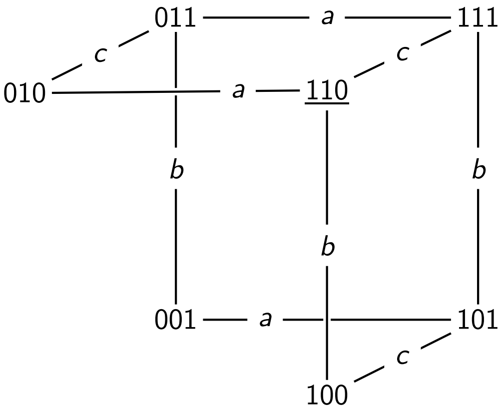
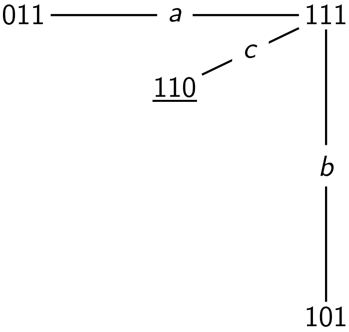
Muddy Children



- The children can see each other.
- At least one of you has mud on his or her forehead.
- Will those who know whether they are muddy step forward?

Muddy Children



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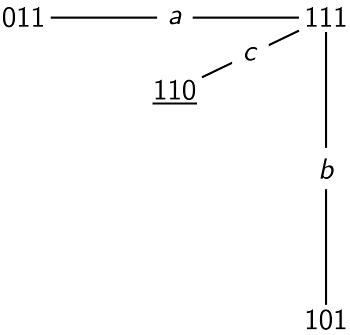
Muddy Children



– The children can see each other.
– At least one of you has mud on his or her forehead.
– Will those who know whether they are muddy step forward?
– Will those who know whether they are muddy step forward?

Muddy Children

110

– The children can see each other.
– At least one of you has mud on his or her forehead.
– Will those who know whether they are muddy step forward?
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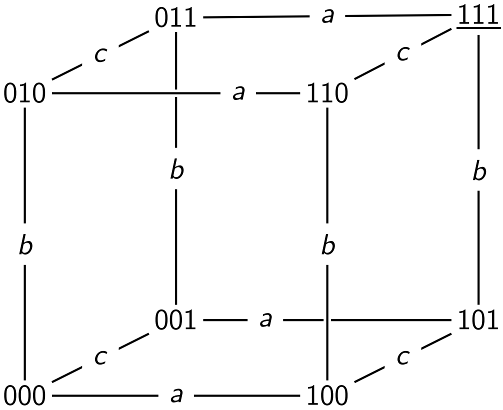
Muddy Children



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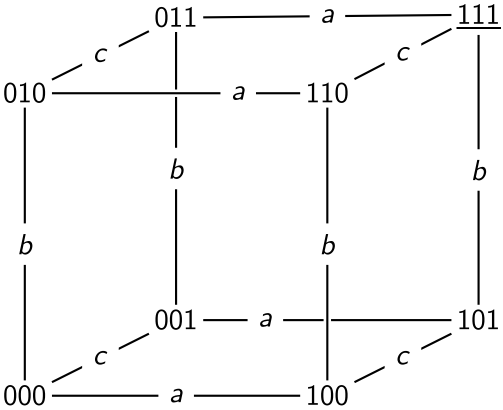
Muddy Children



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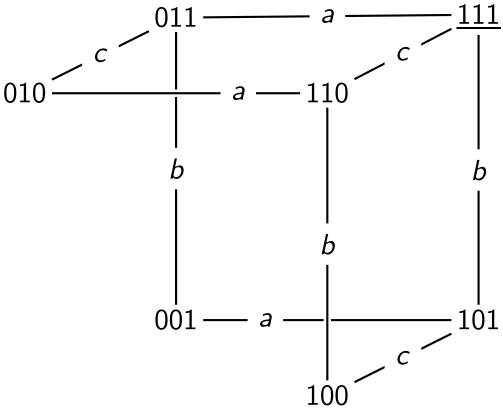
Muddy Children



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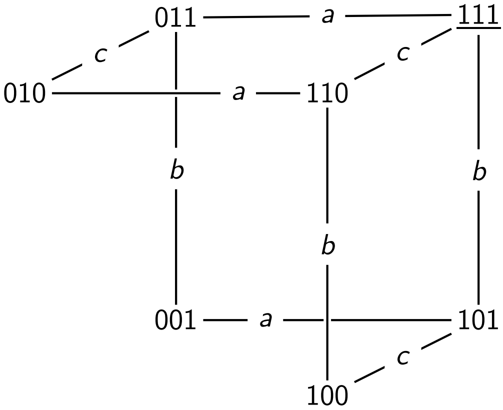
Muddy Children



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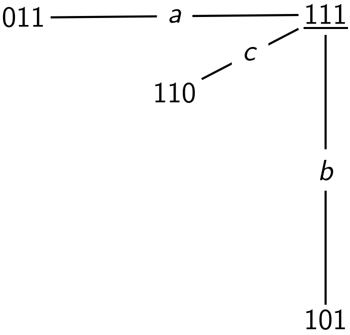
Muddy Children



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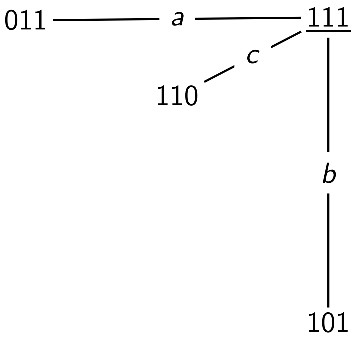
Muddy Children



– The children can see each other.
– At least one of you has mud on his or her forehead.
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Muddy Children

111

– The children can see each other.
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Muddy Children

111

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Muddy Children – proof by natural induction

Natural induction: given a statement $S(n)$ involving natural numbers $n \in \mathbb{N}$, we can prove that it holds for all $n \in \mathbb{N}$ if (i) it holds for $n = 1$ and if (ii) on the assumption that it holds for n it also holds for $n + 1$.

Let $S(n)$ be the following statement:

If father makes his statement 'Will those who know whether they are muddy step forward?' n times and nobody steps forward, then there are at least $n + 1$ muddy children.

We can prove this statement with natural induction.

Muddy Children – proof by natural induction

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Let $S(n)$ be the following statement:

If father makes his statement 'Will those who know whether they are muddy step forward?' n times and nobody steps forward, then there are at least $n + 1$ muddy children.

We can prove this statement with natural induction.

And so can the children, as they are perfect logicians.

Why is this relevant for logic? Unsuccessful updates!

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You don't know that I came to Latvia by bicycle in 1998.

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Liar!

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Liar!

- There are announcements that become false because you announce them.
- So, you cannot truthfully announce them twice.
- These are known as unsuccessful updates.
- In the Muddy Children problem, the unsuccessful update is "Nobody knows whether he/she is muddy".

Variations of Muddy Children

- Instead of not stepping forward after father's request "Will those who know whether they are muddy step forward?," the children answer "We already knew!" What happens?

Variations of Muddy Children

- Instead of not stepping forward after father's request "Will those who know whether they are muddy step forward?," the children answer "We already knew!" What happens?

Other variation:

- All children have a white or a black hat. The children do not stand in a circle, but in a line. They can only see the hats ahead of them. And they can see who steps forward. (This one is a bit boring.)

Variations of Muddy Children

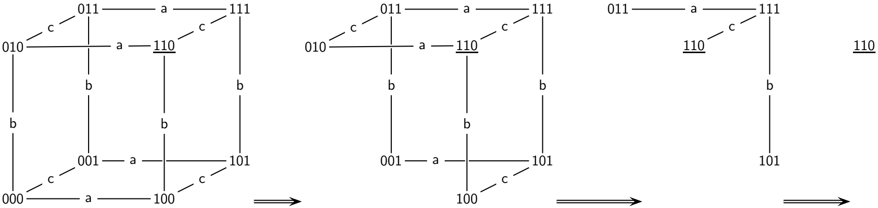
- All children have a white or a black hat. The children do not stand in a circle, but in a line. They can only see the hats ahead of them. From back to front, they consecutively announce the colour of their hat, “white” or “black”. Is there a protocol so that they all correctly announce the colour of their hat?

Variations of Muddy Children

- All children have a white or a black hat. The children do not stand in a circle, but in a line. They can only see the hats ahead of them. From back to front, they consecutively announce the colour of their hat, "white" or "black". Is there a protocol so that they all correctly announce the colour of their hat?

The children agree upon the following protocol: the child at the back says "white" if it sees an even number of white hats, and "black" if it sees an odd number of white hats. This announcement has a 50% chance of being correct. All other children now correctly announce the colour of their hat. (E.g., if the announcement was "white" and the next in line sees an even number of white hats, it knows that its hat is black, and announces "black". Otherwise, it is white, and it announces "white". And so on.)

Muddy Children — overview

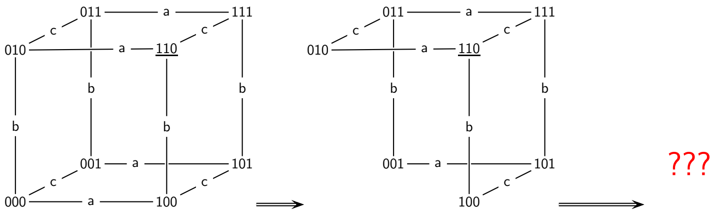


- At least one of you has mud on his or her forehead.
- Will those who know whether they are muddy step forward?
- Will those who know whether they are muddy step forward?

Muddy Children — with cleaning!!

Suppose that after telling the children that at least one of them is muddy... father empties a bucket of water over Anne (splash!).

Muddy Children — with cleaning



- At least one of you has mud on his or her forehead.
- Father empties a bucket of water over Anne (splash!). ?
- Will those who know whether they are muddy step forward? ?
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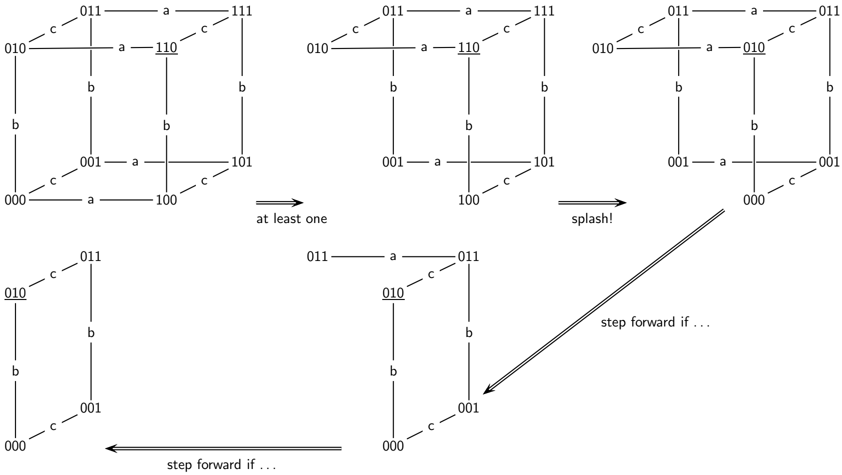


- Last step Cath learns that Anne knows that she **was** muddy.
- Relevant for logic? Factual and informational change interact!
- Relevant for artificial intelligence? Computational frame problem!

Muddy Children with lying

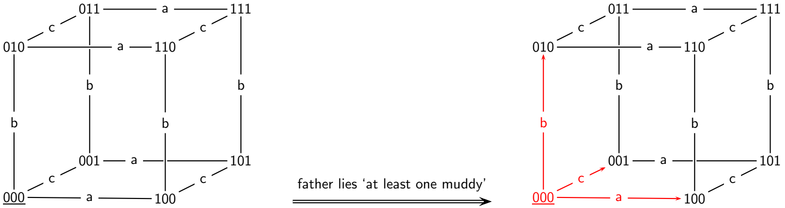
There are three ways to lie in the muddy children problem.

- Father is lying that at least one child is muddy.
- A child incorrectly steps forward when it should not.
- A child incorrectly does not step forward when it should have.

Various issues here.

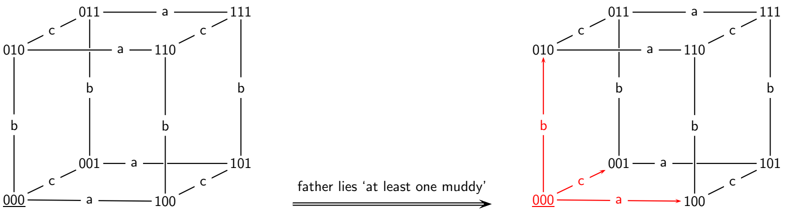
- After a lie children may no longer consider the actual state of affairs possible.
- What if a child does not step forward by mistake or because of remaining doubt, even when it should have?
- What is the difference between lying and being mistaken?

Muddy Children — father is lying



- If nobody was muddy, all children now have incorrect beliefs.
- What will happen next?

Muddy Children — father is lying



- If nobody was muddy, all children now have incorrect beliefs.
- What will happen next?

Other scenarios:

- Anne and Bill are muddy. Anne steps forward the first time.
  What does Bill conclude? What does Cath conclude?
- Anne and Bill are muddy. Bill doesn't step forward 2nd time.
  What does Anne conclude? What does Cath conclude?

On the origin of Muddy Children

- Moses, Dolev and Halpern, *Cheating husbands and other stories: a case study in knowledge, action, and communication*, Distributed Computing, 1986
- van Tilburg, *Doe wel en zie niet om (Do well and don't look back)*, Katholieke Illustratie (Catholic Illustrated Journal), 1956
- Littlewood, *A Mathematician's Miscellany*, 1953
- British Magazines, 1940s
- . . . ?

On the origin of Muddy Children

German translation of Rabelais' *Gargantua et Pantagruel*:
Gottlob Regis, *Meister Franz Rabelais der Arzeney Doctoren Gargantua und Pantagruel, usw.*, Barth, Leipzig, 1832.

Ungelacht pfetz ich dich. Gesellschaftsspiel. Jeder zwickt seinen rechten Nachbar an Kinn oder Nase; wenn er lacht, giebt er ein Pfand. Zwei von der Gesellschaft sind nämlich im Complot und haben einen verkohlten Korkstöpsel, woran sie sich die Finger, und mithin denen, die sie zupfen, die Gesichter schwärzen. Diese werden nun um so lächerlicher, weil jeder glaubt, man lache über den anderen.

I pinch you without laughing. Parlour game. Everybody pinches his right neighbour into chin or nose; if one laughs, one must give a pledge. Two in the round have secretly blackened their fingers on a charred piece of cork, and hence will blacken the faces of their neighbours. These neighbours make a fool of themselves, since they both think that everybody is laughing about the other one.

Sum and product

- Sum and Product

Sum and product

A says to S and P: I have chosen two integers x, y such that $1 < x < y$ and $x + y \leq 100$. In a moment, I will inform S only of $s = x + y$, and P only of $p = xy$. These announcements remain private. You are required to determine the pair (x, y).
He acts as said. The following conversation now takes place:

1. P says: "I do not know it."
2. S says: "I knew you didn't."
3. P says: "I now know it."
4. S says: "I now also know it."

Determine the pair (x, y).

Sum and product — history

Originally stated, in Dutch, by Hans Freudenthal.
Nieuw Archief voor Wiskunde 3(17):152, 1969.
Became popular in AI by way of John McCarthy, Martin Gardner.

Towards a solution: first announcement

A says to S and P: I have chosen two integers x, y such that $1 < x < y$ and $x + y \leq 100$. In a moment, I will inform S only of $s = x + y$, and P only of $p = xy$. These announcements remain private. You are required to determine the pair (x, y).
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1. ~~P says: "I do not know it."~~
2. S says: "I knew you didn't."
3. P says: "I now know it."
4. S says: "I now also know it."

Determine the pair (x, y).

All four announcements are informative.
The second announcement implies the first announcement.

Towards a solution: $(2, 3)$ and $(14, 16)$

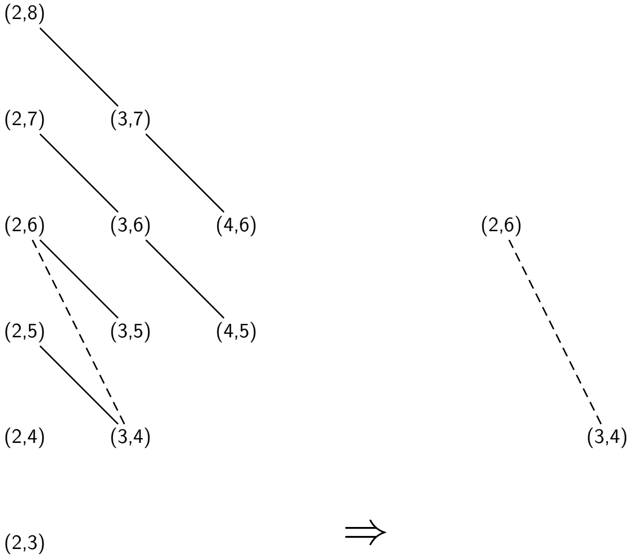
If the numbers were 2 and 3, then P deduces the pair from their product: $6 = 2 \cdot 3$ and $6 = 1 \cdot 6$, but the numbers are larger than 1 (two integers x, y such that $1 < x < y$ and $x + y \leq 100$).

If the numbers were prime, then P deduces the pair, because of the unique factorization of the product.

If the numbers were 14 and 16, then their sum 30 is also the sum of 7 and 23. If they had been 7 and 23 P would know the numbers. But if they had been 14 and 16, P would not know the numbers because this is also the product of 7 and 32, or of 28 and 8 (and also of 2 and 102: but that's out, because $2 + 102 > 100$!). Therefore, S considers is possible that P knows the numbers and that P does not know the numbers. In other words: S does not know that P does not know the numbers.

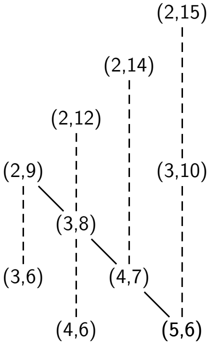
Towards a solution: $1 < x < y$ and $x + y \leq 10$

P: "I do not know it." S: "I now know it." P: "I do not know it."



For sum 11 S knows that P does not know

sum 11	*product*	*other sum, same product*
$(2,9)$	18	$(3,6)$
$(3,8)$	24	$(4,6),(2,12)$
$(4,7)$	28	$(2,14)$
$(5,6)$	30	$(2,15),(3,10)$



Second announcement: S says: “I knew you didn’t.”

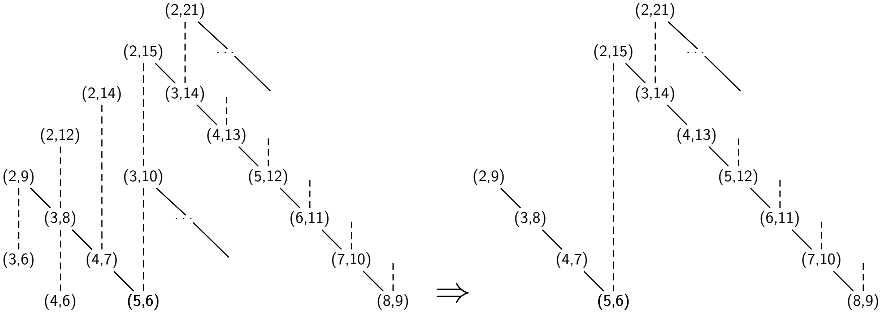
Remaining sums are

$$11, 17, 23, 27, 29, 35, 37, 41, 47, 53$$

For sum 11, see previous slide. For another example, sum 17:

sum 17	*product*	*other sum, same product*
$(2, 15)$	30	$(3, 10), (5, 6)$
$(3, 14)$	42	$(2, 21), (6, 7)$
$(4, 13)$	52	$(2, 26)$
$(5, 12)$	60	$(2, 30), (3, 20), (4, 15), (6, 10)$
$(6, 11)$	66	$(2, 33), (3, 22)$
$(7, 10)$	70	$(2, 35), (5, 14)$
$(8, 9)$	72	$(2, 36), (3, 24), (4, 18)$

Second announcement: S says: “I knew you didn’t.”



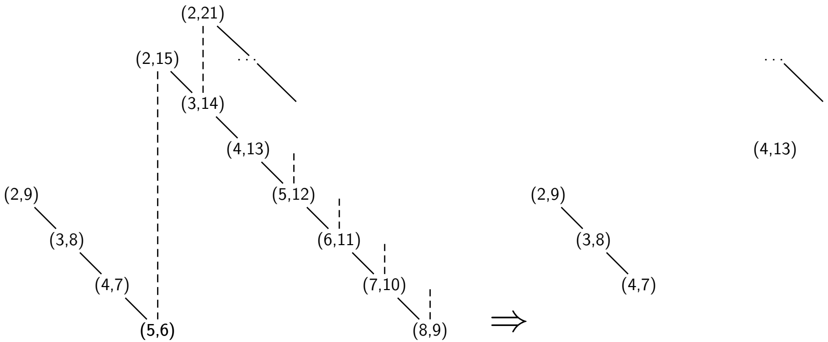
Third announcement: P says: "I now know it."

Call a number pair **closed** if P knows what the numbers are. Otherwise, it is **open**. Because P says he now knows, all open pairs can be eliminated. Example for sum 17:

sum 17	*product*	*other sum, same product*
$(2, 15)$	30	$(5, 6)$
$(3, 14)$	42	$(2, 21)$
$(4, 13)$	52	—
$(5, 12)$	60	$(3, 20)$
$(6, 11)$	66	$(2, 33)$
$(7, 10)$	70	$(2, 35)$
$(8, 9)$	72	$(3, 24)$

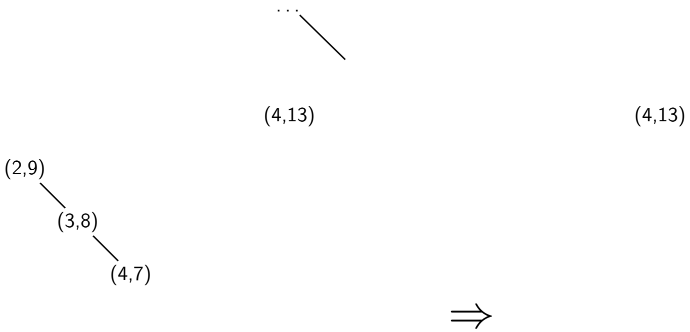
Third announcement: *P* says: “I now know it.”

Remove open pairs.



Fourth announcement: S says: "I now also know it."

Of the remaining lines with the same sum, the one for sum 17 is the only one that contains a single pair, $(4, 13)$.



Gossip

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- The minimum of calls required is $2n - 4$.
- Let ij mean that friend i and friend j make a call.
- For $n = 4$, the four required calls are 12 34 13 24.
- For $n = k + 4$, first friend 1 makes k calls to all friends > 4: 15 16 . . . , then friends 1 to 4 make calls 12 34 13 24 as above, then friend 1 again makes the same k calls to all friends > 4.
- For $n = 6$, this makes $12 - 4 = 8$ calls.
- Cor Hurkens proved that $2n - 4$ is also the minimum.

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If gossip is the goal, prolonging gossip is better! The *maximum* number of calls to spread all the news is $\binom{n}{2} = (n \cdot n - 1)/2$.

For example:

12 13 14 15 16   23 24 25 26   34 35 36   45 46   56

This is also the maximum number of different calls.

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This is also the maximum number of different calls.

What is the maximum in case they need not exchange all the secrets? Suppose exactly one secret is exchanged in one call.

Gossip

- The minimum number of calls $2n - 4$ is $O(n)$.
- The maximum number of calls $n(n-1)/2$ is $O(n^2)$.
- The average number of calls, if all calls take place randomly, is $O(n \log n)$.
- This is also known as the complexity of the *coupon collector* problem.
- Every day, or even more than once every day, I go to MacDonalds in Nancy to buy a coffee and I get an order number between 00 and 99. How many coffees do I have to order before I received all different order numbers?

Variations of gossip:

- *knowledge* conditions for calls ('only call someone whose secret you do not know', etc.)
- exchange of secrets *and knowledge* ('I know these secrets and Claire knows them too')
- multi-cast, broadcast, ...

One hundred prisoners and a light bulb

- One hundred prisoners and a light bulb

One hundred prisoners and a lightbulb

A group of 100 prisoners, all together in the prison dining area, are told that they will be all put in isolation cells and then will be interrogated one by one in a room containing a light with an on/off switch. The prisoners may communicate with one another by toggling the light-switch (and that is the only way in which they can communicate). The light is initially switched off. There is no fixed order of interrogation, or interval between interrogations, and the same prisoner may be interrogated again at any stage. When interrogated, a prisoner can either do nothing, or toggle the light-switch, or announce that all prisoners have been interrogated. If that announcement is true, the prisoners will (all) be set free, but if it is false, they will all be executed. While still in the dining room, and before the prisoners go to their isolation cells (forever), can the prisoners agree on a protocol that will set them free?

100 prisoners — not a solution

Let there be one prisoner:

Protocol: *If a prisoner enters the interrogation room, he announces that all prisoners have been interrogated.*

Let there be two prisoners:

Protocol: *If a prisoner enters the interrogation room and the light is off, he turns it on, if a prisoner enters the interrogation room and the light is on, he announces that all prisoners have been interrogated.*

Let there be three prisoners:

Protocol: ...

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Protocol: ... Prisoners may perform different roles.

100 prisoners — solution

Protocol for $n > 3$ prisoners:

The n prisoners appoint one amongst them as the counter. All non-counting prisoners follow the following protocol: the first time they enter the room when the light is off, they turn it on; on all other occasions, they do nothing. The counter follows a different protocol. The first $n-2$ times that the light is on when he enters the interrogation room, he turns it off. Then the next time he enters the room when the light is on, he (truthfully) announces that everybody has been interrogated.

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What if it is not known whether the light is initially on?
Same count, you may get hanged (namely if light was on).
One higher, you may never terminate (namely if light was off).
???

100 prisoners — solution if light may be on or off

The n prisoners appoint one amongst them as the counter. All non-counting prisoners follow the following protocol: the ~~first time~~ first two times they enter the room when the light is off, they turn it on; on all other occasions, they do nothing. The counter follows a different protocol. The first ~~$n-2$~~ $2n-3$ times that the light is on when he enters the interrogation room, he turns it off. Then the next time he enters the room when the light is on, he (truthfully) announces that everybody has been interrogated.

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For $n = 100$, the next entry (198) after 197 switches:

light was off and 99 non-counters have been interrogated twice
light was on and 98 non-counters twice and one once only.

Either way is fine!

100 prisoners — knowing before the counter

After a non-counter has turned the light on, he counts the number of times he sees the sequence ‘light off – light on’.

If this is 98 times, all have been interrogated.

His announcement will then be before the counter’s.

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How likely is this to happen? (three prisoners, 100 prisoners, ...)

If there are three prisoners, the probability that the non-counters find out before the counter is 50%.

If there are 100 prisoners, the probability that the non-counters find out before the counter is (upper bound) $5.63 \cdot 10^{-72}$.

Three prisoners — a uniform role protocol

Each prisoner holds a token initially worth one point. Turning the light on if it is off, means dropping one point. Leaving the light on if it is on, means not being able to drop one point. Turning the light off if it is on, means collecting one point. Leaving the light off if it is off, means not being able to collect one point.

> **Protocol**: Let your token be m points. If the light is on, add one. Let a function $Pr : \{0, ..., n\} \to [0, 1]$ be given, with $Pr(0) = Pr(1) = 1$, $0 < Pr(x) < 1$ for $x \neq 0, 1, n$, and $Pr(n) = 0$. Drop your point with probability $Pr(m+1)$, otherwise, collect it. The protocol terminates once a prisoner has collected n points.

Consider 4 prisoners a, b, c, d. Choose $Pr(0) = Pr(1) = 1$, $Pr(2) = 0.5$, $Pr(3) = 0$, $Pr(4) = 0$. An interrogation sequence:

$$- : a_1 + : b_2^1 + : c_2^0 - : d_1 + : b_2^0 - : c_2^0 - : c_2^1 + : b_3 - : c_1 + : b_4$$

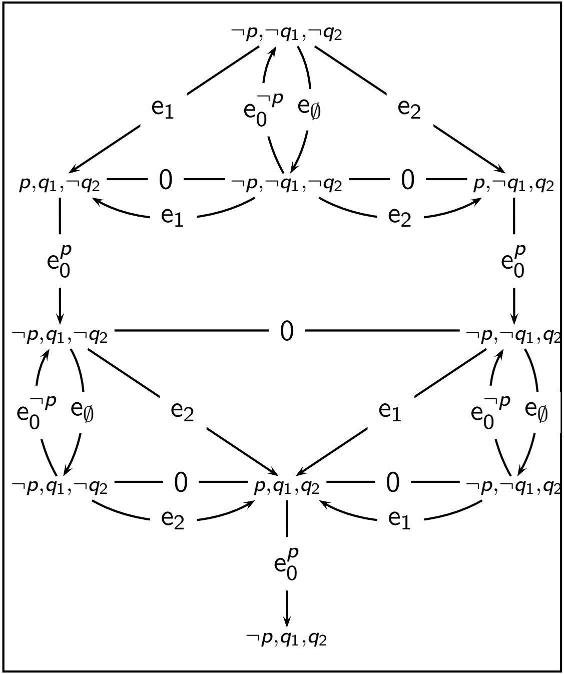
Three prisoners — a dash of logic

- Prisoner 0 is counter. Prisoners 1 and 2 are non-counters.
- p stands for 'the light is on'.
- q_1 stands for 'prisoner 1 has turned on the light'.
- q_2 stands for 'prisoner 2 has turned on the light'.
- $\neg p$ stands for 'not p', and $p \rightarrow q$ stands for 'p implies q'.

event	*description*
$e_\emptyset$	nothing happens
e_1	p becomes $q_1 \rightarrow p$, and q_1 becomes $p \rightarrow q_1$
e_2	p becomes $q_2 \rightarrow p$, and q_2 becomes $p \rightarrow q_2$
$e_0^{\neg p}$	if $\neg p$, nothing happens
e_0^{p}	if p, p becomes false

How the events appear to prisoner 0:

- he cannot distinguish e_1, e_2, and $e_\emptyset$
- he can distinguish e_0^{p} from the rest
- he can distinguish $e_0^{\neg p}$ from the rest



¬p,¬q1,¬q2
e1
e0¬p
e∅
e2
p,q1,¬q2
0
¬p,¬q1,¬q2
0
p,¬q1,q2
e1
e2
e0p
e0p
¬p,q1,¬q2
0
¬p,¬q1,q2
e0¬p
e∅
e2
e1
e0¬p
e∅
¬p,q1,¬q2
0
p,q1,q2
0
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100 prisoners — synchronization

Assume a single interrogation per day takes place.
When can the prisoners expect to be set free from prison?

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non-counter / counter / another non-counter / counter / etc.

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Summation:

$$\sum_{i=1}^{99}\left(\frac{100}{i}+\frac{100}{1}\right) = 99 \cdot 100 + 100 \cdot \sum_{i=1}^{99}\frac{1}{i} = 9,900 + 518 \text{ days } \approx 28.5 \text{ years}$$

100 prisoners — improvements given synchronization

Dynamic counter assignment (protocol in two stages):

- stage 1, 99 days: the first prisoner to enter the room twice turns on the light. (Expectation: 13 days.)
- stage 1, day 100: if light off, done; otherwise, turn light off.
- stage 2, from day 101: as before, except that:
  counter twice interrogated on day n counts until $100 - n$ only;
  non-counters who only saw light off in stage 1: do nothing;
  non-counters who saw light on in stage 1: do the usual. (24 y)

Head counter and assistant counters (iterated protocol, 2 stages):

- stage 1: head and assistant counters count to agreed max. n;
- stage 2: head counter collects from *successful* assistants;
- repeat stage 1 (unsuccessful assistants continue counting to n) and stage 2 (not yet collected successful assistants, and newly successful assistants) until termination. (9 years)

Minimum not known!

Reading

A source for many puzzles is:

- Hans van Ditmarsch and Barteld Kooi.
  One Hundred Prisoners and a Light Bulb. Copernicus, 2015.
- Available in Dutch, English, Japanese, Chinese editions

See http://personal.us.es/hvd/lightbulb.html