Aggregating Judgements: Logical and Probabilistic Approaches

Lecture 1

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August 6, 2018

Plan

Monday Representing judgements; Introduction to judgement aggregation; Aggregation paradoxes

Tuesday Axiomatic characterizations of aggregation methods I

Wednesday Axiomatic characterizations of aggregation methods II, Distance-based characterizations

Thursday Opinion pooling; Merging of probabilistic opinions (Blackwell-Dubins Theorem); Aumann's agreeing to disagree theorem and related results

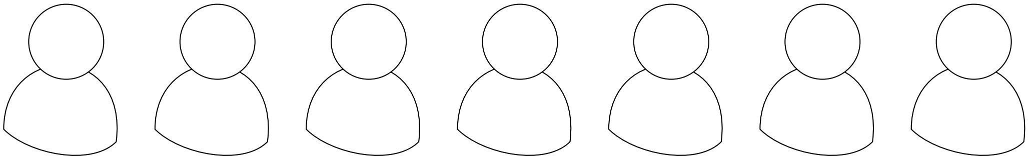
Friday Belief polarization; Diversity trumps ability theorem (The Hong-Page Theorem)

Judgement aggregation model

- Group of experts
- Agenda
- Judgement
- Aggregation method

Group of experts

- Evidence: shared or independent
- Communication: Allow communication/sharing of opinions
- Opinionated
- Coherent: logically and/or probabilistically



Agenda

- Single issue/proposition
- Set of independent issues/propositions
- Set of logically connected issues/propositions

Is P true?

Do you accept P_1?
Do you accept P_2?
$\vdots$
Do you accept P_n?

Do you accept P?
Do you accept $P \rightarrow Q$?
Do you accept Q?

Agenda

- Single issue/proposition
- Set of independent issues/propositions
- Set of logically connected issues/propositions
- Value from some range (quantity/chance)
- Causal relationships between variables

What is the chance that E will happen?

What is the value of x?

Which intervention will be most effective?

Judgements

- Expressions of judgement vs. expressions of preference
- Qualitative: Accept/Reject; Orderings; Grades

Accept P

P_1	P_2	$\cdots$	P_n
Y	N	$\cdots$	Y

$P \geq Q \geq R \geq \cdots$

Reject P

P	$P \to Q$	Q
Y	N	N

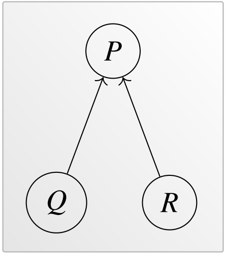
P is very likely
Q is very likely
R is very unlikely
$\vdots$

Judgements

- Expressions of judgement vs. expressions of preference
- Qualitative: Accept/Reject; Orderings; Grades
- Quantitative: Probabilities; Imprecise probabilities
- Causal models
- Do the experts provide their reasons/arguments/confidence?

$$\boxed{Pr(P) = p}$$

$$\boxed{Pr(P) = [l, h]}$$

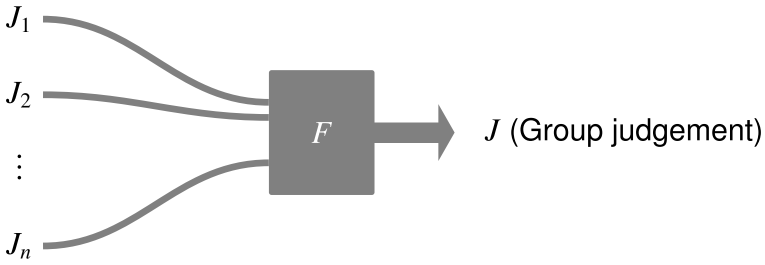


Judgements

localhost
Tutorial: What is the chance that humans will land on Mars by 2024?
This question is in round 1, and round 2 will open on 31 Dec, 2022 at 06:59 (EDT).
DETAILS
ANSWER
DISCUSS
RESULTS
Estimate
2 Develop
3 Think
4 Submit
75%
20%
5%
0%
60%
75%
90%
0%
50%
100%
?

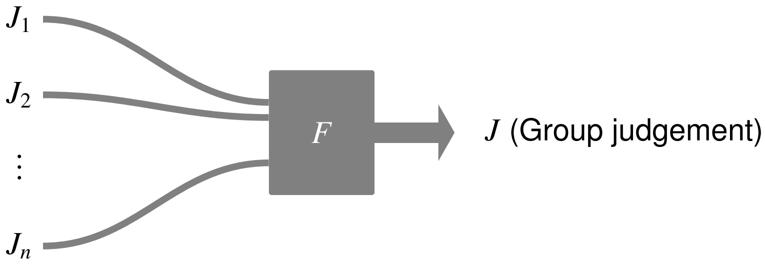
Aggregation method

- Functions from *profiles* of judgements to judgements.
- Is the group judgement the same type as the individual judgements?
- Hides *disagreement* among the experts.

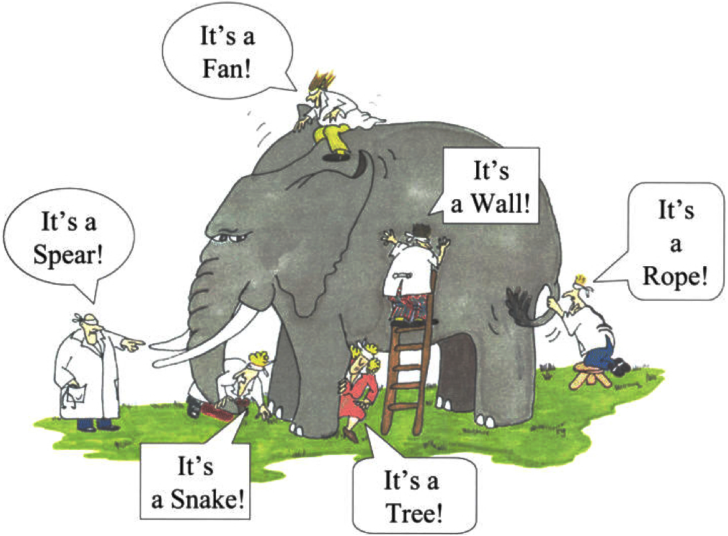


Aggregation method

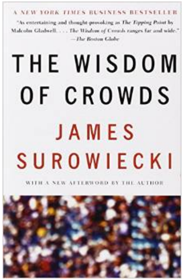
- Epistemic considerations: How likely is it that the group judgement is *correct*?
- Procedural/fairness considerations: Does the group judgement *reflect* the individual judgements?

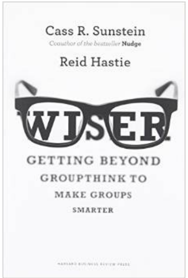


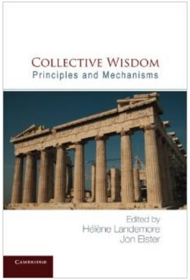
Wisdom of the Crowds

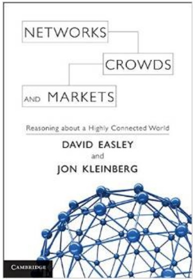


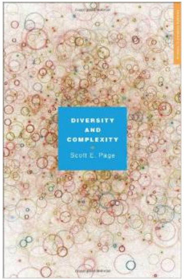
Collective Intelligence











Collective wisdom

A. Lyon. *Collective wisdom*. forthcoming, Journal of Philosophy, `http://aidanlyon.com/media/publications/WoCC.pdf`.

A. Lyon and EP. *The wisdom of crowds: Methods of human judgement aggregation*. Handbook of Human Computation, pp. 599 - 614, 2013.

C. Sunstein. *Deliberating groups versus prediction markets (or Hayek's challenge to Habermas)*. Episteme, 3:3, pgs. 192 - 213, 2006.

A. B. Kao and I. D. Couzin. *Decision accuracy in complex environments is often maximized by small group sizes*. Proceedings of the Royal Society: Biological Sciences, 281(1784), 2014.

Group of experts

Assume that there are an odd number of experts

Agenda:

A single proposition P

Judgements:

Accept P/Judge that P is true

Reject P/Judge that P is false

Suspend judgement about P

Aggregation method

Majority rule: Accept P if more people accept P than reject P; Reject P if more people reject P than accept P

Group of voters

Assume that there are an odd number of experts

Candidates:

Two candidates A and B

Preferences:

Rank A above B

Rank B above A

Indifferent between A and B

Aggregation method

Majority rule: A wins if more voters rank A above B than B above A; B wins if more voters rank B above A than A above B;

Characterizing Majority Rule

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n experts/voters.

Aggregation function: $F : \{1, 0, -1\}^n \rightarrow \{1, 0, -1\}$,
where

- 1 means Accept P or rank A above B
- -1 means Reject P or rank B above A
- 0 means Suspend judgement about P or A and B are tied

May's Theorem: Details

Aggregation function: $F : \{1, 0, -1\}^n \to \{1, 0, -1\}$

For $\mathbf{v} \in \{1, 0, -1\}^n$ and $x \in \{1, 0, -1\}$, let $\mathbf{N_v}(x) = \{i \in N \mid \mathbf{v}_i = x\}$

$$F_{Maj}(\mathbf{v}) = \begin{cases} 1 & \text{if } |\mathbf{N_v}(1)| > |\mathbf{N_v}(-1)| \\ 0 & \text{if } |\mathbf{N_v}(1)| = |\mathbf{N_v}(-1)| \\ -1 & \text{if } |\mathbf{N_v}(-1)| > |\mathbf{N_v}(1)| \end{cases}$$

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there?

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there? $19,683$

- There are three possible rankings for 2 candidates.
- When there are two voters there are $3^2 = 9$ possible profiles:

$$\{(1,1),(1,0),(1,-1),(0,1),(0,0),(0,-1),(-1,1),(-1,0),(-1,-1)\}$$

- Since there are 9 profiles and 3 rankings, there are $3^9 = 19,683$ possible preference aggregation functions.

May's Theorem: Details

- **Anonymity**: all voters should be treated equally.

- **Neutrality**: all candidates should be treated equally.

May's Theorem: Details

- **Anonymity**: all voters should be treated equally.

  $F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where $v_i \in \{1, 0, -1\}$ and π is a permutation of the voters.

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- **Neutrality**: all candidates should be treated equally.

  $F(-\mathbf{v}) = -F(\mathbf{v})$ where $-\mathbf{v} = (-v_1, \ldots, -v_n)$.

May's Theorem: Details

- **Positive Responsiveness** (Monotonicity): unidirectional shift in the voters' opinions should help the alternative toward which this shift occurs.

For any $\mathbf{v}, \mathbf{v}' \in \{1, 0, -1\}$ with $\mathbf{v}'_i \geq \mathbf{v}_i$ for all $i \in N$ and $\mathbf{v}'_j > \mathbf{v}_j$ for some $j \in N$ we have $F(\mathbf{v}) \in \{0, 1\}$ implies that $F(\mathbf{v}') = 1$.

Similarly, for any $\mathbf{v}, \mathbf{v}' \in \{1, 0, -1\}$ with $\mathbf{v}'_i \leq \mathbf{v}_i$ for all $i \in N$ and $\mathbf{v}'_j < \mathbf{v}_j$ for some $j \in N$ we have $F(\mathbf{v}) \in \{0, -1\}$ implies that $F(\mathbf{v}') = -1$.

Warm-up Exercise

Suppose that there are two experts/voters. How many aggregation functions satisfy anonymity?

Anonymity: all voters should be treated equally.

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

Warm-up Exercise

Suppose that there are two experts/voters. How many aggregation functions satisfy anonymity? 729

Anonymity: all voters should be treated equally.

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- Imposing anonymity reduces the number of preference aggregation functions.
- If F satisfies anonymity, then $F(1,0) = F(0,1)$, $F(1,-1) = F(-1,1)$ and $F(-1,0) = F(0,-1)$.
- This means that there are essentially 6 elements of the domain. So, there are $3^6 = 729$ preference aggregation functions.

Anonymity

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

Alternative definition of anonymity:

For all $\mathbf{v}$, $F(\mathbf{v}) = \text{sgn}(\sum_{i \in N} v_i)$ where,

$$\text{sgn}(r) = \begin{cases} 1 & r > 0 \\ 0 & r = 0 \\ -1 & r < 0 \end{cases}$$

May's Theorem (1952) A social decision method F satisfies neutrality, anonymity and positive responsiveness iff F is majority rule.

Proof Idea

For any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| = |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 0$.

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If $(1, 0, -1)$ is assigned 1 or -1 then

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If $(1, 0, -1)$ is assigned 1 or -1 then

✓ Anonymity implies $(-1, 0, 1)$ is assigned 1 or -1

Proof Idea

For any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| = |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 0$.

If $(1, 0, -1)$ is assigned 1 or -1 then

- ✓ Anonymity implies $(-1, 0, 1)$ is assigned 1 or -1
- ✓ Neutrality implies $(1, 0, -1)$ is assigned -1 or 1
  Contradiction.

Proof Idea

For any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| > |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 1$.

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If $(1, 1, -1)$ is assigned 0 or −1 then

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Proof Idea

For any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| > |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 1$.

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Proof Idea

For any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| > |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 1$.

If $(1, 1, -1)$ is assigned 0 or -1 then

- ✓ Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1

- ✓ Anonymity implies $(1, -1, -1)$ is assigned 0 or 1

- ✓ Positive Responsiveness implies $(1, 0, -1)$ is assigned 1

Proof Idea

For any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| > |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 1$.

If $(1, 1, -1)$ is assigned 0 or -1 then

- ✓ Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- ✓ Anonymity implies $(1, -1, -1)$ is assigned 0 or 1
- ✓ Positive Responsiveness implies $(1, 0, -1)$ is assigned 1
- ✓ Positive Responsiveness implies $(1, 1, -1)$ is assigned 1
  Contradiction.

Other characterizations

G. Asan and R. Sanver. *Another characterization of the majority rule*. Economics Letters, 75 (3), pgs. 409-413, 2002.

D. E. Campbell and J.S. Kelly. *A simple characterization of majority rule*. Economic Theory 15, pgs. 689 - 700, 2000.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Positive Responsiveness

For any $\mathbf{v}, \mathbf{v}' \in \{1, 0, -1\}$ with $\mathbf{v}'_i \geq \mathbf{v}_i$ for all $i \in N$ and $\mathbf{v}'_j > \mathbf{v}_j$ for some $j \in N$ we have $F(\mathbf{v}) \in \{0, 1\}$ implies that $F(\mathbf{v}') = 1$.

Similarly, for any $\mathbf{v}, \mathbf{v}' \in \{1, 0, -1\}$ with $\mathbf{v}'_i \leq \mathbf{v}_i$ for all $i \in N$ and $\mathbf{v}'_j < \mathbf{v}_j$ for some $j \in N$ we have $F(\mathbf{v}) \in \{0, -1\}$ implies that $F(\mathbf{v}') = -1$.

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Similarly, for any $\mathbf{v}, \mathbf{v}' \in \{1, 0, -1\}$ with $\mathbf{v}'_i \leq \mathbf{v}_i$ for all $i \in N$ and $\mathbf{v}'_j < \mathbf{v}_j$ for some $j \in N$ we have $F(\mathbf{v}) \in \{0, -1\}$ implies that $F(\mathbf{v}') = -1$.

Pareto Optimality: For any $\mathbf{v}$, if $\mathbf{v}_i \geq 0$ for all $i \in N$ and $\mathbf{v}_j = 1$ for some $j \in N$, then $F(\mathbf{v}) = 1$. Similarly, for any $\mathbf{v}$, if $\mathbf{v}_i \leq 0$ for all $i \in N$ and $\mathbf{v}_j = -1$ for some $j \in N$, then $F(\mathbf{v}) = -1$.

Monotonicity: For all $\mathbf{v}, \mathbf{v}'$, if $\mathbf{v} \leq \mathbf{v}'$ (i.e., $\mathbf{v}_i \leq \mathbf{v}'_i$ for all $i \in N$), then $F(\mathbf{v}) \leq F(\mathbf{v}')$

Aggregation function with variable domain: $F : \bigcup_{n \in \mathbb{N}} \{-1, 0, 1\}^n \to \{-1, 0, 1\}$.

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For $n, n' \in \mathbb{N}$, let $N = \{1, \ldots, n\}$ and $N' = \{n + 1, \ldots, n + n'\}$ be disjoint populations. If $\mathbf{v} \in \{-1, 0, 1\}^n$ and $\mathbf{v}' \in \{-1, 0, 1\}^{n'}$, then let:

$$\mathbf{v} \oplus \mathbf{v}' = (v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+n'})$$

Aggregation function with variable domain: $F : \bigcup_{n\in\mathbb{N}}\{-1,0,1\}^n \to \{-1,0,1\}$.

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Path Independence: For all $\mathbf{v} \in \{-1,0,1\}^n$ and $\mathbf{v}' \in \{-1,0,1\}^{n'}$, we have $F(\mathbf{v} \oplus \mathbf{v}') = F(F(\mathbf{v}) \oplus F(\mathbf{v}'))$.

Weak Path Independence: For all $\mathbf{v} \in \{-1,0,1\}^n$ and $\mathbf{v}' \in \{-1,0,1\}^{n'}$ with $|F(\mathbf{v}) - F(\mathbf{v}')| \neq 2$, we have $F(\mathbf{v} \oplus \mathbf{v}') = F(F(\mathbf{v}) \oplus F(\mathbf{v}'))$.

Theorem (Asan and Sanver). An aggregation function

$$F : \bigcup_{n \in \mathbb{N}} \{-1, 0, 1\}^n \to \{-1, 0, 1\}$$

satisfies Anonymity, Neutrality, Pareto Optimality and Weak Path Independence if and only if it is the majority rule.

Proof Sketch

If F satisfies Anonymity and Neutrality, then for any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| = |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 0$.

Proof Sketch

If F satisfies Anonymity and Neutrality, then for any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| = |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 0$.

Fix a $\mathbf{v}$ with $|N_{\mathbf{v}}(1)| > |N_{\mathbf{v}}(-1)|$. Let $k = |N_{\mathbf{v}}(1)| - |N_{\mathbf{v}}(-1)|$.

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Define a coalition $K \subseteq \{i \mid \mathbf{v}_i = 1\}$ with $|K| = k$.

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Let $\mathbf{v}' \in \{-1, 0, 1\}^k$ be a profile with $\mathbf{v}'_i = \mathbf{v}_i$ for all $i \in K$ and $\mathbf{v}'' \in \{-1, 0, 1\}^{n-k}$ be a profile with $\mathbf{v}''_i = \mathbf{v}_i$ for all $i \in N - K$.

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Then, $\mathbf{v} = \mathbf{v}' \oplus \mathbf{v}''$, $F(\mathbf{v}') = 1$ and $F(\mathbf{v}'') = 0$.

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Then, $\mathbf{v} = \mathbf{v}' \oplus \mathbf{v}''$, $F(\mathbf{v}') = 1$ and $F(\mathbf{v}'') = 0$.

By Weak Path Independence, $F(\mathbf{v}) = F(\mathbf{v}' \oplus \mathbf{v}'') = F(F(\mathbf{v}') \oplus F(\mathbf{v}'')) = F(1, 0)$.

Proof Sketch

If F satisfies Anonymity and Neutrality, then for any $\mathbf{v} \in \{1, 0, -1\}$, if $|N_{\mathbf{v}}(1)| = |N_{\mathbf{v}}(-1)|$, then $F(\mathbf{v}) = 0$.

Fix a $\mathbf{v}$ with $|N_{\mathbf{v}}(1)| > |N_{\mathbf{v}}(-1)|$. Let $k = |N_{\mathbf{v}}(1)| - |N_{\mathbf{v}}(-1)|$.

Define a coalition $K \subseteq \{i \mid \mathbf{v}_i = 1\}$ with $|K| = k$.

Let $\mathbf{v}' \in \{-1, 0, 1\}^k$ be a profile with $\mathbf{v}'_i = \mathbf{v}_i$ for all $i \in K$ and $\mathbf{v}'' \in \{-1, 0, 1\}^{n-k}$ be a profile with $\mathbf{v}''_i = \mathbf{v}_i$ for all $i \in N - K$.

Then, $\mathbf{v} = \mathbf{v}' \oplus \mathbf{v}''$, $F(\mathbf{v}') = 1$ and $F(\mathbf{v}'') = 0$.

By Weak Path Independence, $F(\mathbf{v}) = F(\mathbf{v}' \oplus \mathbf{v}'') = F(F(\mathbf{v}') \oplus F(\mathbf{v}'')) = F(1, 0)$.

By Pareto Optimality, $F(1, 0) = 1$.

Preference Aggregation vs. Judgement Aggregation

W. Rabinowicz. *Aggregation of value judgments differs from aggregation of preferences*. Uncovering Facts and Values: Studies in Contemporary Epistemology and Political Philosophy. Poznań Studies in the Philosophy of the Sciences and the Humanities (107), 2016, pp. 9-40.

Preference Aggregation vs. Judgement Aggregation

Preferring one alternative to another is *not* the same as judging it to be better. Judgments of betterness, and in general value judgments, often accompany preferences and the latter might often be based on the former. But it is possible to prefer a to b even though one lacks a clear view about their relative value. Indeed, it is even possible to judge b to be better than a and still prefer a to b; perhaps because one thinks that a is better *for oneself*, even though one considers b to be better overall; or perhaps because one is simply irrational. Consequently, aggregation of preferences is not reducible to aggregation of value judgments.
(Rabinowicz, pg. 11)

Preference Aggregation vs. Judgement Aggregation

Pareto: If every individual ranks a at least as highly as b and some individuals rank a higher than b, then a is ranked higher than b by the collective.

"This condition is intuitively plausible for preference aggregation, if we think of collective preferences as primarily guides to choice and if we in addition take it to be important that the collective in its choices endeavors to satisfy individual preferences. ... By opting for a rather than b, it will satisfy the preferences of some and frustrate the preferences of no one. "

Preference Aggregation vs. Judgement Aggregation

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”When it comes to the aggregation of value rankings, things are different. In this aggregation process it is important to require that the collective judgment as far as possible approximates the judgments of the individuals. ... if some individuals believe a to be better than b, but the overwhelming majority believes a and b to be equally good, then — it would seem — the collective value judgment should follow the majority view: a and b should be considered by the collective to be of equal value. ”

Preference Aggregation vs. Judgement Aggregation

Indifference: The collective ranking of the alternatives doesn't change if voters who rank all the alternatives equally are removed from consideration (as long as some voters still remain to be considered).

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Proceduralist Justifications

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J. Coleman and J. Ferejohn. *Democracy and social choice*. Ethics, 97(1): 6-25, 1986..

Epistemic Justifications

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J. Cohen. *An epistemic conception of democracy*. Ethics, 97(1): 26-38, 1986.

The Condorcet Jury Theorem

Suppose that P takes on two values: $P = 1$ (i.e., P is true) or $P = 0$ (i.e., P is false).

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Given a profile $\mathbf{p}$ of competences and an aggregation method F, let $\pi(F, \mathbf{p})$ be the probability that F identifies the correct answer.

Expert rule

Suppose that all competences are the same $\mathbf{p} = (p_1, \ldots, p_n)$ with $p_i = p_j$ for all i, j. Then,

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In general, for a profile $\mathbf{p} = (p_1, \dots, p_n)$, we have that

$$\pi(F^e, \mathbf{p}) = \sum_i^n Pr(\text{“choosing } i\text{”}) p_i$$

In particular, if each expert is equally likely to be chosen:

$$\pi(F^e, \mathbf{p}) = \sum_i^n \frac{1}{n} p_i$$

Majority Rule

$$\pi(F^m, \mathbf{p})$$

Majority Rule

$$\pi(F^m, \mathbf{p}) = p^3$$

The probability everyone is correct is p^3

Majority Rule

$$\pi(F^m, \mathbf{p}) = p^3 + 3p^2(1-p)$$

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Condorcet Jury Theorem tutorial.

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Independence The reports of the voters are independent conditional on the state of the world: $R_1, R_2, \ldots$ are *independent conditional* on P

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Condorcet Jury Theorem. Suppose Independence and Competence. As the group size increases, the probability that majority opinion is correct (i) increases and (ii) converges to one.

Literature

D. Austen-Smith and J. Banks. *Aggregation, Rationality and the Condorcet Jury Theorem*. The American Political Science Review, 90:1, pgs. 34 - 45, 1996.

D. Estlund. *Opinion Leaders, Independence and Condorcet's Jury Theorem*. Theory and Decision, 36, pgs. 131 - 162, 1994.

F. Dietrich. *The premises of Condorcet's Jury Theorem are not simultaneously justified*. Episteme,5:1, pgs. 56 - 73, 2008.

S. Nitzan. *Collective Preference and Choice (Part III)*. Cambridge University Press, 2010.

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Multiple Elections Paradox

Voters are asked to give their opinion on three yes/no issues:

YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
1	1	1	3	1	3	3	0

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But there is no support for NNN*!*

Complete Reversal

$YYYN$	$YYNY$	$YNYY$	$NYYY$	$NNNN$
2	2	2	2	3

Outcome by majority vote

Proposition 1: Y (6 - 5)
Proposition 2: Y (6 - 5)
Proposition 3: Y (6 - 5)
Proposition 4: Y (6 - 5)

$YYYY$ wins proposition-wise voting, but the “opposite” outcome NNN has the *most* overall support!

S. Brams, M. Kilgour and W. Zwicker. *Voting on referenda: the separability problem and possible solutions*. Electoral Studies, 16(3), pp. 359 - 377, 1997.

D. Lacy and E. Niou. *A problem with referenda*. Journal of Theoretical Politics 12(1), pp. 5 - 31, 2000.

J. Lang and L. Xia. *Sequential composition of voting rules in multi-issue domains*. Mathematical Social Sciences 57(3), pp. 304 - 324, 2009.

L. Xia, V. Conitzer and J. Lang. *Strategic Sequential Voting in Multi-Issue Domains and Multiple-Election Paradoxes*. In Proceedings of the Twelfth ACM Conference on Electronic Commerce (EC-11), pp. 179-188, 2010.

A decision has to be made about whether or not to build a new swimming pool (S or $\overline{S}$) and a new tennis court (T or $\overline{T}$). Consider 5 voters with rankings over $\{S\,T, \overline{S}\,T, S\,\overline{T}, \overline{S}\,\overline{T}\}$:

rank	2 voters	2 voters	1 voter
1	$S\,\overline{T}$	$\overline{S}\,T$	$S\,T$
2	$\overline{S}\,T$	$S\,\overline{T}$	$S\,\overline{T}$
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The preferences of voters 1-4 are not *separable*. So, they will have a hard time voting on S vs. $\overline{S}$ and T vs. $\overline{T}$.

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Assume that the voters are *optimistic*: They vote for the options that are top on their list.

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When voting on the individual issues, S wins (3-2) and T wins (3-2), but the outcome $S\,T$ is a *Condorcet loser*.

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