Puzzles and Paradoxes from Decision and Game Theory

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Plan

- ✓ Day 1: Rational Choice Theory, Decision Theory
- Day 2: Expected Utility Theory, Evidential and Causal Decision Theory
- Day 3: Decisions over Time, Introduction to Game Theory
- Day 4: Common Knowledge, Backward Induction and Epistemic Game Theory
- Day 5: Paradoxes of Interactive Epistemology, Framing in Games and Decisions

Brief review from yesterday...

$$X = \{M, C, P, L\}$$

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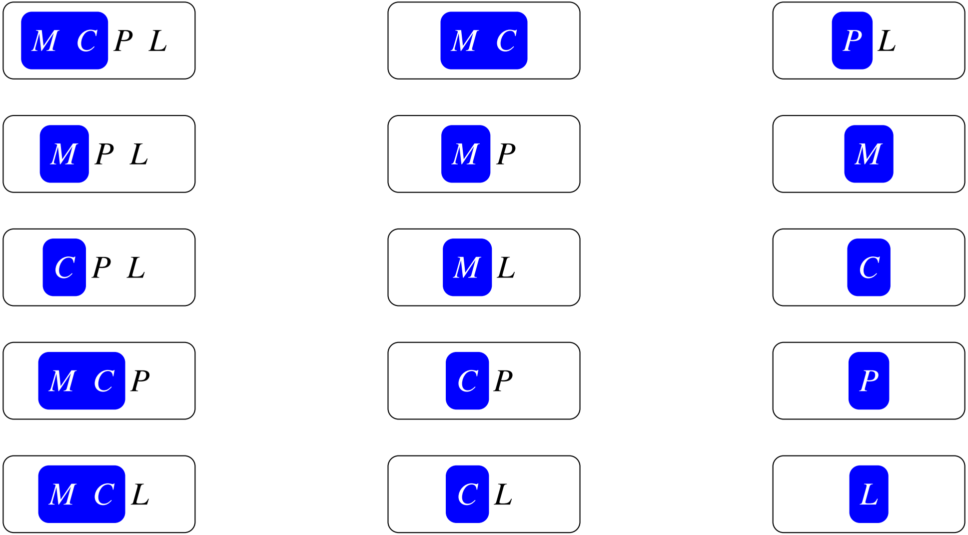
$M\ C\ P\ L$	$M\ C$	$P\ L$
$M\ P\ L$	$M\ P$	M
$C\ P\ L$	$M\ L$	C
$M\ C\ P$	$C\ P$	P
$M\ C\ L$	$C\ L$	L

$$X = \{M, C, P, L\}$$



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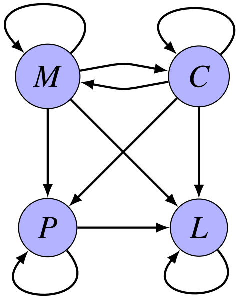

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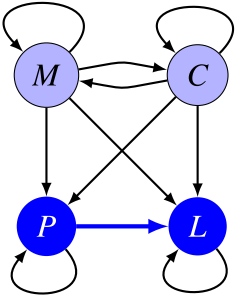

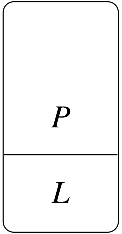

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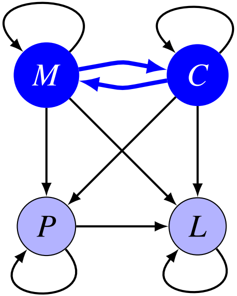

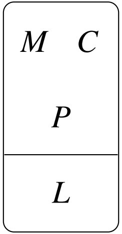

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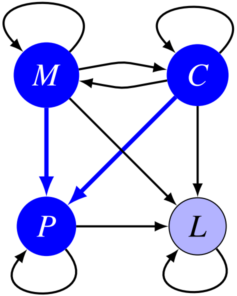

$M \quad C$
P
L

$$\succeq = \{(M,C),(C,M),(M,P),(M,L),(C,P),(C,L),(P,L),$$
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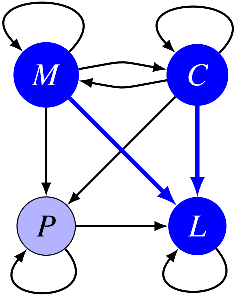

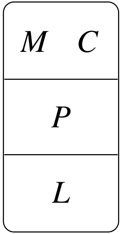

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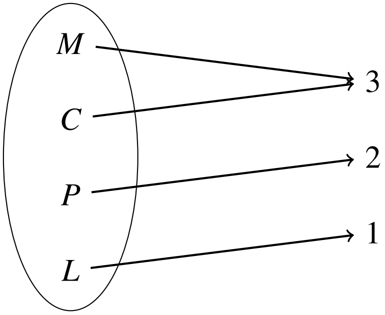

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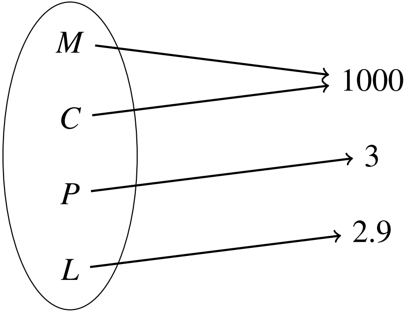

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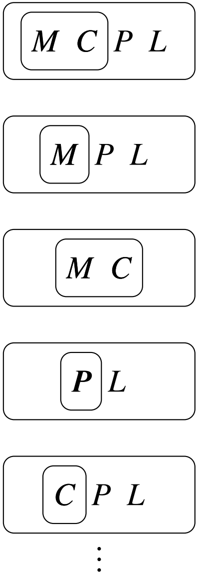
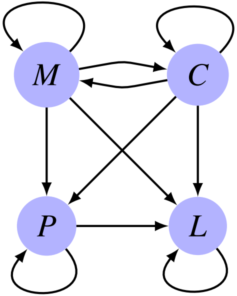
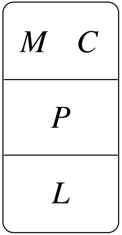
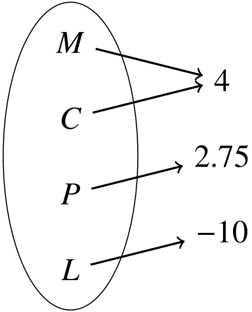








Decision Problems

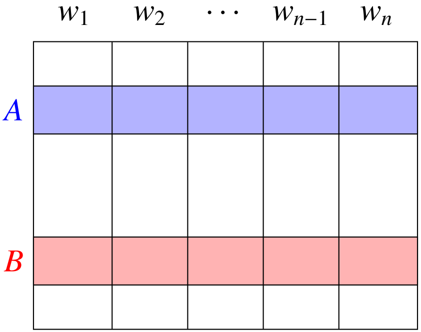
A

B

Decision Problems

	w_1	w_2	$\cdots$	w_{n-1}	w_n
A					
B					

Decision Problems



An **act** is a function $F : W \to O$

Making an omelet

States: {the sixth egg is good, the sixth egg is rotten}

Consequences: { six-egg omelet, no omelet and five good eggs destroyed, six-egg omelet and a cup to wash....}

Acts: { break egg into bowl, break egg into a cup, throw egg away}

Making an omelet

	Good egg (s_1)	Bad egg (s_2)
Break into a bowl (A_1)	six egg omelet (o_1)	no omelet and five good eggs destroyed (o_2)
Break into a cup (A_2)	six egg omelet and a cup to wash (o_3)	five egg omelet and a cup to wash (o_4)
Throw away (A_3)	five egg omelet and one good egg destroyed (o_5)	five egg omelet (o_6)

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$$o_1 \succ o_6 \succ o_3 \succ o_4 \succ o_5 \succ o_2$$

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$o_1 \succ o_6 \succ o_3 \succ o_4 \succ o_5 \succ o_2$ How should A_1, A_2 and A_3 be ranked?

Subjective Expected Utility

Probability: Suppose that $W = \{w_1, \ldots, w_n\}$ is a finite set of states. A probability function on W is a function $P : W \to [0, 1]$ where $\sum_{w \in W} P(w) = 1$ (i.e., $P(w_1) + P(w_2) + \cdots + P(w_n) = 1$).

Suppose that A is an act for a set of outcomes O (i.e., $A : W \to O$). The **expected utility** of A is:

$$\sum_{w \in W} P(w) * u(A(w))$$

Making an omelet

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Making an omelet

	Good egg (s_1) **0.8**	Bad egg (s_2) **0.2**
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$$o_1 \succ o_6 \succ o_3 \succ o_4 \succ o_5 \succ o_2 \qquad P(s_1) = 0.8, P(s_2) = 0.2$$

$$u(o_1) = 6, u(o_6) = 5, u(o_3) = 4, u(o_4) = 3, u(o_5) = 2, u(o_2) = 1$$

Making an omelet

	Good egg (s_1) **0.8**	Bad egg (s_2) **0.2**
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$$o_1 \succ o_6 \succ o_3 \succ o_4 \succ o_5 \succ o_2 \qquad P(s_1) = 0.8, P(s_2) = 0.2$$

$$EU(A_1) = P(s_1) * u(A_1(s_1)) + P(s_2) * u(A_1(s_2)) = 0.8 * 6 + 0.2 * 1 = 5.0$$

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$$EU(A_3) = P(s_1) * u(A_3(s_1)) + P(s_2) * u(A_3(s_2)) = 0.8 * 2 + 0.2 * 5 = 2.6$$

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	Good egg (s_1) **0.8**	Bad egg (s_2) **0.2**
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$$o_1 \succ o_6 \succ o_3 \succ o_4 \succ o_5 \succ o_2 \qquad P(s_1) = 0.8, P(s_2) = 0.2$$

$$EU(A_1) = 5 > EU(A_2) = 3.8 > EU(A_3) = 2.6$$

Making an omelet

	Good egg (s_1) **0.8**	Bad egg (s_2) **0.2**
Break into a bowl (A_1)	six egg omelet (o_1) **9**	no omelet and five good eggs destroyed (o_2) **0**
Break into a cup (A_2)	six egg omelet and a cup to wash (o_3) **8**	five egg omelet and a cup to wash (o_4) **7**
Throw away (A_3)	five egg omelet and one good egg destroyed (o_5) **1**	five egg omelet (o_6) **8.5**

$$o_1 \succ o_6 \succ o_3 \succ o_4 \succ o_5 \succ o_2 \qquad P(s_1) = 0.8, P(s_2) = 0.2$$

$$u(o_1) = 9, u(o_6) = 8.5, u(o_3) = 8, u(o_4) = 7, u(o_5) = 1, u(o_2) = 0$$

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$$EU(A_2) = 7.8 > EU(A_1) = 7.2 > EU(A_3) = 2.7$$

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Utility of outcome o

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Expected utility of action *A*

Utility of outcome *o*

Probability of outcome *o* conditional on *A*

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Classical: $P_A(o) = \sum_{s \in S} P(s) f_{A,s}(o)$, where

$$f_{A,s}(o) = \begin{cases} 1 & A(s) = o \\ 0 & A(s) \neq o \end{cases}$$

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Causal: $P_A(o) = P(A \mathrel{\Box\!\!\rightarrow} o)$

P("if A were performed, outcome o would ensue")

(Lewis, 1981)

Three Immediate Issues with Expected Utility

1. St. Petersburg Game
2. Pasadena Game
3. Two Envelop Paradox

St. Petersburg Paradox

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Utility is bounded

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Intuitively, Moscow game should be preferred to the St. Petersburg game, but both have infinite expected utility.

The Pasadena Game

Toss a fair coin until it lands heads up for the first time. Suppose this happens at toss number n. If n is odd, you win $(2^n)/n$ units, but if n is even you pay $(2^n)/n$.

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$$\frac{1}{2}\cdot\frac{2}{1}-\frac{1}{4}\cdot\frac{4}{2}+\frac{1}{8}\cdot\frac{8}{3}-\frac{1}{16}\cdot\frac{16}{4}+\cdots=\sum_n \frac{(-1)^{n-1}}{n}$$

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The Two Envelop Paradox

Suppose that you have a choice between two envelops, each containing some money. A trustworthy informant tells you that one of the envelops contains exactly twice as much as the other, but not which is which. Since this is all you know, you pick an envelop at random. Just before you open the envelop, you are given the opportunity to switch envelops. Should you swap?

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Yes: Suppose the chosen envelop has \$$x$. The other envelop has either $\frac{1}{2} \cdot x$ dollars or $2 \cdot x$ dollars. Each is equally likely, so the expected utility of switching is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot x + \frac{1}{2} \cdot 2 \cdot x = 1.25 \cdot x$$

Cardinal Utility Theory

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$$u : X \to \mathbb{R}$$

Which comparisons are meaningful?

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Cardinal Utility Theory

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Which comparisons are meaningful?

1. $u(x)$ and $u(y)$? (ordinal utility)
2. $u(x) - u(y)$ and $u(a) - u(b)$?
3. $u(x)$ and $2 * u(z)$?

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

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Cardinal scales:

> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
>
> E.g., the difference between 75°F and 70°F is the same as the difference between 30°F and 25°F However, 70°F (= 21.11°C) is **not** twice as hot as 35°F (= 1.67°C).

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

Cardinal scales:

Interval scale: Quantitative comparisons of objects, accurately reflects differences between objects.

E.g., the difference between 75°F and 70°F is the same as the difference between 30°F and 25°F However, 70°F (= 21.11°C) is **not** twice as hot as 35°F (= 1.67°C).

Ratio scale: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb (= 4.53592kg) is twice as much as 5lb (= 2.26796kg).

Cardinal Utility Theory

$x \succ y \succ z$ is represented by both $(3, 2, 1)$ and $(1000, 999, 1)$, so we cannot say y whether is "closer" to x than to z.

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Key idea: Ordinal preferences over *lotteries* allows us to infer a cardinal scale (with some additional axioms).

John von Neumann and Oskar Morgenstern. *The Theory of Games and Economic Behavior*. Princeton University Press, 1944.

A Choice

R

B

W

S

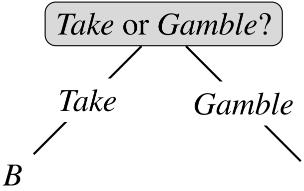
A Choice

R

B

W

S



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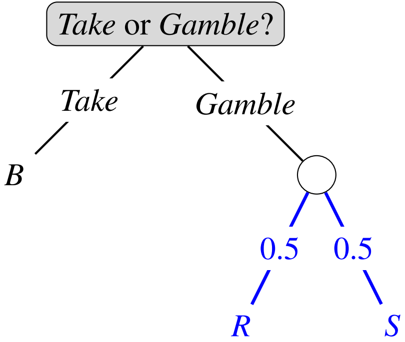

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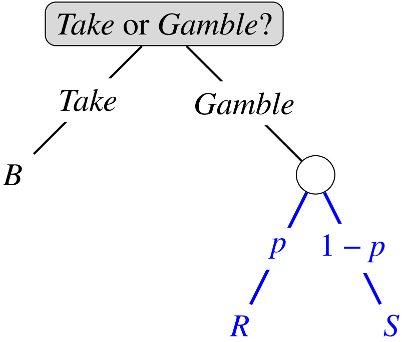

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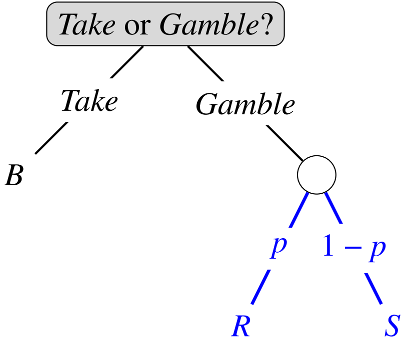

$$[1 : B] \sim [p : R, 1 - p : S]$$

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R

B

W

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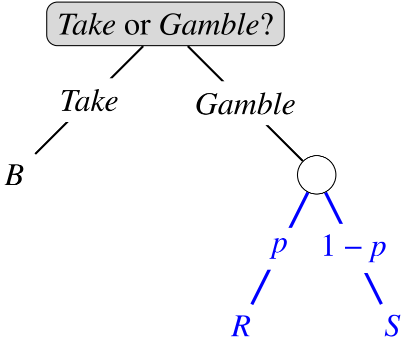

$$1 * u(B) = p * u(R) + (1 - p) * u(S)$$

A Choice

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W

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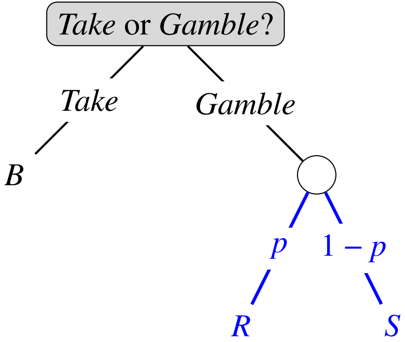

$$u(B) = p * 1 + (1 - p) * 0 = p$$

Suppose that X is a set of outcomes.

A **(simple) lottery** over X is denoted $[x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$ where for $i = 1, \ldots, n$, $x_i \in X$ and $p_i \in [0,1]$, and $\sum_i p_i = 1$.

Let $\mathcal{L}$ be the set of (simple) lotteries over X. We identify elements $x \in X$ with the lottery $[x : 1]$.

Suppose that $\succeq$ is a relation on $\mathcal{L}$.

Lotteries

Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of outcomes. A **lottery** over X is a tuple $[p_1 : x_1, \ldots, p_n : x_n]$ where $\sum_i p_i = 1$.

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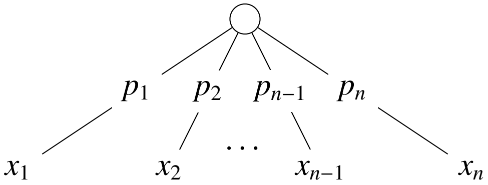

Let $\mathcal{L}$ be the set of lotteries. Suppose that $\succeq \subseteq \mathcal{L} \times \mathcal{L}$ is a preference ordering on $\mathcal{L}$.

Axioms

Preference $\succeq$ is reflexive, transitive and complete

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Independence For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0, 1]$, $L_1 \succ L_2$

if, and only if,

$[L_1 : a, L_3 : (1 - a)] \succ [L_2 : a, L_3 : (1 - a)]$.

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if $L_1 \succ L_2 \succ L_3$, then there exists $a, b \in (0, 1)$
such that $[L_1 : a, L_3 : (1 - a)] \succ L_2$ and
$L_2 \succ [L_1 : b, L_3 : (1 - b)]$.

$u : \mathcal{L} \to \Re$ is linear provided for all $L = [L_1 : p_1, \ldots, L_n : p_n] \in \mathcal{L}$,

$$u(L) = \sum_{i=1}^{n} p_i u(L_i)$$

von Neumann-Morgenstern Representation Theorem A binary relation $\succeq$ on $\mathcal{L}$ satisfies Preference, Compound Lotteries, Independence and Continuity iff $\succeq$ is representable by a linear utility function $u : \mathcal{L} \to \Re$.

Moreover, $u' : \mathcal{L} \to \Re$ represents $\succeq$ iff there exists real numbers $c > 0$ and d such that $u'(\cdot) = cu(\cdot) + d$. ("$u$ is unique up to linear transformations.")

Axioms of Cardinal Utility

Suppose that X is a set of outcomes and consider **lotteries over** X (i.e., probability distributions over X)

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Running example: Suppose Ann prefers pizza (p) over taco (t) over yogurt (y) ($p \succ t \succ y$) and consider the different lotteries where the prizes are p, t and y.

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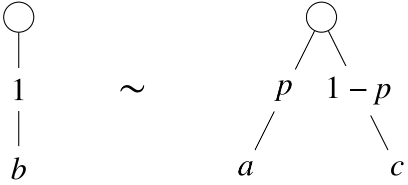

$$p \succ t \succ y$$

Suppose Ann has t.

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Would Ann trade t for L?

Consider the lottery $L' = 0.99$ get p and 0.01 get y
Would Ann trade t for L'?

Continuity says that there is must be some lottery where Ann is indifferent between keeping t and playing the lottery.

Better Prizes

Better Prizes: suppose L_1 is a lottery over (w, x) and L_2 is over (y, z) suppose that L_1 and L_2 have the same probability over prizes. The lotteries each have an equal prize in one position they have unequal prizes in the other position then if L_1 is the lottery with the better prize then $L_1 \succ L_2$; if neither lottery has a better prize then $L_1 \approx L_2$.

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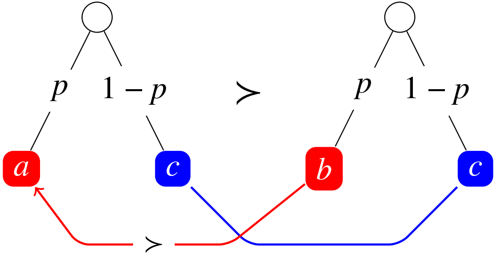

$$p \succ t \succ y$$

Lottery 1 (L_1) is 0.6 chance for p and 0.4 chance for y

Lottery 2 (L_2) is 0.6 chance for t and 0.4 chance for y

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Since Ann prefers p to t, this axiom says that Ann prefers L_1 to L_2

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Better Chances: Suppose L_1 and L_2 are two lotteries which have the same prizes, then if L_1 offers a better chance of the better prize, then $L_1 \succ L_2$

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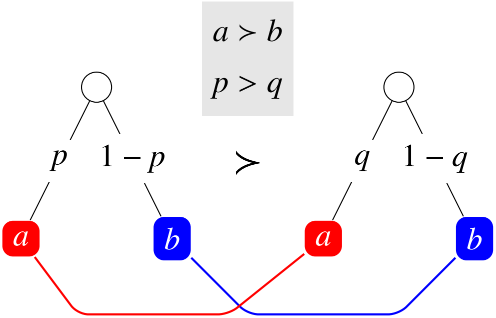

$$p \succ t \succ y$$

Lottery 1 (L_1) is 0.7 chance for p and 0.3 chance for y

Lottery 2 (L_2) is 0.6 chance for p and 0.4 chance for y

$$p \succ t \succ y$$

Lottery 1 (L_1) is 0.7 chance for p and 0.3 chance for y
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This axioms states that Ann must prefer L_1 to L_2

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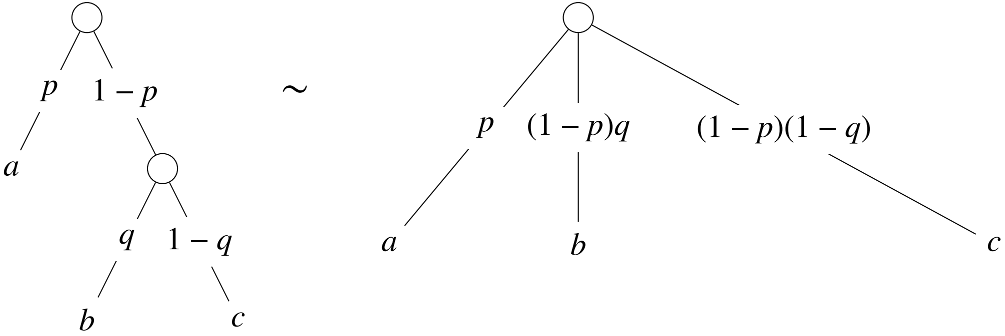
This eliminates utility from the thrill of gambling and so the only ultimate concern is the prizes.


p
1 − p
a
q
1 − q
b
c
∼
p
(1 − p)q
(1 − p)(1 − q)
a
b
c

Von Neumann-Morgenstern Theorem

Suppose that $\mathcal{L}$ is the set of lotteries. A function $u : \mathcal{L} \to \Re$ is linear provided for all $L = [L_1 : p_1, \ldots, L_n : p_n] \in \mathcal{L}$,

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Von Neumann-Morgenstern Theorem. A binary relation $\succeq$ on $\mathcal{L}$ is transitive, complete, and satisfies Continuity, Better Prizes, Better Chances, Reduction of Compound Lotteries iff $\succeq$ is representable by a linear utility function $u : \mathcal{L} \to \Re$.

Moreover, $u' : \mathcal{L} \to \Re$ represents $\succeq$ iff there exists real numbers $c > 0$ and d such that $u'(\cdot) = cu(\cdot) + d$. ("$u$ is unique up to linear transformations.")

Independence

Independence For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0, 1]$,

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Preference	$\succeq$ is reflexive, transitive and complete
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Von Neumann-Morgenstern

S is a finite set of states and X is a set of outcomes.

A **lottery** is a function $p : X \to [0, 1]$ satisfying the usual probability axioms. Let $\mathcal{L}_X$ be the set of lotteries on X.

Theorem. $\succeq \subseteq \mathcal{L}_X \times \mathcal{L}_X$ satisfies the vNM-axioms iff there is a probability a utility function $u : X \to \mathbb{R}$ such that

$$p \succeq q \text{ iff } \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x)$$

Aumann-Anscombe

S is a finite set of states and X is a set of outcomes.

A **horse lottery** is a function $h : S \to \mathcal{L}_X$ (write h_s for $h(s)$). Let $\mathcal{H}$ be the set of horse lotteries.

Theorem. $\succeq \subseteq \mathcal{H} \times \mathcal{H}$ satisfies the AA-axioms iff there is a probability distribution μ on S and a utility function $u : X \to \mathbb{R}$ such that

$$h \succeq g \text{ iff } \sum_{s \in S} \mu(s) \sum_{x \in X} h_s(x)u(x) \geq \sum_{s \in S} \mu(s) \sum_{x \in X} g_s(x)u(x)$$

Savage

S is a countable set of states, $\mathcal{F}$ some algebra on S and X is a set of outcomes.

An **act** is a function $f : S \to X$. Let $\mathcal{A}$ be the set of acts.

Theorem. $\geq \subseteq \mathcal{A} \times \mathcal{A}$ satisfies the S-axioms iff there is a probability distribution μ on $(S, \mathcal{F})$ and a utility function $u : X \to \mathbb{R}$ such that

$$f \geq g \text{ iff } \int_S u[f(s)]d\mu \geq \int_S u[g(s)]d\mu$$

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- Important issues about how to identify correct descriptions of the outcomes and options.

Issue with Better Prizes

Suppose you have a kitten, which you plan to give away to either Ann or Bob. Ann and Bob both want the kitten very much. Both are deserving, and both would care for the kitten. You are sure that giving the kitten to Ann (x) is at least as good as giving the kitten to Bob (y) (so $x \geq y$). But you think that would be unfair to Bob. You decide to flip a fair coin: if the coin lands heads, you will give the kitten to Bob, and if it lands tails, you will give the kitten to Ann.

(J. Drier, "Morality and Decision Theory" in *Handbook of Rationality*)

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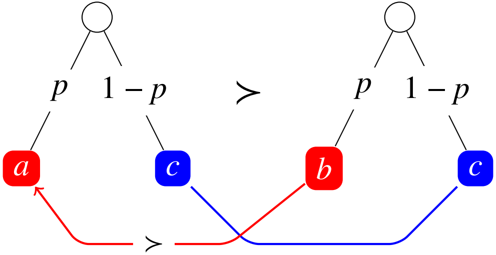
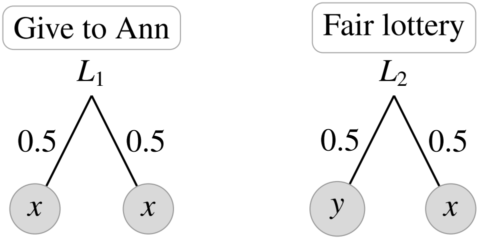
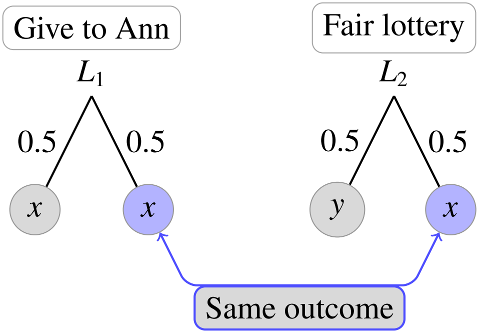




- x is the outcome “Ann gets the kitten”
- y is the outcome “Bob gets the kitten”



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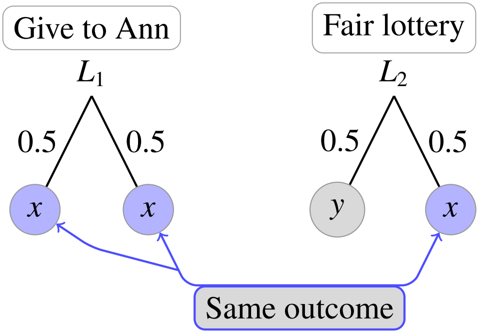

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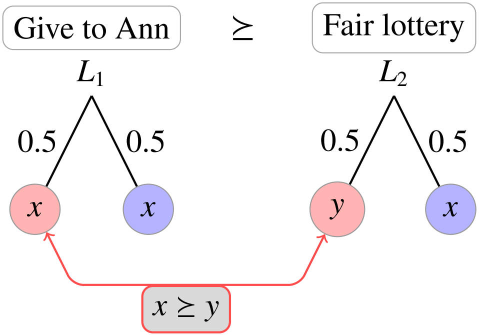

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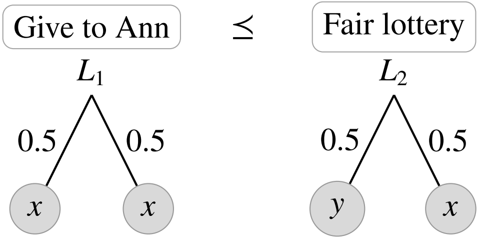
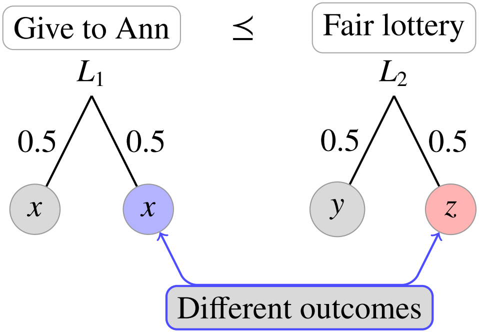
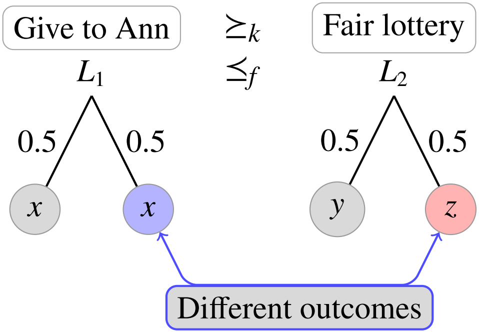


- x is the outcome "Ann gets the kitten, *in a fair way*"
- y is the outcome "Bob gets the kitten"



- x is the outcome “Ann gets the kitten”
- z is the outcome “Ann gets the outcome, *fairly*
- y is the outcome “Bob gets the kitten, *fairly*”



If all the agent cares about is who gets the kitten, then $L_1 \succeq L_2$

If all the agent cares about is being fair, then $L_1 \preceq L_2$

Allais Paradox

	Options	Red (1)	White (89)	Blue (10)
S_1	A	$1M$	$1M$	$1M$
	B	0	$1M$	$5M$

Allais Paradox

	Options	Red (1)	White (89)	Blue (10)
S_2	C	$1M$	0	$1M$
	D	0	0	$5M$

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$$A \succeq B \text{ iff } C \succeq D$$