Social Choice Theory for Logicians
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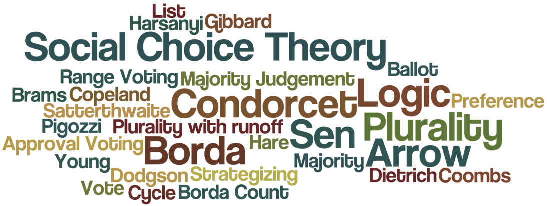

Plan

- Introduction, Background, Voting Theory, May's Theorem, Arrow's Theorem
- Social Choice Theory: Variants of Arrow's Theorem, Weakening Arrow's Conditions (Domain Conditions), Harsanyi's Theorem, Characterizing Voting Methods
- Strategizing (Gibbard-Satterthwaite Theorem) and Iterative Voting/ Introduction to Judgement Aggregation
- Aggregating Judgements (linear pooling, wisdom of the crowds, prediction markets), Probabilistic Social Choice.
- Logics for Social Choice Theory (Preference Logic, Modal Logic, Dependence/Independence Logic, First Order Logic)

The Propositions

Propositions: Let $\mathcal{L}$ be a logical language (called **propositions** in the literature) with the usual boolean connectives.

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Consistency: The standard notion of logical consistency.

Aside: We actually need

1. $\{p, \neg p\}$ are inconsistent
2. all subsets of a consistent set are consistent
3. $\emptyset$ is consistent and each $S \subseteq \mathcal{L}$ has a consistent maximal extension (not needed in all cases)

The Agenda

Definition The **agenda** is a non-empty set $X \subseteq \mathcal{L}$, interpreted as the set of propositions on which judgments are made (note: X is a union of proposition-negation pairs $\{p, \neg p\}$).

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Example: In the discursive dilemma: $X = \{p, \neg p, q, \neg q, p \to q, \neg(p \to q)\}$.

The Judgement Sets

Definition: Given an agenda X, each individual i's judgement set is a subset $A_i \subseteq X$.

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Rationality Assumptions:

1. A_i is **consistent**
2. A_i is **complete**, if for each $p \in X$, either $p \in A_i$ or $\neg p \in A_i$

Aggregation Rules

Let X be an agenda, $N = \{1, \ldots, n\}$ a set of voters, a **profile** is a tuple $(A_i, \ldots, A_n)$ where each A_i is a judgement set. An **aggregation function** is a map from profiles to judgment sets. I.e., $F(A_1, \ldots, A_n)$ is a judgement set.

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Examples:

- **Propositionwise majority voting**: for each $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = \{p \in X \mid |\{i \mid p \in A_i\}| \geq |\{i \mid p \notin A_i\}|\}$$

- **Dictator of** i: $F(A_1, \ldots, A_n) = A_i$
- **Reverse Dictator of** i: $F(A_1, \ldots, A_n) = \{\neg p \mid p \in A_i\}$

Input

Universal Domain: The domain of F is the set of all possible profiles of consistent and complete judgement sets.

Output

Collective Rationality: F generates consistent and complete collective judgment sets.

Anonymity: For all profiles $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = F(A_{\pi(1)}, \ldots, A_{\pi(n)})$ where π is a permutation of the voters.

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Unanimity: For all profiles $(A_1, \ldots, A_n)$ if $p \in A_i$ for each i then $p \in F(A_1, \ldots, A_n)$

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Monotonicity: For any $p \in X$ and all $(A_1, \ldots A_i, \ldots, A_n)$ and $(A_1, \ldots, A_i^*, \ldots, A_n)$ in the domain of F,

$$\text{if } [p \notin A_i, p \in A_i^* \text{ and } p \in F(A_1, \ldots, A_i, \ldots A_n)]$$
$$\text{then } [p \in F(A_1, \ldots, A_i^*, \ldots A_n)].$$

Systematicity: For any $p, q \in X$ and all $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ in the domain of F,

$$\text{if [for all } i \in N, p \in A_i \text{ iff } q \in A_i^*]$$
$$\text{then } [p \in F(A_1, \ldots, A_n) \text{ iff } q \in F(A_1^*, \ldots A_n^*)\ ].$$

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- independence
- neutrality

Independence: For any $p \in X$ and all $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ in the domain of F,

$$\text{if [for all } i \in N, p \in A_i \text{ iff } p \in A_i^*]$$
$$\text{then } [p \in F(A_1, \ldots, A_n) \text{ iff } p \in F(A_1^*, \ldots A_n^*)\ ].$$

Non-dictatorship: There exists no $i \in N$ such that, for any profile $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = A_i$

Baseline Result

Theorem (List and Pettit, 2001) If $X \subseteq \{a, b, a \wedge b\}$, there exists no aggregation rule satisfying universal domain, collective rationality, systematicity and anonymity.

Agenda Richness

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Definition A set $Y \subseteq \mathcal{L}$ is **minimally inconsistent** if it is inconsistent and every proper subset $X \subsetneq Y$ is consistent.

Agenda Richness

Definition An agenda X is **minimally connected** if

1. (*non-simple*) it has a minimal inconsistent subset $Y \subseteq X$ with $|Y| \geq 3$
2. (*even-number-negatable*) it has a minimal inconsistent subset $Y \subseteq X$ such that

$$Y - Z \cup \{\neg z \mid z \in Z\} \text{ is consistent}$$

   for some subset $Z \subseteq Y$ of even size.

Impossibility Theorems

Theorem (Dietrich and List, 2007) If (and only if) an agenda is non-simple and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, systematicity and unanimity is a dictatorship (or inverse dictatorship).

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Theorem (Nehring and Puppe, 2002) If (and only if) an agenda is non-simple, every aggregation rule satisfying universal domain, collective rationality, systematicity, unanimity, and monotonicity is a dictatorship.

Characterization Result

$p \in X$ **conditionally entails** $q \in X$, written $p \vdash^* q$ provided there is a subset $Y \subseteq X$ consistent with each of p and $\neg q$ such that $\{p\} \cup Y \vdash q$.

Totally Blocked: X is totally blocked if for any $p, q \in X$ there exists $p_1, \ldots, p_k \in X$ such that

$$p = p_1 \vdash^* p_2 \vdash^* \cdots \vdash^* p_k = q$$

Characterization Result

Theorem (Dietrich and List, 2007, Dokow Holzman 2010) If (and only if) an agenda is totally blocked and even-number negatable, every aggregation rule satisfying universal domain, collective rationality, independence and unanimity is a dictatorship.

Theorem (Nehring and Puppe, 2002, 2010) If (and only if) an agenda is totally blocked, every aggregation rule satisfying universal domain, collective rationality, independence, unanimity, and monotonicity is a dictatorship.

$C \subseteq N$ is **winning for** p if for all profiles $\mathbf{A} = (A_1, \ldots, A_n)$, if $p \in A_i$ for all $i \in C$ and $p \notin A_j$ for all $j \notin C$, then $p \in F(\mathbf{A})$

$\mathcal{C}_p = \{C \mid C \text{ is winning for } p\}$

Proof Sketch

1. If the agenda is totally blocked, then $C_p = C_q$ for all p, q. Let $C = C_p$ for some p (hence for all p).

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3. If $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2 \in \mathcal{C}$.

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Proof Sketch

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4. $N \in \mathcal{C}$.

5. For all $C \subseteq N$, either $C \in \mathcal{C}$ or $\overline{C} \in \mathcal{C}$.

6. There is an $i \in N$ such that $\{i\} \in \mathcal{C}$.

Many Variants!

C. List. *The theory of judgment aggregation: An introductory review*. Synthese 187(1), pgs. 179-207, 2012.

D. Grossi and G. Pigozzi. *Judgement Aggregation: A Primer*. Morgan & Claypol, 2014.

Logic and Social Choice

An Email

An Email

“Interesting

An Email

"Interesting...but what does logic have to do with group decision making??? I've never seen logic prevail at any of our faculty meetings."

Setting the Stage: Logic and Games

M. Pauly and W. van der Hoek. *Modal Logic form Games and Information*. Handbook of Modal Logic (2006).

G. Bonanno. *Modal logic and game theory: Two alternative approaches*. Risk Decision and Policy **7** (2002).

J. van Benthem. *Extensive games as process models*. Journal of Logic, Language and Information **11** (2002).

J. Halpern. *A computer scientist looks at game theory*. Games and Economic Behavior **45:1** (2003).

R. Parikh. *Social Software*. Synthese **132: 3** (2002).

What do the (Im)possibility results say?

M. Pauly. *On the Role of Language in Social Choice Theory*. Synthese, 163, 2, pgs. 227 - 243, 2008.

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Fix a language $\mathcal{L}$ and a *satisfaction relation* $\models \subseteq \mathcal{D} \times \mathcal{L}$

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Δ **absolutely axiomatizes** $\mathcal{T}$ iff for all $M \in \mathcal{D}$, $M \in \mathcal{T}$ iff $M \models \Delta$ (i.e., Δ *defines* $\mathcal{T}$)

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Δ **relatively axiomatizes** $\mathcal{T}$ iff for all $\varphi \in \mathcal{L}$, $\mathcal{T} \models \varphi$ iff $\Delta \models \varphi$ (i.e., Δ axiomatizes the theory of $\mathcal{T}$)

What do the (Im)possibility results say?

May's Theorem: Δ is the set of aggregation functions w.r.t. 2 candidates, $\mathcal{T}$ is majority rule, $\mathcal{L}$ is the language of set theory, Δ is the properties of May's theorem, then Δ absolutely axiomatizes $\mathcal{T}$.

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May's Theorem: Δ is the set of aggregation functions w.r.t. 2 candidates, $\mathcal{T}$ is majority rule, $\mathcal{L}$ is the language of set theory, Δ is the properties of May's theorem, then Δ absolutely axiomatizes $\mathcal{T}$.

Arrow's Theorem: Δ is the set of aggregation functions w.r.t. 3 or more candidates, $\mathcal{T}$ is a dictatorship, $\mathcal{L}$ is the language of set theory, Δ is the properties of May's theorem, then Δ absolutely axiomatizes $\mathcal{T}$.

A Minimal Language

M. Pauly. *Axiomatizing Collective Judgement Sets in a Minimal Logical Language*. 2006.

Let Φ_I be the set of **individual formulas** (standard propositional language)

V_I the set of individual valuations

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Φ_C the set of **collective formulas**: $\Box\alpha \mid \varphi \wedge \psi \mid \neg\varphi$

$\Box\alpha$: *The group collectively accepts* α.
V_C the set of collective valuations: $v : \Phi_C \to \{0, 1\}$

A Minimal Language

Let $\mathcal{CON}_n = \{v \in V_C \mid v(\Box\alpha) = 1 \text{ iff } \forall i \leq n,\ v_i(\alpha) = 1\}$

E. $\Box\varphi \leftrightarrow \Box\psi$ provided $\varphi \leftrightarrow \psi$ is a tautology

M. $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C. $(\Box\varphi \wedge \Box\psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

N. $\Box\top$

D. $\neg\Box\bot$

Theorem [Pauly, 2005] $V_C(\mathsf{KD}) = \mathcal{CON}_n$, provided $n \geq 2^{|\Phi_0|}$.

($\mathcal{D} = V_C$, $\mathcal{T} = \mathcal{CON}_n$, $\Delta = EMCND$, then Δ absolutely axiomatizes $\mathcal{T}$.)

A Minimal Language

Let $\mathcal{MAJ}_n = \{v \in \mathcal{V}_C \mid v([>]\alpha) = 1 \text{ iff } |\{i \mid v_i(\alpha) = 1\}| > \frac{n}{2}\}$

STEM contains all instances of the following schemes

S. $[>]\varphi \to \neg[>]\neg\varphi$

T. $([\geq]\varphi_1 \wedge \cdots \wedge [\geq]\varphi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \to \bigwedge_{1\leq i\leq k}([=]\varphi_i \wedge [=]\psi_i)$ where $\forall v \in V_I : |\{i \mid v(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

E. $[>]\varphi \leftrightarrow [>]\psi$ provided $\varphi \leftrightarrow \psi$ is a tautology

M. $[>](\varphi \wedge \psi) \to ([>]\varphi \wedge [>]\psi)$

Theorem [Pauly, 2005] $V_C(\mathsf{STEM}) = \mathcal{MAJ}$.

($\mathcal{D} = V_C$, $\mathcal{T} = \mathcal{MAJ}_n$, $\Delta = STEM$, then Δ absolutely axiomatizes $\mathcal{T}$.)

- Compare principles in terms of the language used to express them

M. Pauly. *On the Role of Language in Social Choice Theory*. Synthese, 163, 2, pgs. 227 - 243, 2008.

T. Daniëls. *Social choice and logic of simple games*. Journal of Logic and Computation, 21, 6, pgs. 883 - 906, 2011.

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T. Daniëls. *Social choice and logic of simple games*. Journal of Logic and Computation, 21, 6, pgs. 883 - 906, 2011.

- How much "classical logic" is "needed" for the judgement aggregation results?

T. Daniëls and EP. *A general approach to aggregation problems*. Journal of Logic and Computation, 19, 3, pgs. 517 - 536, 2009.

F. Dietrich. *A generalised model of judgment aggregation*. Social Choice and Welfare 28(4): 529 - 565, 2007.

G. Ciná and U. Endriss. *Proving Classical Theorems of Social Choice Theory in Modal Logic*. Journal of Autonomous Agents and Multiagent Systems, forthcoming.

N. Troquard, W. van der Hoek, and M. Wooldridge. *Reasoning about social choice Functions*. Journal of Philosophical Logic 40(4), 473 - 498 (2011).

T. Agotnes, W. van der Hoek, and M. Wooldridge. *On the logic of preference and judgment aggregation*. Journal of Autonomous Agents and Multiagent Systems 22(1), 4 - 30 (2011).

Language

Atomic Propositions:

- $Pref[N, X] := \{p^i_{x \leq y} \mid i \in N, x, y \in X\}$ is the set of preference atomic propositions, where $p^i_{x \leq y}$ means i prefers y to x.
- Each $x \in X$ is an atomic proposition.

Modality:

- $\Diamond_C \varphi$: C can *ensure* the truth of φ.

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Modality:

- $\Diamond_C \varphi$: C can *ensure* the truth of φ.

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond_C \varphi$$

Model

A **model** is a triple $M = \langle N, X, F \rangle$, consisting of a finite set of agents N (with $n = |N|$), a finite set of alternatives X, and a SCF $F : L(X)^n \rightarrow X$.

A **world** is a profile $(\succeq_1, \ldots, \succeq_n)$

Truth

Let $w = (\succeq_1, \dots, \succeq_n)$

- $M, w \models p^i_{x \preceq y}$ iff $x \preceq_i y$
- $M, w \models x iff F(\preceq_1, \dots, \preceq_n) = x$ $M, w \models \neg\varphi$ iff $M, w \not\models \varphi$
- $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$
- $M, w \models \Diamond_C \varphi$ iff $M, w' \models \varphi$ for some $w' = (\succeq'_1, \dots, \succeq'_n)$ with $\succeq_j = \succeq'_j$ for all $j \in N - C$.

(1) $p^i_{x \geq x}$

(2) $p^i_{x \geq y} \leftrightarrow \neg p^i_{y \geq x}$ for $x \neq y$

(3) $p^i_{x \geq y} \wedge p^i_{y \geq y} \rightarrow p^i_{x \geq z}$

(1) $p^i_{x \succeq x}$
(2) $p^i_{x \succeq y} \leftrightarrow \neg p^i_{y \succeq x}$ for $x \neq y$
(3) $p^i_{x \succeq y} \wedge p^i_{y \succeq y} \rightarrow p^i_{x \succeq z}$

$$ballot_i(w) \;=\; p^i_{x_1 \succeq x_2} \wedge \cdots \wedge p^i_{x_{m-1} \succeq x_m}$$

$$profile(w) \;=\; ballot_1(w) \wedge \cdots \wedge ballot_n(w)$$

(4) all propositional tautologies

(5) $\Box_i(\varphi \to \psi) \to (\Box_i \varphi \to \Box_i \psi)$ (K(i))

(6) $\Box_i \varphi \to \varphi$ (T(i))

(7) $\varphi \to \Box_i \Diamond_i \varphi$ (B(i))

(8) $\Diamond_i \Box_j \varphi \leftrightarrow \Box_j \Diamond_i \varphi$ (confluence)

(9) $\Box_{C_1} \Box_{C_2} \varphi \leftrightarrow \Box_{C_1 \cup C_2} \varphi$ (union)

(10) $\Box_\emptyset \varphi \leftrightarrow \varphi$ (empty coalition)

(11) $(\Diamond_i p \wedge \Diamond_i \neg p) \to (\Box_j p \vee \Box_j \neg p)$, where $i \neq j$ (exclusiveness)

(12) $\Diamond_i ballot_i(w)$ (ballot)

(13) $\Diamond_{C_1} \delta_1 \wedge \Diamond_{C_2} \delta_2 \to \Diamond_{C_1 \cup C_2}(\delta_1 \wedge \delta_2)$ (cooperation)

(14) $\bigvee_{x \in X}(x \wedge \bigwedge_{y \in X \setminus \{x\}} \neg y)$ (resoluteness)

(15) $(profile(w) \wedge \varphi) \to \Box_N(profile(w) \to \varphi)$ (functionality)

Theorem (Cinà and Endriss) The logic $L[N, X]$ is sound and complete w.r.t. the class of models of social choice functions.

Universal Domain

Lemma For every possible profile $w \in L(X)^n$, $\vdash \Diamond_N \mathit{profile}(w)$

Pareto

$$Par \ := \ \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} \left[\left(\bigwedge_{i \in N} p^i_{x \succeq y} \right) \rightarrow \neg y \right]$$

IIA

$$IIA \;:=\; \bigwedge_{w\in L(X)^n} \bigwedge_{x\in X} \bigwedge_{y\in X-\{x\}} [\Diamond_N(\mathit{profile}(w) \wedge x) \rightarrow (\mathit{profile}(w)(x,y) \rightarrow \neg y)]$$

- $N^w_{x\succeq y} = \bigwedge\{p^i_{x\succeq y} \mid x \succeq_i y \text{ in } w\}$
- $\mathit{profile}(w)(x,y) := N^w_{x\succeq y} \wedge N^w_{y\succeq x}$

Dictatorship

$$Dic := \bigvee_{i \in N} \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} (p^i_{x \succeq y} \rightarrow \neg y)$$

Theorem (Cinà and Endriss) Consider a logic $L[N, X]$ with a language parameterised by X such that $|X| > 3$. Then we have:

$$\vdash Par \wedge IIA \rightarrow Dic$$

Verification existing proofs of Arrow's Theorem in higher-order logic proof assistants.

T. Nipkow. *Social choice theory in HOL: Arrow and Gibbard-Satterthwaite*. Journal of Automated Reasoning 43(3), 289304, 2009.

F. Wiedijk. *Arrow's Impossibility Theorem*. Formalized Mathematics 15(4), 171 - 174, 2007.

Classical first-order logic is sufficiently expressive to express all aspects of Arrows Theorem (except that the set of agents in finite).

U. Grandi and U. Endriss. *First-order logic formalisation of impossibility theorems in preference aggregation*. Journal of Philosophical Logic 42(4), 595 - 618 (2013).

Arrow's Theorem for a fixed set of alternatives (e.g., $|N| = 2, |X| = 3$) can be embedded into classical propositional logic and automatically checked as a SAT problem. (The full theorem is proved by mathematical induction).

P. Tang and F. Lin. *Computer-aided proofs of Arrows and other impossibility theorems*. Artificial Intelligence 173(11), 1041 - 1053 (2009).

U. Endriss. *Logic and social choice theory*. In: A. Gupta, J. van Benthem (eds.) Logic and Philosophy Today, vol. 2, pp. 333377. College Publications (2011).

1. Does the approach require us to fix the sets of agents and alternatives upfront?
2. Is the universal domain assuming expressed in an elegant manner?
3. Does the approach facilitate automation?

Does the approach offer a new perspective on Arrow’s Theorem (and Social Choice Theory more generally)?

Competing desiderata

1. The voters' inputs (rankings, judgements) should *completely determine* the group decision.

2. The group decision should depend *in the right way* on the voters' inputs.

3. The voters' inputs are not constrained in any way (unless there is good reason to think otherwise).

Competing desiderata

1. The voters' inputs (rankings, judgements) should *completely determine* the group decision. [Dependence]

2. The group decision should depend *in the right way* on the voters' inputs. [Dependence]

3. The voters' inputs are not constrained in any way (unless there is good reason to think otherwise). [Independence]

A primer on dependence logic

Jouko Väänänen: Dependence and independence concepts are ubiquitous. What are the fundamental principles governing them?

J. Väänämen. *Dependence Logic: A New Approach to Independence Friendly Logic*. Cambridge University Press, 2007.

- Logic/Math: $\forall x \exists y R(x, y)$
- Independence Friendly Logic: $\forall x \exists y_{|x} R(x, y)$
  $\forall x \exists y (x = y)$ vs. $\forall x \exists y_{|x} (x = y)$ vs. $\forall x \exists z \exists y_{|x} (x = y)$
- Database: functional dependence
- Probability/statistics
- Quantum Mechanics: No-Go Theorems
- Social Choice
- $\cdots$

Notation

- Fix a first-order language with equality
- A first order structure $\mathcal{M}$ consists of a domain D and an interpretation for the non-logical symbols
- Let $\mathcal{V}$ be the set of variables. $\vec{x}$ denotes a finite sequence of variables

Substitution

A substitution is a function $s : \mathcal{V} \to D$

For any $d \in D$, let $s[d/x] : \mathcal{V} \to D$ be the substitution

$$s[d/x](y) = \begin{cases} s(y) & \text{if } y \in \mathcal{V} - \{x\} \\ d & \text{if } x = y \end{cases}$$

Teams

Dependence/independence can only be observed when there is more than one substitution.

A **team** S is a set of substitutions.

Formulas of dependence/independence logic are interpreted at teams:

$$\mathcal{M}, S \models \varphi$$

Equality, Dependence, Independence

- $x = y$: x equals y

  $\mathcal{M}, X \models x = y$ iff for all $s \in S$, $s(x) = s(y)$

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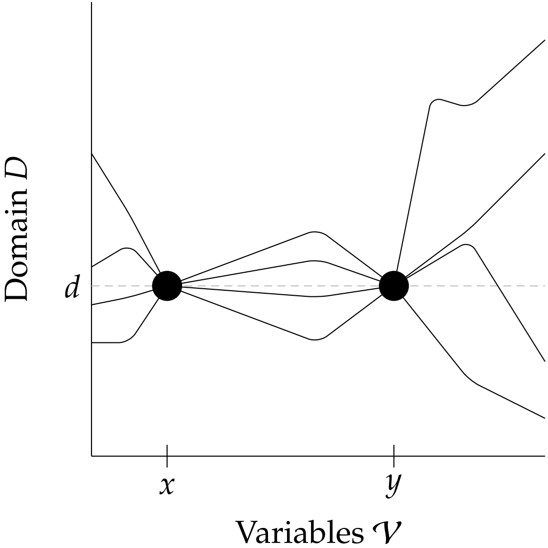
  $\mathcal{M}, X \models\ =\!(x, y)$ iff for all $s, s' \in S$, if $s(x) = s'(x)$ then $s(y) = s'(y)$

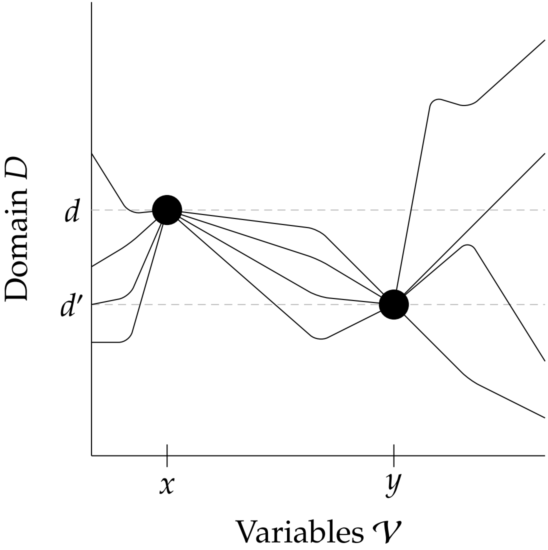
- $x \perp y$: x and y are *completely independent*

  $\mathcal{M}, X \models x \perp y$ iff for all $s, s' \in S$ there exists $s'' \in S$ such that $s''(x) = s(x)$ and $s''(y) = s'(y)$

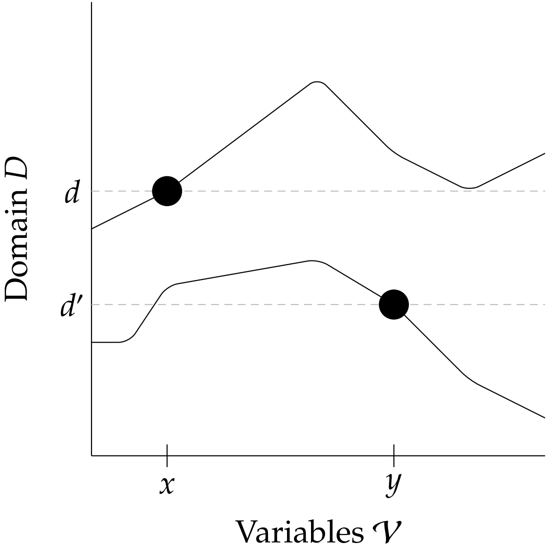
$x = y$



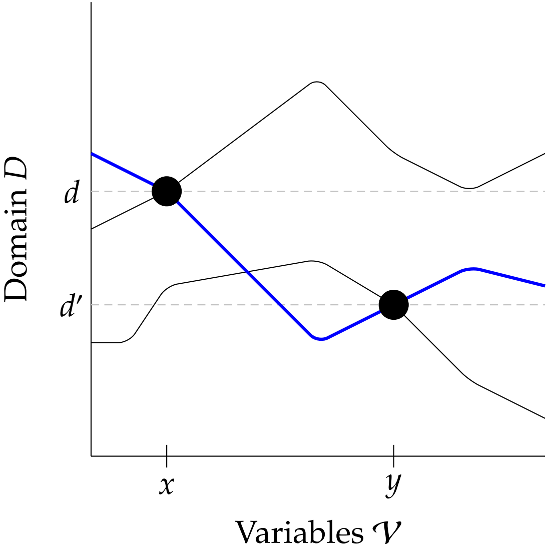
$=(x, y)$



$x \perp y$



$x \perp y$



Truth

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- $\mathcal{M}, S \models \varphi \vee \psi$ iff there exists $Y, Z \subseteq X$ such that $S_1 \cup S_2 = S$, $\mathcal{M}, S_1 \models \varphi$ and $\mathcal{M}, S_2 \models \psi$

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- $\mathcal{M}, S \models \exists x \varphi$ iff there exists S' such that $\mathcal{M}, S' \models \varphi$ such that for all $s \in S$ there is $d \in D$ such that $s[d/x] \in S'$

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- $\mathcal{M}, S \models \forall x \varphi$ iff there is some X' such that $\mathcal{M}, S' \models \varphi$ and for all $s \in X$ and $d \in D$, $s[d/x] \in S'$.

A Few Observations

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- $=\!(x)$ means that x is constant
- $=\!(x) \lor =\!(x)$ is not equivalent to $=\!(x)$
- $\exists z \forall x \exists y (=\!(y, x) \land \neg y = z)$ is true in a model iff the domain is infinite
- $\forall x \exists y \forall u \exists v (=\!(u, v) \land (x = v \leftrightarrow y = u) \land \neg x = y)$ is true in a model iff the domain has even cardinality

Armstrong Axioms

1. $=\!(x, x)$
2. $=\!(y, x)$ and $y \subseteq z$, then $=\!(z, x)$
3. If y is a permutation of x and u a permutation of x and $=\!(z, x)$, then $=\!(y, u)$
4. If $=\!(y, z)$ and $=\!(z, x)$, then $=\!(y, x)$

Theorem. If T is a finite set of dependence atoms of the form $=\!(u, v)$ for various u and v, then $=\!(y, x)$ follows from T according to the above rules if and only if every team that satisfies T also satisfies $=\!(y, x)$

Geiger-Paz-Pearl Axioms

1. $x \perp \emptyset$
2. If $x \perp y$ then $y \perp x$
3. If $x \perp yz$ then $x \perp y$
4. If $x \perp y$ and $xy \perp z$ then $x \perp yz$

Theorem. If T is a finite set of dependence atoms of the form $u \perp v$ for various u and v, then $y \perp x$ follows from T according to the above rules if and only if every team that satisfies T also satisfies $y \perp x$

Dependency Atoms

- $\mathcal{M}, S \models {=}(x, y)$ iff for all $s, s' \in S$, if $s(x) = s'(x)$ then $s(y) = s'(y)$

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- $\mathcal{M}, S \models x \mid y$ iff for all $s, s' \in S$ $s(x) \neq s'(y)$ (i.e., $\{s(x) \mid s \in S\} \cap \{s(y) \mid s \in S\} = \emptyset$)

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- $\mathcal{M}, S \models x \mid y$ iff for all $s, s' \in S$ $s(x) \neq s'(y)$ (i.e., $\{s(x) \mid s \in S\} \cap \{s(y) \mid s \in S\} = \emptyset$)
- $\mathcal{M}, S \models x \perp_z y$ iff for all $s, s' \in S$ if $s(z) = s'(z)$ then there exists $s'' \in S$ such that $s(z) = s''(z)$, $s(x) = s''(x)$ and $s'(y) = s''(y)$

- Reflexivity: $x \perp_x y$
- Symmetry: If $y \perp_x z$, then $z \perp_x y$
- Weakening: If $yy' \perp_x zz'$, then $y \perp_x z$
- First Transitivity: If $x \perp_z y$ and $u \perp_{zx} y$, then $x \perp_{zx} y$
- Second Transitivity: If $y \perp_z y$ and $zx \perp_y u$, then $x \perp_z u$
- Exchange: If $x \perp_z y$ and $xy \perp_z u$, then $x \perp_z yu$

Other Connectives

- Contradictory Negation: $\mathcal{M}, S \models {\sim}\varphi$ iff $\mathcal{M}, S \not\models \varphi$
- Boolean Disjunction: $\mathcal{M}, S \models \varphi \sqcup \psi$ iff $\mathcal{M}, S \models \varphi$ or $\mathcal{M}, S \models \psi$
- Intuitionistic Implication: $\mathcal{M}, S \models \varphi \rightarrow \psi$ iff for all $S' \subseteq S$, if $\mathcal{M}, S' \models \varphi$ then $\mathcal{M}, S' \models \psi$
- Announcement: $\mathcal{M}, S \models \delta x \varphi$ iff for all $d \in Dom(\mathcal{M})$, $\mathcal{M}, S|_{x=d} \models \varphi$

- $=\!(x, y)$ is equivalent to $=\!(x) \to =\!(y)$
- $=\!(x, y)$ is equivalent to $y \perp_x y$
- $DL = FO(=\!(\cdot,\cdot)) = \Sigma^1_1$ (in terms of expressive power) with respect to sentences.
- $DL = FO(=\!(\cdot,\cdot))$ is not axiomatizable
- The first order consequences is axiomatizable: $T \models \varphi$ where φ is a first order formula.
- $\cdots$

EP and F. Yang. *Dependence and Independence in Social Choice Theory*. 2015.

Variables: $V = \{x_1, \ldots, x_n\}$ is a distinguished set of first-order variables (one for each voter) and y is a fresh first-order variable intended to represent the group decision.

Suppose that $\mathbf{R} = (R_1, \ldots, R_n) \in O(X)^n$ is a profile for V and $F : \mathcal{B} \to \mathcal{O}$ is a preference aggregation function with $\mathbf{R} \in \mathcal{B}$.

The pair $(\mathbf{R}, F)$ induces an assignment on $V^+ = \{x_1, \ldots, x_n, y\}$, denoted $s_{\mathbf{R},F} : V^+ \to \mathcal{B} \cup \mathcal{O}$, defined as follows:

$$s_{\mathbf{R},F}(x_1) = R_1, \ \ldots, \ s_{\mathbf{R},F}(x_n) = R_n \ \text{ and } \ s_{\mathbf{R},F}(y) = F(\mathbf{R}).$$

Then, any group decision function F is associated with a set of assignments:

$$S_F = \{s_{\mathbf{R},F} \mid \mathbf{R} \in dom(F)\}$$

	x_1	x_2	y
s_1	$a\ b\ c$	$c\ b\ a$	$b\ a\ c$
s_2	$a\ c\ b$	$b\ c\ a$	$c\ b\ a$
s_3	$c\ a\ b$	$b\ a\ c$	$a\ c\ b$
s_4	$b\ c\ a$	$a\ c\ b$	$c\ a\ b$
s_5	$a\ b\ c$	$b\ c\ a$	$b\ a\ c$

Table: An example of a team for 2 voters.

$$
\begin{aligned}
\varphi ::= {} & \alpha \mid \neg\alpha \mid \bot \mid {=}(w_1, \ldots, w_k, u) \mid w_1 \ldots w_k \perp u_1 \ldots u_m \mid w_1 \ldots w_k \subseteq u_1 \ldots u_k \\
& \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x \varphi \mid \exists x \varphi,
\end{aligned}
$$

- $M \models_S {=\!(}w_1, \ldots, w_k, u)$ iff for all $s, s' \in S$,

$$\text{if } \langle s(w_1), \ldots, s(w_k)\rangle = \langle s'(w_1), \ldots, s'(w_k)\rangle, \text{ then } s(u) = s'(u);$$

- $M \models_S w_1 \ldots w_k \perp u_1 \ldots u_m$ iff for all $s, s' \in S$, there is $s'' \in S$ such that

$$\langle s''(w_1), \ldots, s''(w_k)\rangle = \langle s(w_1), \ldots, s(w_k)\rangle$$

  and

$$\langle s''(u_1), \ldots, s''(u_m)\rangle = \langle s'(u_1), \ldots, s'(u_m)\rangle;$$

- $M \models_S w_1 \ldots w_k \subseteq u_1 \ldots u_k$ iff for all $s \in S$, there is $s' \in S$ such that

$$\langle s'(w_1), \ldots, s'(w_k)\rangle = \langle s(u_1), \ldots, s(u_k)\rangle;$$

Our formalization of Arrow's Theorem requires that the domain contains all linear rankings of (at least three) candidates.

An **intended $\mathcal{L}_X$-model** is a $\mathcal{L}_X$-model M where $dom(M) = L(X)$. The set of intended models is first-order definable using the unary predicates E_R.

For any $e \in dom(M)$ and any linear ranking $R \in L(X)$, the intended interpretation of $E_R^M(e)$ is that e is the linear ranking R, i.e., $E_R^M = \{e \in dom(M) \mid e = R\}$.

For each $e \in dom(M)$, the intended interpretation of $R_{ab}^M(e)$ is that the ranking associated with the element e ranks a above b: For $a, b \in X$, $R_{ab}^M = \{R \in L(X) \mid a\ R\ b\}$.

(*Strict preference*) For each $a, b \in X$, let $P_{ab}(w) \ := \ R_{ab}(w) \wedge \neg R_{ba}(w)$

(*Indifference*) For each $a, b \in X$, let $I_{ab}(w) \ := \ R_{ab}(w) \wedge R_{ba}(w)$

$$(\textit{Unanimity}) \qquad \theta_U := \bigwedge\{(P_{ab}(x_1) \wedge \cdots \wedge P_{ab}(x_n)) \supset P_{ab}(y) \mid a, b \in X\}$$

$$(\textit{Functionality of Preference Aggregation Rule}) \qquad \theta_F := \; =\!(x_1, \ldots, x_n, y)$$

IIA

- $M \models_S {=}(\varphi_1, \ldots, \varphi_k, \psi)$ iff for all $s, s' \in S$, if $s \sim_{\{\varphi_1, \ldots, \varphi_k\}} s'$, then $s \sim_{\{\psi\}} s'$.

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(*Independence of Irrelevant Alternatives*)

$$\theta_{IIA} := \bigwedge \{ =\!(R_{ab}(x_1), R_{ba}(x_1) \ldots, R_{ab}(x_n), R_{ba}(x_n), R_{ab}(y)) \mid a, b \in X\}.$$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$C\ B\ A$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$C\ A\ B$
$C\ B\ A$	$A\ B\ C$	$B\ C\ A$
$C\ A\ B$	$B\ A\ C$	$A\ B\ C$

$S_{A,B} \not\models =\!(x_1, x_2, y)$

	x_1			x_2			y	
A	B			B	A	B	A	
A		B	B		A		B	A
B	A			A	B	A	B	
B		A	A		B		A	B
	B	A	A	B		B		A
	A	B	B	A		A	B	

$S_{A,B} \models =\!(x_1, x_2, y)$

x_1	x_2	y
$A\ B$	$B\ A$	$B\ A$
$A \quad B$	$B \quad A$	$B\ A$
$B\ A$	$A\ B$	$A\ B$
$B \quad A$	$A \quad B$	$A\ B$
$B\ A$	$A\ B$	$A \quad B$
$A\ B$	$B\ A$	$B\ A$

$S_{B,C} \not\models =\!(x_1, x_2, y)$

x_1			x_2			y		
	B	C	C	B		B		C
	C	B	B	C		C	B	
B		C	C		B		B	C
B	C			C	B	C		B
C	B			B	C		C	B
C		B	B		C	B		C

$S_{B,C} \models {=}(x_1, x_2, y)$

x_1			x_2			y		
	B	C	C	B		B		C
	C	B	B	C		C	B	
B		C	C		B		B	C
B	C			C	B	B		C
C	B			B	C		C	B
C		B	B		C	C		B

$S_{A,C} \not\models {=}(x_1, x_2, y)$

x_1	x_2	y
A C	C A	A C
A C	C A	C A
A C	C A	A C
C A	A C	A C
C A	A C	A C
C A	A C	A C

$S_{A,C} \models {=}(x_1, x_2, y)$

x_1	x_2	y
$A \quad C$	$C \quad A$	$A\,C$
$A\,C$	$C\,A$	$A \quad C$
$A\,C$	$C\,A$	$A \quad C$
$C\,A$	$A\,C$	$A\,C$
$C \quad A$	$A \quad C$	$A \quad C$
$C\,A$	$A\,C$	$A\,C$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$

Dictatorship

- $\theta_{D_0}(x_d) := \bigwedge_{a,b \in X} (P_{ab}(x_d) \supset P_{ab}(y)).$

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Dictatorship

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- $M \models_S \varphi \Vee \psi$ iff $M \models_S \varphi$ or $M \models_S \psi$.

$$(\textit{Dictator}) \;\; \theta_D := \mathop{\Vee}_{i=1}^{n} \theta_{D_0}(x_i).$$

Independence

$$(\textit{All Rankings})\ \ \theta_{AR} := \bigwedge\{\forall u(u \subseteq x_i) : 1 \leq i \leq n\}$$

$$(\textit{Independence})\ \ \theta_I := \bigwedge\{\langle x_j\rangle_{j\neq i} \perp x_i : 1 \leq i \leq n\}$$

$S \models \mathsf{all}(x_1) \wedge \mathsf{all}(x_2)$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$

$S \not\models x_1 \perp x_2$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$

$S \models x_1 \perp x_2$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	$?\ ?\ ?$
$\vdots$	$\vdots$	$\vdots$

$$S \models [P_{CA}(x_1) \wedge P_{CA}(x_2)] \supset P_{CA}(y)$$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	$C\ A$
$\vdots$	$\vdots$	$\vdots$

$$S_{\{A,B\}} \models {=}(x_1, x_2, y)$$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	$C\ A$ $A\ B$
$\vdots$	$\vdots$	$\vdots$

$S_{\{B,C\}} \models {=}(x_1, x_2, y)$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	$C\ A$ $A\ B$ $B\ C$
$\vdots$	$\vdots$	$\vdots$

$$S \not\models [P_{AB}(y) \wedge P_{BC}(y)] \supset P_{AC}(y)$$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$A\ B\ C$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	$C\ A$ $A\ B$ $B\ C$
$\vdots$	$\vdots$	$\vdots$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	? ? ?
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	? ? ?
$\vdots$	$\vdots$	$\vdots$

x_1	x_2	y
$A\ B\ C$	$C\ B\ A$	$B\ A\ C$
$A\ C\ B$	$B\ C\ A$	$B\ C\ A$
$B\ A\ C$	$C\ A\ B$	$A\ B\ C$
$B\ C\ A$	$A\ C\ B$	$B\ A\ C$
$C\ B\ A$	$A\ B\ C$	$A\ B\ C$
$C\ A\ B$	$B\ A\ C$	$B\ A\ C$
$C\ A\ B$	$B\ C\ A$	$B\ C\ A$
$\vdots$	$\vdots$	$\vdots$

Theorem (Arrow's Theorem, semantic version)
$\Gamma_{Arrow} \models \theta_D$, where $\Gamma_{Arrow} = \Gamma_{DM} \cup \Gamma_{RK} \cup \{\theta_U, \theta_F, \theta_{IIA}, \theta_{AR}, \theta_I\}$.

Current work: Derivations of Arrow's Theorem and related results.

1. $=(w_1, \ldots, w_k, u)$: The value assigned to v is completely determined by the values assigned to the w_i.
2. $=(\varphi(w_1), \ldots, \varphi(w_k), \varphi(u))$: The truth value of $\varphi(u)$ is completely determined by the truth values of the $\varphi(w_i)$.
3. $(\bigwedge_{i=1}^{k} \varphi(w_i)) \supset \varphi(u)$: If each of the w_i satisfy φ, then u must also satisfy φ.

Concluding Remarks, I

Social choice theory = Preference Logic + ???

Concluding Remarks, I

Social choice theory = Preference Logic + ???

D. Makinson. *Combinatorial versus decision-theoretic components of impossibility theorems*. Theory and Decision 40, 1996, 181-190.

Concluding Remarks, II

Group decision making from a logicians perspective...

1. Logical (and algebraic) methods can be used to prove/generalize various results.
2. Two aspects of judgement aggregation: (1) logically connected agendas and (2) use methods that are more likely to get the answer "correct".
3. Logics for *social epistemology*

General Aggregation Theory

F. Dietrich and C. List. *The aggregation of propositional attitudes: Towards a general theory*. Oxford Studies in Epistemology, Vol. 3, pgs. 215 - 234, 2010.

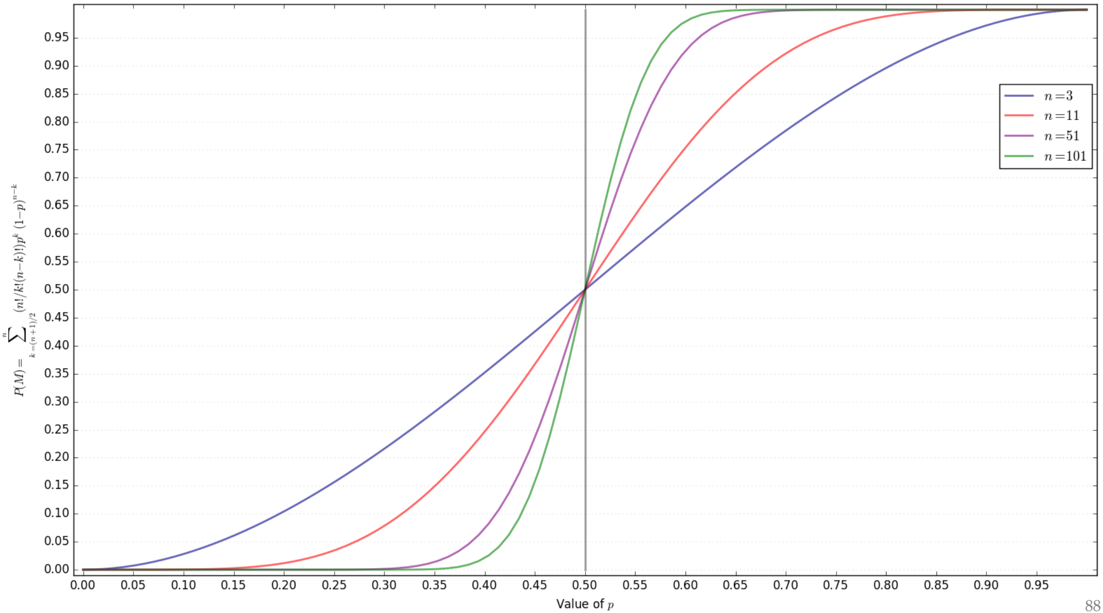
F. Herzberg. *Universal algebra for general aggregation theory: Many-valued propositional-attitude aggregators as MV-homomorphisms*. Journal of Logic and Computation, 2013.

S. Abramsky. *Arrow's Theorem by Arrow Theory*. arxiv, 2013.

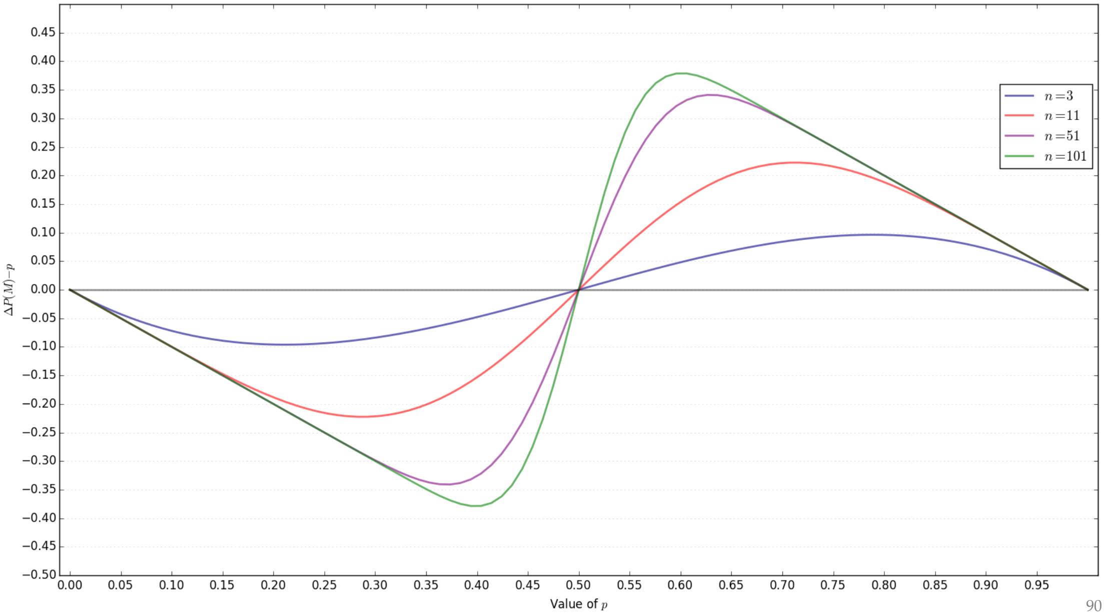
T. Daniëls and EP. *A general approach to aggregation problems*. Journal of Logic and Computation, 19, pgs. 517 - 536, 2009.

L. Bovens and W. Rabinowicz. *Voting Procedures for Complex Collective Decisions. An Epistemic Perspective*. Ratio Juris, 17:2, pp. 241-258, 2004.

$$P(M) = \sum_{k=(n+1)/2}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$


$P(M) = \sum_{k=(n+1)/2}^{n} (n!/k!(n-k)!) p^k (1-p)^{n-k}$
Value of p
$n = 3$
$n = 11$
$n = 51$
$n = 101$
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95

$$\Delta = P(M) - p$$



0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35
−0.40
−0.45
−0.50
$\Delta P(M) - p$
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
Value of p
$n = 3$
$n = 11$
$n = 51$
$n = 101$

	S	F	$D \leftrightarrow (F \wedge S)$
C1	T	T	T
C2	T	F	F
C3	F	T	F
C4	F	F	F

	S	F	$D \leftrightarrow (F \wedge S)$
$C1$	T	T	T
$C2$	T	F	F
$C3$	F	T	F
$C4$	F	F	F

$$P(C1) = q^2$$

$$P(C2) = P(C3) = q(1 - q)$$

$$P(C4) = (1 - q)^2$$

$$P(V \mid C1) = p^2$$

$$P(V \mid C2) = p^2 + p(1 - p) + (1 - p)^2$$

$$P(V \mid C4) = p^2 + 2p(1 - p)$$

$$P(V) = \sum_{i=1}^{4} P(V \mid Ci)P(Ci)$$

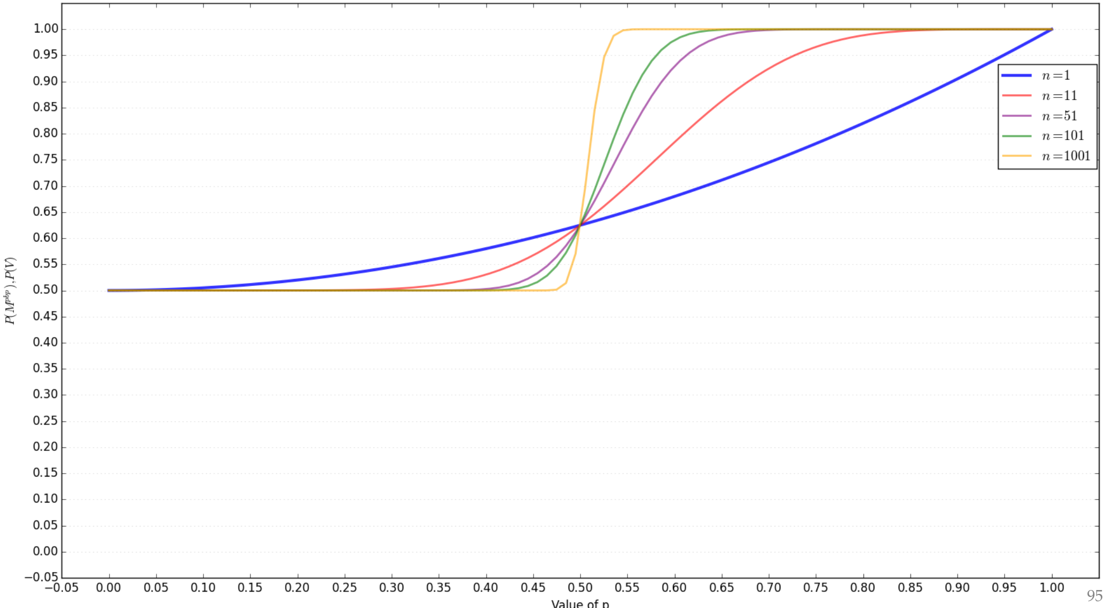
$$P(M^{pbp} \mid C1) = P(M)^2$$

$$P(M^{pbp} \mid C2) = P(M^{pbp} \mid C3) = P(M)^2 + P(M)(1 - P(M)) + (1 - P(M))^2$$

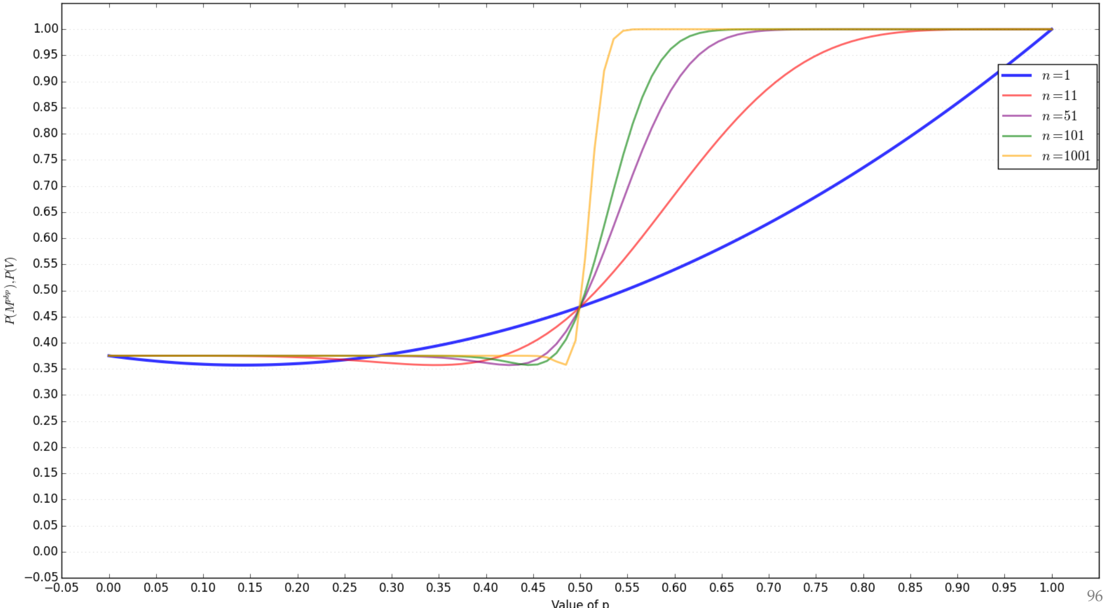
$$P(M^{pbp} \mid C4) = P(M)^2 + 2P(M)(1 - P(M))$$

$$P(M^{pbp}) = \sum_{i=1}^{4} P(M^{pbp} \mid Ci)P(Ci)$$

$q = 0.5$



$q = 0.75$

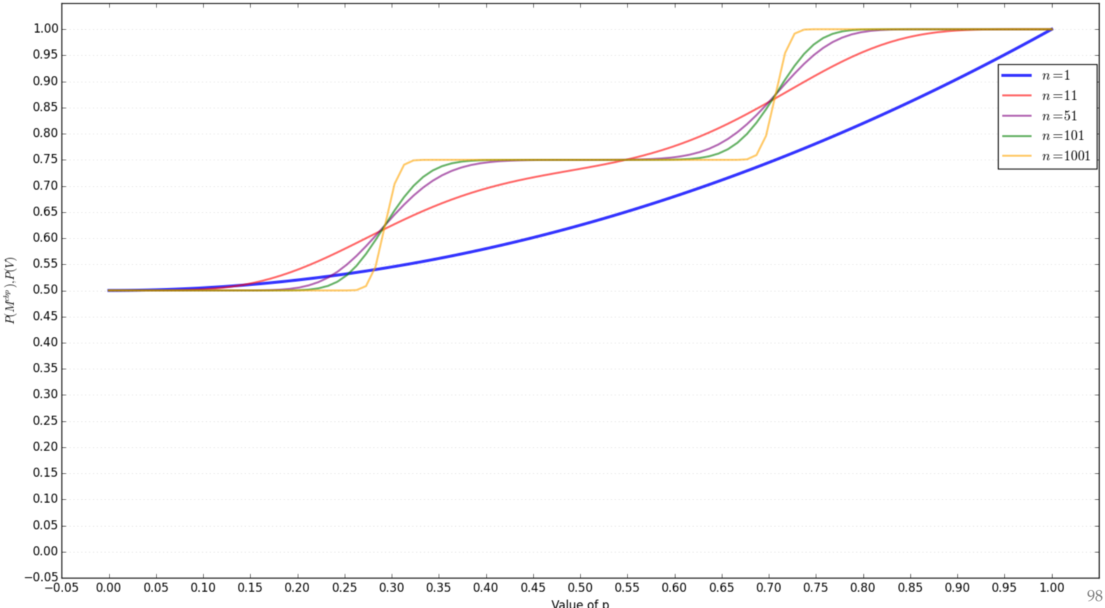


$n=1$
$n=11$
$n=51$
$n=101$
$n=1001$
$P(M^{pbp}), P(V)$
Value of p
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.05 0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00

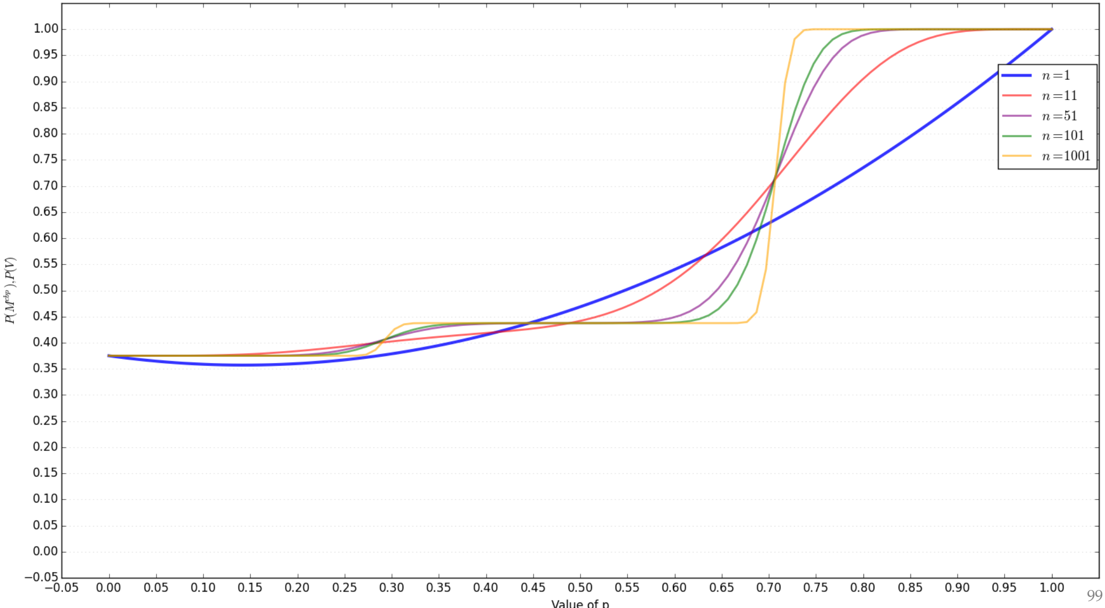
$$P(M^{cbp} \mid Ci) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} P(V \mid Ci)^k (1 - P(V \mid Ci))^{n-k}$$

$$P(M^{cbp}) = \sum_{i=1}^{4} P(M^{cbp} \mid Ci) P(Ci)$$

$q = 0.5$



$q = 0.75$



1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
$P(M^{cbp}), P(V)$
−0.05 0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
Value of p
$n = 1$
$n = 11$
$n = 51$
$n = 101$
$n = 1001$

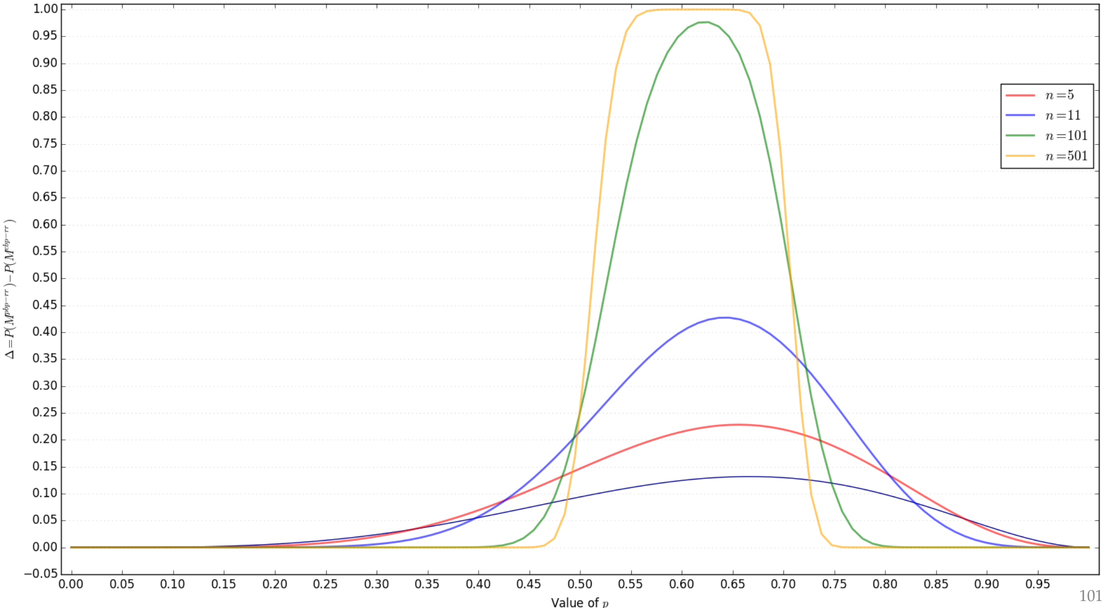
$$P(M^{pbp}) = \sum_{i=1}^{4} P(M^{pbp} \mid Ci)P(Ci)$$

$$P(M^{pbp-rr}) = P(M)^2$$

$$P(M^{cbp}) = \sum_{i=1}^{4} P(M^{cbp} \mid Ci)P(Ci)$$

$$P(M^{cbp-rr}) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p^2(1-p^2)^{n-k}$$

$q = 0.5$



$n = 5$
$n = 11$
$n = 101$
$n = 501$
$\Delta = P(M^{pbp-rr}) - P(M^{cbp-rr})$
Value of p
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95

Topics

- Monday: Introduction, Background, Voting Theory, May's Theorem, Arrow's Theorem
- Tuesday: Social Choice Theory: May's Theorem, Arrow's Theorem, Variants of Arrow's Theorem,
- Wednesday: Weakening Arrow's Conditions (Domain Conditions), Harsanyi's Theorem,
- Thursday: Strategizing (Gibbard-Satterthwaite Theorem) and Iterative Voting/ Introduction to Judgement Aggregation
- Friday: Logics for Social Choice Theory (Modal Logic, Dependence/Independence Logic)
- (Aggregating Judgements: (linear pooling, wisdom of the crowds, prediction markets), Probabilistic Social Choice.)

Thank you!!