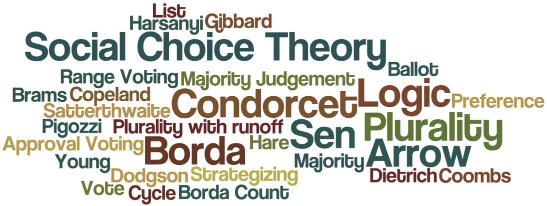
Social Choice Theory for Logicians
ESSLLI 2016

Eric Pacuit
University of Maryland, College Park
pacuit.org



Plan

- Introduction, Background, Voting Theory, May's Theorem, Arrow's Theorem
- Social Choice Theory: Variants of Arrow's Theorem, Weakening Arrow's Conditions (Domain Conditions), Harsanyi's Theorem, Characterizing Voting Methods
- Strategizing (Gibbard-Satterthwaite Theorem) and Iterative Voting/ Introduction to Judgement Aggregation
- Aggregating Judgements (linear pooling, wisdom of the crowds, prediction markets), Probabilistic Social Choice.
- Logics for Social Choice Theory (Preference Logic, Modal Logic, Dependence/Independence Logic, First Order Logic)

Arrow's Theorem

K. Arrow. *Social Choice and Individual Values*. John Wiley & Sons, 1951.

Arrow's Theorem

Let X be a finite set with *at least three elements* and N a finite set of n voters.

Social Welfare Function: $F : \mathcal{D} \to O(X)$ where $\mathcal{D} \subseteq O(X)^n$

Arrow's Theorem

Let X be a finite set with *at least three elements* and N a finite set of n voters.

Social Welfare Function: $F : \mathcal{D} \to O(X)$ where $\mathcal{D} \subseteq O(X)^n$

Reminders:

- $O(X)$ is the set of transitive and complete relations on X
- For $R \in O(X)$, let P_R denote the strict subrelation and I_R the indifference subrelation:
  - $A\ P_R\ B$ iff $A\ R\ B$ and not $B\ R\ A$
  - $A\ I_R\ B$ iff $A\ R\ B$ and $B\ R\ A$

Unanimity

$F : \mathcal{D} \to O(X)$

If each agent ranks A above B, then so does the social ranking.

Unanimity

$F : \mathcal{D} \to O(X)$

If each agent ranks A above B, then so does the social ranking.

For all profiles $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{D}$:

$$\text{If for each } i \in N,\ A\ P_i\ B \text{ then } A\ P_{F(\mathbf{R})}\ B$$

Universal Domain

$F : \mathcal{D} \to O(X)$

Voter's are free to choose any preference they want.

Universal Domain

$F : \mathcal{D} \to O(X)$

Voter's are free to choose any preference they want.

The domain of F is the set of *all* profiles, i.e., $\mathcal{D} = O(X)^n$.

Independence of Irrelevant Alternatives

$F : \mathcal{D} \to O(X)$

The social ranking (higher, lower, or indifferent) of two alternatives A and B depends only the relative rankings of A and B for each voter.

Independence of Irrelevant Alternatives

$F : \mathcal{D} \to O(X)$

The social ranking (higher, lower, or indifferent) of two alternatives A and B depends only the relative rankings of A and B for each voter.

For all profiles $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{R}' = (R'_1, \ldots, R'_n)$:

$$\text{If } R_{i\{A,B\}} = R'_{i\{A,B\}} \text{ for all } i \in N, \text{ then } F(\mathbf{R})_{\{A,B\}} \text{ iff } F(\mathbf{R}')_{\{A,B\}}.$$

where $R_{\{X,Y\}} = R \cap \{X, Y\} \times \{X, Y\}$

IIA For all profiles $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{R}' = (R'_1, \ldots, R'_n)$:

$$\text{If } R_{i\{A,B\}} = R'_{i\{A,B\}} \text{ for all } i \in N, \text{ then } F(\mathbf{R})_{\{A,B\}} \text{ iff } F(\mathbf{R}')_{\{A,B\}}.$$

IIA* For all profiles $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{R}' = (R'_1, \ldots, R'_n)$:

$$\text{If } A\ R_i\ B \text{ iff } A\ R'_i\ B \text{ for all } i \in N, \text{ then } A\ F(\mathbf{R})\ B \text{ iff } A\ F(\mathbf{R}')\ B.$$

Dictatorship

$F : \mathcal{D} \to O(X)$

A voter $d \in N$ is a **dictator** if society strictly prefers A over B *whenever* d strictly prefers A over B.

Dictatorship

$F : \mathcal{D} \to O(X)$

A voter $d \in N$ is a **dictator** if society strictly prefers A over B *whenever* d strictly prefers A over B.

There is a $d \in N$ such that for each profile $\mathbf{R} = (R_1, \ldots, R_d, \ldots, R_n)$, if $A \; P_d \; B$, then $A \; P_{F(\mathbf{R})} \; B$

M. Morreau. *Arrow's Theorem*. Stanford Encyclopedia of Philosophy, 2014.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Arrow's Theorem

D. Campbell and J. Kelly. *Impossibility Theorems in the Arrovian Framework*. Handbook of Social Choice and Welfare Volume 1, pgs. 35 - 94, 2002.

W. Gaertner. *A Primer in Social Choice Theory*. Oxford University Press, 2006.

J. Geanakoplos. *Three Brief Proofs of Arrow's Impossibility Theorem*. Economic Theory, **26**, 2005.

P. Suppes. *The pre-history of Kenneth Arrow's social choice and individual values*. Social Choice and Welfare, 25, pgs. 319 - 326, 2005.

Arrow’s Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\mathbf{R}|_S$ denote the restriction of the profile to candidates in S.

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\mathbf{R}|_S$ denote the restriction of the profile to candidates in S.

Binary Independence: For all profiles $\mathbf{R}, \mathbf{R}'$ and candidates $A, B \in X$:

$$\text{If } \mathbf{R}|_{\{A,B\}} = \mathbf{R}'|_{\{A,B\}}, \text{ then } F(\mathbf{R})|_{\{A,B\}} = F(\mathbf{R}')|_{\{A,B\}}$$

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\mathbf{R}|_S$ denote the restriction of the profile to candidates in S.

Binary Independence: For all profiles $\mathbf{R}, \mathbf{R}'$ and candidates $A, B \in X$:

$$\text{If } \mathbf{R}|_{\{A,B\}} = \mathbf{R}'|_{\{A,B\}}, \text{ then } F(\mathbf{R})|_{\{A,B\}} = F(\mathbf{R}')|_{\{A,B\}}$$

***m*-Ary Independence**: For all profiles $\mathbf{R}, \mathbf{R}'$ and for all $S \subseteq X$ with $|S| = m$:

$$\text{If } \mathbf{R}|_S = \mathbf{R}'|_S, \text{ then } F(\mathbf{R})|_S = F(\mathbf{R}')|_S$$

Weakening IIA

Theorem. (Blau) Suppose that $m = 2, \dots, |X| - 1$. If a social welfare function F satisfies m-ary independence, then it also satisfies binary independence.

J. Blau. *Arrow's theorem with weak independence*. Economica, 38, pgs. 413 - 420, 1971.

S. Cato. *Independence of Irrelevant Alternatives Revisited*. Theory and Decision, 2013.

Let $\mathcal{S} \subseteq \wp(X)$. F is **$\mathcal{S}$-independent** if for all profiles $\mathbf{R}, \mathbf{R}'$, and all $S \in \mathcal{S}$,

$$\text{if } \mathbf{R}|S = \mathbf{R}'|S, \text{ then } F(\mathbf{R})|S = F(\mathbf{R}')|S$$

$\mathcal{S} \subseteq \wp(X)$ is **connected** provided for all $x, y \in X$ there is a finite set $S^1, \ldots, S^k \in \mathcal{S}$ such that

$$\{x, y\} = \bigcap_{j \in \{1,\ldots,k\}} S^j$$

Let $\mathcal{S} \subseteq \wp(X)$. F is **$\mathcal{S}$-independent** if for all profiles $\mathbf{R}, \mathbf{R}'$, and all $S \in \mathcal{S}$,

$$\text{if } \mathbf{R}|S = \mathbf{R}'|S, \text{ then } F(\mathbf{R})|S = F(\mathbf{R}')|S$$

$\mathcal{S} \subseteq \wp(X)$ is **connected** provided for all $x, y \in X$ there is a finite set $S^1, \ldots, S^k \in \mathcal{S}$ such that

$$\{x, y\} = \bigcap_{j \in \{1,\ldots,k\}} S^j$$

Theorem (Sato). (i) Suppose that $\mathcal{S} \subseteq \wp(X)$ is connected. If a collective choice rule F satisfies $\mathcal{S}$-independence, then it also satisfies binary independence.

(ii) Suppose that $\mathcal{S} \subseteq \wp(X)$ is not connected. Then, there exists a social welfare function F that satisfies $\mathcal{S}$-independence and weak Pareto but does not satisfy binary independence.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Weakening Unanimity

$F : \mathcal{D} \to O(X)$

Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A \; P_d \; B$, then $A \; P_{F(\mathbf{R})} \; B$

Inversely Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A \; P_d \; B$, then $B \; P_{F(\mathbf{R})} \; A$

Weakening Unanimity

$F : \mathcal{D} \to O(X)$

Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $A\ P_{F(\mathbf{R})}\ B$

Inversely Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $B\ P_{F(\mathbf{R})}\ A$

Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A\ I_{F(\mathbf{R})}\ B$

Weakening Unanimity

$F : \mathcal{D} \to O(X)$

Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $A\ P_{F(\mathbf{R})}\ B$

Inversely Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $B\ P_{F(\mathbf{R})}\ A$

Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A\ I_{F(\mathbf{R})}\ B$

Non-Imposition: For all $A, B \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $A\ F(\mathbf{R})\ B$

Weakening Unanimity

Theorem (Wilson) Suppose that N is a finite set. If a social welfare function satisfies universal domain, independence of irrelevant alternatives and non-imposition, then it is either null, dictatorial or inversely dictatorial.

R. Wilson. *Social Choice Theory without the Pareto principle*. Journal of Economic Theory, 5, pgs. 478 - 486, 1972.

Y. Murakami. *Logic and Social Choice*. Routledge, 1968.

S. Cato. *Social choice without the Pareto principle: A comprehensive analysis*. Social Choice and Welfare, 39, pgs. 869 - 889, 2012.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Social Choice Functions

$F : \mathcal{D} \to \wp(X) - \emptyset$

Resolute: For all profiles $\mathbf{R} \in \mathcal{D}, |F(\mathbf{R})| = 1$

Non-Imposed: For all candidates $A \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $F(\mathbf{R}) = \{A\}$.

Monotonicity: For all profiles $\mathbf{R}$ and $\mathbf{R}'$, if $A \in F(\mathbf{R})$ and for all $i \in N$, $\mathbf{N}_{\mathbf{R}}(A\ P_i\ B) \subseteq \mathbf{N}_{\mathbf{R}'}(A\ P'_i\ B)$ for all $B \in X - \{A\}$, then $A \in F(\mathbf{R}')$.

Dictator: A voter d is a dictator if for all $\mathbf{R} \in \mathcal{D}$, $F(\mathbf{R}) = \{A\}$, where A is d's top choice.

Social Choice Functions

Muller-Satterthwaite Theorem. Suppose that there are more than three alternatives and finitely many voters. Every resolute social choice function $F : L(X)^n \rightarrow X$ that is monotonic and non-imposed is a dictatorship.

E. Muller and M.A. Satterthwaite. *The Equivalence of Strong Positive Association and Strategy-Proofness*. Journal of Economic Theory, 14(2), pgs. 412 - 418, 1977.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

- Infinitely many voters.
- Domain restrictions.
- Richer ballots.

Universal Domain

Universal Domain: The domain of the social welfare (choice) function is $\mathcal{D} = L(X)^n$ (or $O(X)^n$)

Universal Domain

Universal Domain: The domain of the social welfare (choice) function is $\mathcal{D} = L(X)^n$ (or $O(X)^n$)

Epistemic Rationale: “If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings.” (Arrow, 1963, pg. 24)

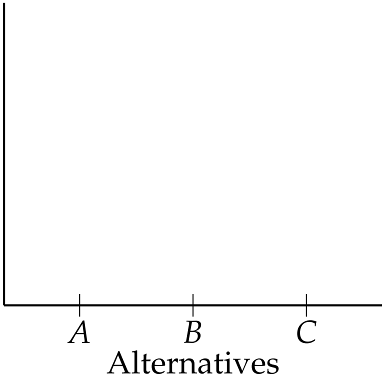
Domain Restrictions

- Single-Peaked preferences
- Sen's Value Restriction
- Assumptions about the distribution of preferences

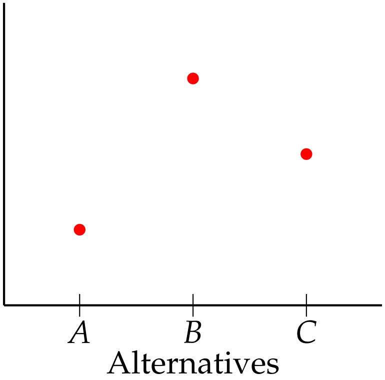
W. Gaertner. *Domain Conditions in Social Choice Theory*. Cambridge University Press, 2001.

1	1	1
A	B	C
B	C	A
C	A	B

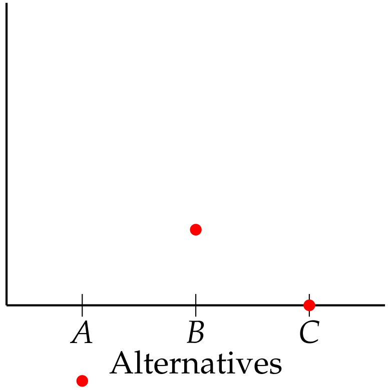
1	1	1
A	B	C
B	C	A
C	A	B



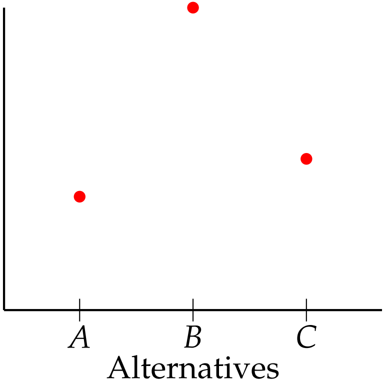
1	1	1
A	B	C
B	C	A
C	A	B



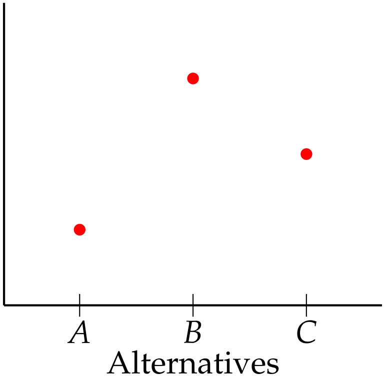
1	1	1
A	B	C
B	C	A
C	A	B



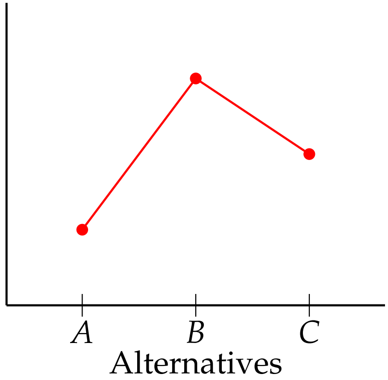
1	1	1
A	B	C
B	C	A
C	A	B



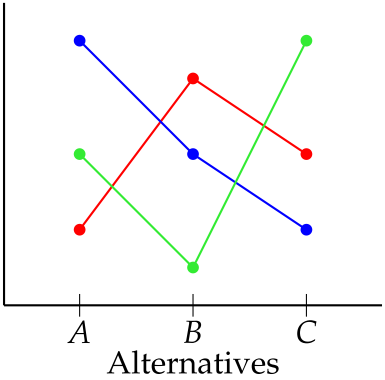
1	1	1
A	B	C
B	C	A
C	A	B



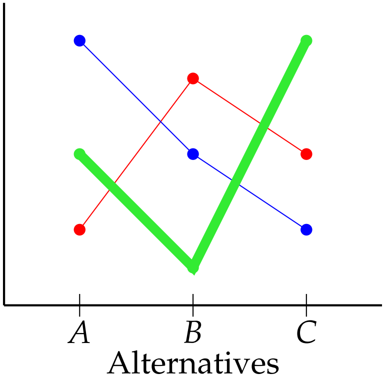
1	1	1
A	B	C
B	C	A
C	A	B



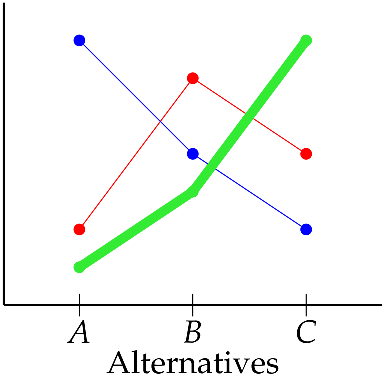
1	1	1
A	B	C
B	C	A
C	A	B



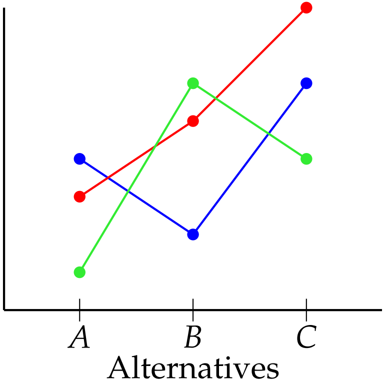
1	1	1
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*



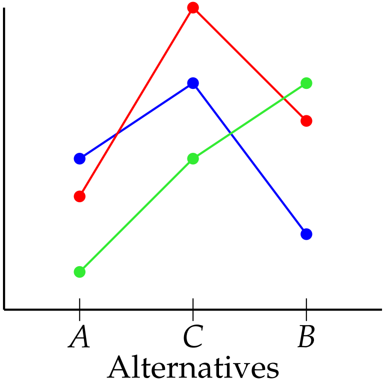
1	1	1
A	B	C
B	C	B
C	A	A



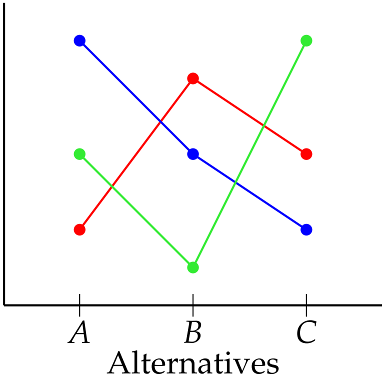
1	1	1
C	*C*	*B*
A	*B*	*C*
B	*A*	*A*



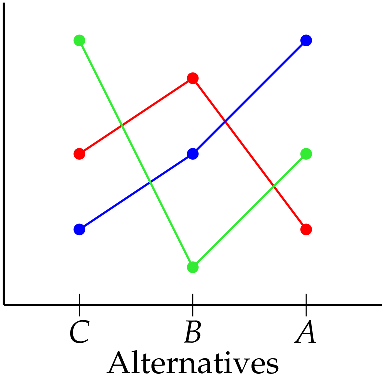
1	1	1
C	C	B
A	B	C
B	A	A



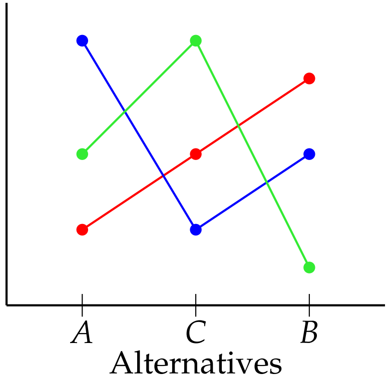
1	1	1
A	B	C
B	C	A
C	A	B



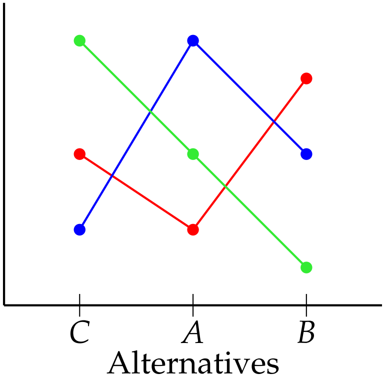
1	1	1
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*



1	1	1
A	B	C
B	C	A
C	A	B



1	1	1
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*



D. Black. *On the rationale of group decision-making*. Journal of Political Economy, 56:1, pgs. 23 - 34, 1948.

Single-Peakedness: the preferences of group members are said to be single-peaked if the alternatives under consideration can be represented as points on a line and each of the utility functions representing preferences over these alternatives has a maximum at some point on the line and slopes away from this maximum on either side.

Single-Peakedness: the preferences of group members are said to be single-peaked if the alternatives under consideration can be represented as points on a line and each of the utility functions representing preferences over these alternatives has a maximum at some point on the line and slopes away from this maximum on either side.

Theorem. If there is an odd number of voters that display single-peaked preferences, then a Condorcet winner exists.

D. Miller. *Deliberative Democracy and Social Choice*. Political Studies, 40, pgs. 54 - 67, 1992.

C. List, R. Luskin, J. Fishkin and I. McLean. *Deliberation, Single-Peakedness, and the Possibility of Meaningful Democracy: Evidence from Deliberative Polls*. Journal of Politics, 75(1), pgs. 80 - 95, 2013.

Sen's Value Restriction

A. Sen. *A Possibility Theorem on Majority Decisions*. Econometrica 34, 1966, pgs. 491 - 499.

Sen's Theorem

Assume n voters (n is odd).

Sen's Theorem

Assume n voters (n is odd).

Triplewise value-restriction: For every triple of distinct candidates A, B, C there exists an $x_i \in \{A, B, C\}$ and $r \in \{1, 2, 3\}$ such that no voter ranks x_i has her rth preference among A, B, C.

Sen's Theorem

Assume n voters (n is odd).

Triplewise value-restriction: For every triple of distinct candidates A, B, C there exists an $x_i \in \{A, B, C\}$ and $r \in \{1, 2, 3\}$ such that no voter ranks x_i has her rth preference among A, B, C.

Theorem (Sen, 1966). For every profile satisfying triplewise value-restriction, pairwise majority voting generates a transitive group preference ordering.

Restrict the *distribution* of preferences

M. Regenwetter, B. Grofman, A.A.J. Marley and I. Tsetlin. *Behavioral Social Choice*. Cambridge University Press, 2006.

Let $\mathbb{P}$ be a probability on $L(X)$.

Let $\mathbb{P}$ be a probability on $L(X)$. For any pair $A, B \in X$, let $\mathbb{P}_{AB}$ be the marginal pairwise ranking probability for A over B:

$$\mathbb{P}_{AB} = \sum_{R \in L(X), ARB} \mathbb{P}(P)$$

Let $\mathbb{P}$ be a probability on $L(X)$. For any pair $A, B \in X$, let $\mathbb{P}_{AB}$ be the marginal pairwise ranking probability for A over B:

$$\mathbb{P}_{AB} = \sum_{R \in L(X), ARB} \mathbb{P}(P)$$

For any triple A, B, C:

$$\mathbb{P}_{ABC} = \sum_{R \in L(X), ARBRC} \mathbb{P}(P)$$

Let $\mathbb{P}$ be a probability on $L(X)$. For any pair $A, B \in X$, let $\mathbb{P}_{AB}$ be the marginal pairwise ranking probability for A over B:

$$\mathbb{P}_{AB} = \sum_{R \in L(X), ARB} \mathbb{P}(P)$$

For any triple A, B, C:

$$\mathbb{P}_{ABC} = \sum_{R \in L(X), ARBRC} \mathbb{P}(P)$$

The **net probability** induced by $\mathbb{P}$ is: $N\!P(R) = \mathbb{P}(R) - \mathbb{P}(R^{-1})$, where $R \in L(X)$ and R^{-1} in the inverse of R ($A\ R^{-1}\ B$ iff $B\ R\ A$).

$N\!P_{ABC} = \mathbb{P}_{ABC} - \mathbb{P}_{CBA}$

Fix three candidates $\{A, B, C\}$

$\mathcal{NP}$ satisfies $NW(C)$ iff $\mathcal{NP}_{ABC} \leq 0$ and $\mathcal{NP}_{BAC} \leq 0$

Fix three candidates $\{A, B, C\}$

$\mathcal{NP}$ satisfies $NW(C)$ iff $\mathcal{NP}_{ABC} \le 0$ and $\mathcal{NP}_{BAC} \le 0$

$\mathcal{NP}$ satisfies $NM(C)$ iff $\mathcal{NP}_{ACB} \le 0$ and $\mathcal{NP}_{BCA} \le 0$ ($\Leftrightarrow \mathcal{NP}_{ACB} = 0$)

Fix three candidates $\{A, B, C\}$

$\mathcal{NP}$ satisfies $NW(C)$ iff $\mathcal{NP}_{ABC} \leq 0$ and $\mathcal{NP}_{BAC} \leq 0$

$\mathcal{NP}$ satisfies $NM(C)$ iff $\mathcal{NP}_{ACB} \leq 0$ and $\mathcal{NP}_{BCA} \leq 0$ ($\Leftrightarrow \mathcal{NP}_{ACB} = 0$)

$\mathcal{NP}$ satisfies $NB(C)$ iff $\mathcal{NP}_{CAB} \leq 0$ and $\mathcal{NP}_{CBA} \leq 0$

Fix three candidates $\{A, B, C\}$

$\mathcal{NP}$ satisfies $NW(C)$ iff $\mathcal{NP}_{ABC} \leq 0$ and $\mathcal{NP}_{BAC} \leq 0$

$\mathcal{NP}$ satisfies $NM(C)$ iff $\mathcal{NP}_{ACB} \leq 0$ and $\mathcal{NP}_{BCA} \leq 0$ ($\Leftrightarrow \mathcal{NP}_{ACB} = 0$)

$\mathcal{NP}$ satisfies $NB(C)$ iff $\mathcal{NP}_{CAB} \leq 0$ and $\mathcal{NP}_{CBA} \leq 0$

$\mathcal{NP}$ is marginally value restricted for the triple $\{Y, Z, W\}$ iff there is an element $C \in \{Y, Z, W\}$ such that $\mathcal{NP}$ satisfies $NW(c)$, $NB(c)$ or $NB(c)$. Net value restriction holds on X if marginal net value restrictions holds on each triple.

Consider a probability $\mathbb{P}$ on $L(X)$. A weak majority preference relation $\succeq$ and a strict majority preference relation $\succ$ are defined as follows:

$$A \succeq B \text{ iff } \mathbb{P}_{AB} \geq \mathbb{P}_{BA}$$

$$A \succ B \text{ iff } \mathbb{P}_{AB} > \mathbb{P}_{BA}$$

Theorem (Regenwetter et al.). The weak majority preference relations is transitive iff for each triple $\{A, B, C\} \subseteq X$ at least one of the following two conditions holds:

1. NP is marginally value restricted on $\{A, B, C\}$

Theorem (Regenwetter et al.). The weak majority preference relations is transitive iff for each triple $\{A, B, C\} \subseteq X$ at least one of the following two conditions holds:

1. NP is marginally value restricted on $\{A, B, C\}$ and, in addition, if at least one net preference is nonzero then the following implication is true $NP_{ABC} = 0 \Rightarrow NP_{BAC} \neq NP_{ACB}$ (with possible relabelings).

Theorem (Regenwetter et al.). The weak majority preference relations is transitive iff for each triple $\{A, B, C\} \subseteq X$ at least one of the following two conditions holds:

1. NP is marginally value restricted on $\{A, B, C\}$ and, in addition, if at least one net preference is nonzero then the following implication is true $NP_{ABC} = 0 \Rightarrow NP_{BAC} \neq NP_{ACB}$ (with possible relabelings).
2. There is a $R_0 \in \{ABC, ACB, BAC, BCA, CAB, CBA\}$ such that R_0 has marginal net preference majority.

Theorem (Regenwetter et al.). The weak majority preference relations is transitive iff for each triple $\{A, B, C\} \subseteq X$ at least one of the following two conditions holds:

1. NP is marginally value restricted on $\{A, B, C\}$ and, in addition, if at least one net preference is nonzero then the following implication is true $NP_{ABC} = 0 \Rightarrow NP_{BAC} \neq NP_{ACB}$ (with possible relabelings).
2. There is a $R_0 \in \{ABC, ACB, BAC, BCA, CAB, CBA\}$ such that R_0 has marginal net preference majority.

   We say CDE has net preference majority provided:

$$NP_{CDE} > \sum_{R' \in \{CED, DEC, DCE, ECD, EDC\}, NP(R') > 0} NP_{R'}$$

- Infinitely many voters.
- Domain restrictions.
- Richer ballots.

Approval Voting: Each voter selects a subset of candidates. The candidate with the most “approvals” wins the election.

S. Brams and P. Fishburn. *Approval Voting*. Birkhauser, 1983.

J.-F. Laslier and M. R. Sanver (eds.). *Handbook of Approval Voting*. Studies in Social Choice and Welfare, 2010.

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

Under ranking voting procedures (such as Borda Count), voters are asked to (linearly) rank the candidates.

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

Under ranking voting procedures (such as Borda Count), voters are asked to (linearly) rank the candidates.

The two pieces of information are related, but not derivable from each other

Under Approval Voting (AV), voters are asked to select the candidates that the voter *approves*.

Under ranking voting procedures (such as Borda Count), voters are asked to (linearly) rank the candidates.

The two pieces of information are related, but not derivable from each other

Approving of a candidate is not (necessarily) the same as simply ranking the candidate first.

Why Approval Voting?

www.electology.org/approval-voting

S. Brams and P. Fishburn. *Going from Theory to Practice: The Mixed Success of Approval Voting*. Handbook of Approval Voting, pgs. 19-37, 2010.

Example

Voters	A	B	C	D
1	1	0	1	1
2	0	1	1	0
3	0	1	0	0
4	0	0	0	0
5	1	1	1	1

Example

Voters	A	B	C	D
1	1	0	1	1
2	0	1	1	0
3	0	1	0	0
4	0	0	0	0
5	1	1	1	1

1	2	3	4	5
A	B	D	D	A
B	C	B	C	B
C	A	C	B	D
D	D	A	A	C

Example

Voters	A	B	C	D
1	1	0	1	1
2	0	1	1	0
3	0	1	0	0
4	0	0	0	0
5	1	1	1	1

1	2	3	4	5
A	B	D	D	A
B	C	B	C	B
C	A	C	B	D
D	D	A	A	C

Example

Voters	A	B	C	D
1	1	0	1	1
2	0	1	1	0
3	0	1	0	0
4	0	0	0	0
5	1	1	1	1

1	2	3	4	5
A	B	D	D	A
B	C	B	C	B
C	A	C	B	D
D	D	A	A	C

An AV ballot is **sincere** if, given the lowest-ranked candidate that a voter approves of, he or she also approves of all candidates ranked higher.

Example

Voters	A	B	C	D
1	1	0	1	1
2	0	1	1	0
3	0	1	0	0
4	0	0	0	0
5	1	1	1	1

1	2	3	4	5
A	B	D	D	A
B	C	B	C	B
C	A	C	B	D
D	D	A	A	C

An AV ballot is **sincere** if, given the lowest-ranked candidate that a voter approves of, he or she also approves of all candidates ranked higher.

Example

Voters	A	B	C	D
1	1	0	1	1
2	0	1	1	0
3	0	1	0	0
4	0	0	0	0
5	1	1	1	1

1	2	3	4	5
A	B	D	D	A
B	C	B	C	B
C	A	C	B	D
D	D	A	A	C

An AV ballot is **sincere** if, given the lowest-ranked candidate that a voter approves of, he or she also approves of all candidates ranked higher.

Approval Voting is more flexible

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is *A*.

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$

Approval Voting is more flexible

There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
Vote-for-1 elects $\{A, B\}$, vote-for-2 elects $\{D\}$, vote-for-3 elects $\{A, B\}$.

Approval Voting is more flexible

AV may elect the Condorcet winner

# voters	2	2	1
	A	B	C
	D	D	A
	B	A	B
	C	C	D

The Condorcet winner is A.
$(\{A\}, \{B\}, \{C, A\})$ elects A under AV.

Possible Failure of Unanimity

# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Possible Failure of Unanimity

# voters	1	1	1
	A	C	D
	B	A	A
	C	B	B
	D	D	C

Approval Winners: A, B

Indeterminate or Responsive?

# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: A, Borda and Condorcet winner: C.

Indeterminate or Responsive?

# voters	6	5	4
	A	B	C
	C	C	B
	B	A	A

Plurality winner: *A*, Borda and Condorcet winner: *C*.
Any combination of *A*, *B* and *C* can be an AV winner (or AV winners).

Generalizing Approval Voting

Generalizing Approval Voting

Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

Generalizing Approval Voting

Ask the voters to provide both a linear ranking of the candidates and the candidates that they approve.

Make the ballots more expressive: Dis&Approval voting, RangeVoting, Majority Judgement

Grading

In many group decision situations, people use measures or grades from a **common language of evaluation** to evaluate candidates or alternatives:

- in figure skating, diving and gymnastics competitions;
- in piano, flute and orchestra competitions;
- in classifying wines at wine competitions;
- in ranking university students;
- in classifying hotels and restaurants, e.g., the Michelin $*$

Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)

Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)
- How should we *aggregate* the grades? (e.g., Average or Median)

Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)
- How should we *aggregate* the grades? (e.g., Average or Median)
- Should there be a “no opinion” option?

Voting by Grading: Questions

- What grading language should be used? (e.g., $A - F$, $0 - 10$, $* - ****$)
- How should we *aggregate* the grades? (e.g., Average or Median)
- Should there be a "no opinion" option?

Voting by Grading: Examples

Approval Voting: voters can assign a single grade “approve” to the candidates

Dis&Approval Voting: voters can approve or disapprove of the candidates

Majority Judgement, Score Voting: voters can assign any grade from a fixed set of grades to the candidates

Score Voting/Range Voting

Fixe a common grading language consisting of, for example, the integers $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest *average* grade is declared the winner.

Score Voting/Range Voting

Fixe a common grading language consisting of, for example, the integers $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest *average* grade is declared the winner.

Suppose A's grades are $\{7, 7, 8, 8, 9, 9, 9, 10\}$. The average grade is 8.375

Suppose B's grades are $\{9, 9, 9, 9, 9, 10, 10, 10\}$. The average grade is 9.375

So, Score Vote (Range Vote) ranks B above candidate A.

Score Voting/Range Voting

Fixe a common grading language consisting of, for example, the integers $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest *average* grade is declared the winner.

Suppose A's grades are $\{7, 7, 8, 8, 9, 9, 9, 10\}$. The average grade is 8.375

Suppose B's grades are $\{9, 9, 9, 9, 9, 10, 10, 10\}$. The average grade is 9.375

So, Score Vote (Range Vote) ranks B above candidate A.

www.electology.org/score-voting and rangevoting.org

Majority Judgement

Fix a common grading language. For example, $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest median grade is declared the winner.

The *median* grade is the grade that is in the middle of the list when the grades are ordered (If there is an even number of judges, then the median grade is the lowest grade in the middle interval.)

Majority Judgement

Fix a common grading language. For example, $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest median grade is declared the winner.

The *median* grade is the grade that is in the middle of the list when the grades are ordered (If there is an even number of judges, then the median grade is the lowest grade in the middle interval.)

Majority Judgement

Fix a common grading language. For example, $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest median grade is declared the winner.

The *median* grade is the grade that is in the middle of the list when the grades are ordered (If there is an even number of judges, then the median grade is the lowest grade in the middle interval.)

Suppose that A's grades are $\{6, 6, 7, 7, \mathbf{7}, 8, 9, 10, 10\}$: The median grade is 7.

Majority Judgement

Fix a common grading language. For example, $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest median grade is declared the winner.

The *median* grade is the grade that is in the middle of the list when the grades are ordered (If there is an even number of judges, then the median grade is the lowest grade in the middle interval.)

Suppose that A's grades are $\{6, 6, 7, 7, \mathbf{7}, 8, 9, 10, 10\}$: The median grade is 7.

Suppose B's grades are $\{6, 6, 6, \mathbf{6}, 9, 9, 9, 10\}$: The median grade is 6.

Majority Judgement

Fix a common grading language. For example, $\{0, 1, 2, \ldots, 10\}$

The candidate with the largest median grade is declared the winner.

The *median* grade is the grade that is in the middle of the list when the grades are ordered (If there is an even number of judges, then the median grade is the lowest grade in the middle interval.)

Suppose that A's grades are $\{6, 6, 7, 7, \mathbf{7}, 8, 9, 10, 10\}$: The median grade is 7.

Suppose B's grades are $\{6, 6, 6, \mathbf{6}, 9, 9, 9, 10\}$: The median grade is 6.

Majority Judgement ranks B above A.

Majority Judgement: Tie-breaking rules

What happens when the median grades are the same?

Majority Judgement: Tie-breaking rules

What happens when the median grades are the same?

A's grades: $\{7, 9, \mathbf{9}, 11, 11\}$
B's grades: $\{8, 9, \mathbf{9}, 10, 11\}$

Majority Judgement: Tie-breaking rules

What happens when the median grades are the same?

A's grades: $\{7, 9, \mathbf{9}, 11, 11\}$
B's grades: $\{8, 9, \mathbf{9}, 10, 11\}$

The second median grade is found:
A's grades: $\{7, \mathbf{9}, 9, 11, 11\}$
B's grades: $\{8, \mathbf{9}, 9, 10, 11\}$

Majority Judgement: Tie-breaking rules

What happens when the median grades are the same?

A's grades: $\{7, 9, \mathbf{9}, 11, 11\}$
B's grades: $\{8, 9, \mathbf{9}, 10, 11\}$

The second median grade is found:
A's grades: $\{7, \mathbf{9}, 9, 11, 11\}$
B's grades: $\{8, \mathbf{9}, 9, 10, 11\}$

The third median grade is found:
A's grades: $\{7, 9, 9, \mathbf{11}, 11\}$
B's grades: $\{8, 9, 9, \mathbf{10}, 11\}$

Majority Judgement: Tie-breaking rules

What happens when the median grades are the same?

A's grades: $\{7, 9, \mathbf{9}, 11, 11\}$
B's grades: $\{8, 9, \mathbf{9}, 10, 11\}$

The second median grade is found:
A's grades: $\{7, \mathbf{9}, 9, 11, 11\}$
B's grades: $\{8, \mathbf{9}, 9, 10, 11\}$

The third median grade is found:
A's grades: $\{7, 9, 9, \mathbf{11}, 11\}$
B's grades: $\{8, 9, 9, \mathbf{10}, 11\}$

So, A is ranked above B.

Example

Suppose that there are five voters, $1, \ldots, 5$ and three candidates *I*, *II*, and *III*. The grades are A, B, C, D, or F (from best to worst).

Example

Suppose that there are five voters, $1, \ldots, 5$ and three candidates *I*, *II*, and *III*. The grades are A, B, C, D, or F (from best to worst).

	1	2	3	4	5
I	A	A	C	D	D
II	B	B	F	B	F
III	D	C	B	A	D

Example

Suppose that there are five voters, $1, \ldots, 5$ and three candidates *I*, *II*, and *III*. The grades are A, B, C, D, or F (from best to worst).

	1	2	3	4	5	Median
I	A	A	C	D	D	C
II	B	B	F	B	F	B
III	D	C	B	A	D	C

Candidate II is the majority judgement winner.

Example

Suppose that there are five voters, $1, \ldots, 5$ and three candidates *I*, *II*, and *III*. The grades are A, B, C, D, or F (from best to worst).

	1	2	3	4	5	Median
I	A	A	C	D	D	C
II	B	B	F	B	F	B
III	D	C	B	A	D	C

Candidate II is the majority judgement winner. *If asked about their preference, 4 voters would rank candidate I above candidate II*

Example

Suppose that there are five voters, $1, \ldots, 5$ and three candidates I, II, and III. The grades are $A = 4, B = 3, C = 2, D = 1$, or $F = 0$ (from best to worst).

	1	2	3	4	5	Average
I	4	4	2	1	1	2.4
II	3	3	0	3	0	1.8
III	1	2	3	4	1	2.2

Candidate II is the Majority Judgement winner. Candidate I is the Score Voting winner

More Information

M. Balinski and R. Laraki. *Majority Judgement: Measuring, Ranking and Electing*. The MIT Press, 2010.

W. D. Smith. *www.rangevoting.org*. .

S. Brams and R. Potthoff. *The Paradox of Grading Systems*. Manuscript, 2015.

A grading system is a voting system in which a voter can give any of g grades, $\{w_1, \ldots, w_g\}$, to each candidates.

AG winner: Candidate(s) that receives the largest average grade

SG winner: compare each candidate's grades with the grades of all other candidates. Candidate X beats candidate Y if the number of voters who grade X higher than Y exceed the number of voters that grade Y higher than X. The candidate(s) that beat every other candidate is(are) the SG winner(s).

Weak Paradox of Grading Systems

Grades: $\{0, 1, 2\}$
Candidates: $\{A, B, C\}$
9 Voters

# voters	2	3	4	Avg
A	2	0	1	
B	1	2	0	
C	0	1	2	

Grades: $\{0, 1, 2\}$
Candidates: $\{A, B, C\}$
9 Voters

# voters	2	3	4	Avg
A	2	0	1	8/9
B	1	2	0	8/9
C	0	1	2	11/9

Grades: $\{0, 1, 2\}$

Candidates: $\{A, B, C\}$

9 Voters

# voters	2	3	4	Avg
A	2	0	1	8/9
B	1	2	0	8/9
C	0	1	2	11/9

Average Grade Winner: C

Grades: $\{0, 1, 2\}$
Candidates: $\{A, B, C\}$
9 Voters

# voters	2	3	4	Avg
A	2	0	1	
B	1	2	0	
C	0	1	2	

Average Grade Winner: C

$A \succ B$

Grades: $\{0, 1, 2\}$
Candidates: $\{A, B, C\}$
9 Voters

# voters	2	3	4	Avg
A	2	0	1	
B	1	2	0	
C	0	1	2	

Average Grade Winner: C

$$A \succ B \succ C$$

Grades: $\{0, 1, 2\}$
Candidates: $\{A, B, C\}$
9 Voters

# voters	2	3	4	Avg
A	2	0	1	
B	1	2	0	
C	0	1	2	

Average Grade Winner: C

$$A \succ B \succ C \succ A$$

Grades: $\{0, 1, 2\}$

Candidates: $\{A, B, C\}$

9 Voters

# voters	2	3	4	Avg
A	2	0	1	
B	1	2	0	
C	0	1	2	

Average Grade Winner: C

Superior Grade Winners: A, B, C

Strong Paradox of Grading Systems

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	5/3
B	0	3	1	4/3
C	0	3	1	4/3

Average Grade Winner: A

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$B \succ A$$

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$C \sim B \succ A$$

Grades: $\{0, 1, 2, 3\}$

Candidates: $\{A, B, C\}$

3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

$$C \sim B \succ A$$

Grades: $\{0, 1, 2, 3\}$
Candidates: $\{A, B, C\}$
3 Voters

# voters	1	1	1	Avg
A	3	2	0	
B	0	3	1	
C	0	3	1	

Average Grade Winner: A

Superior Grade Winners: C, B

Grades: $\{0, 1, 2, 3, 4, 5\}$
Candidates: $\{A, B, C\}$
5 Voters

# voters	1	4	Avg
A	5	0	5/5
B	0	1	4/5
C	0	1	4/5

Average Grade Winner: A

Superior Grade Winner: B, C

To conclude, we have identified a paradox of grading systems, which is not just a mirror of the well-known differences that crop up in aggregating votes under ranking systems. Unlike these systems, for which there is no accepted way of reconciling which candidate to choose when, for example, the Hare, Borda and Condorcet winners differ, AV provides a solution when the AG and SG winners differ.

Theorem. When there are two grades, the AG and SG winners are identical.

Harsanyi’s Theorem

Assume that there is a finite number of citizens ($N = \{1, \ldots, n\}$), and a finite set of social states X.

Assume that there is a finite number of citizens ($N = \{1, \dots, n\}$), and a finite set of social states X.

Assume that there is a *Planner*.

- The planner's utility function matches the social utility function
- If the Planner is a citizen, he is required to have two (but not necessarily different) preference orderings — his personal ordering and his moral ordering.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

Subjective Expected Utility

Probability: Suppose that $W = \{w_1, \ldots, w_n\}$ is a finite set of states. A probability function on W is a function $P : W \to [0, 1]$ where $\sum_{w \in W} P(w) = 1$ (i.e., $P(w_1) + P(w_2) + \cdots + P(w_n) = 1$).

Suppose that A is an act for a set of outcomes O (i.e., $A : W \to O$) and $u : O \to \mathbb{R}$ is a **cardinal utility function**. The **expected utility** of A is:

$$\sum_{w \in W} P(w) * u(A(w))$$

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

Cardinal scales:

- **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.

Ordinal vs. Cardinal Utility

Ordinal scale: Qualitative comparisons of objects allowed, no information about differences or ratios.

Cardinal scales:

> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
>
> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects.

Interval scale: E.g., the difference between 75°F and 70°F is the same as the difference between 30°F and 25°F However, 70°F (= 21.11°C) is **not** twice as hot as 35°F (= 1.67°C). The difference between 70°F and 65°F is **not** the same as the difference between 25°C and 20°C.

Ratio scale: E.g., 10lb is twice as much as 5lb. But, 10kg is not twice as much as 5lb.

Suppose that X is a set of outcomes.

A **(simple) lottery** over X is denoted $[x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$ where for $i = 1, \ldots, n$, $x_i \in X$ and $p_i \in [0, 1]$, and $\sum_i p_i = 1$.

Let $\mathcal{L}$ be the set of (simple) lotteries over X. We identify elements $x \in X$ with the lottery $[x : 1]$.

Suppose that $\succeq$ is a relation on $\mathcal{L}$.

Axioms

Preference $\succeq$ is reflexive, transitive and complete

Compound Lotteries The decision maker is indifferent between every compound lottery and the *corresponding* simple lottery.

Independence For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0,1]$, $L_1 \succ L_2$
if, and only if,
$[L_1 : a, L_3 : (1-a)] \succ [L_2 : a, L_3 : (1-a)]$.

Continuity For all $L_1, L_2, L_3 \in \mathcal{L}$ and $a \in (0,1]$,
if $L_1 \succ L_2 \succ L_3$, then there exists $a \in (0,1)$
such that $[L_1 : a, L_3 : (1-a)] \sim L_2$

Cardinal Utility Theory

Von Neumann-Morgenstern Theorem. If an agent satisfies the previous axioms, then the agent's ordinal utility function can be turned into cardinal utility function.

Cardinal Utility Theory

Von Neumann-Morgenstern Theorem. If an agent satisfies the previous axioms, then the agent's ordinal utility function can be turned into cardinal utility function.

- Utility is unique only *up to linear transformations*. So, it still does not make sense to add two different agents cardinal utility functions.

Cardinal Utility Theory

Von Neumann-Morgenstern Theorem. If an agent satisfies the previous axioms, then the agent's ordinal utility function can be turned into cardinal utility function.

- Utility is unique only *up to linear transformations*. So, it still does not make sense to add two different agents cardinal utility functions.
- Issue with continuity: 1EUR $\succ$ 1 cent $\succ$ death, but who would accept a lottery which is p for 1EUR and $(1 - p)$ for death??

Cardinal Utility Theory

Von Neumann-Morgenstern Theorem. If an agent satisfies the previous axioms, then the agent's ordinal utility function can be turned into cardinal utility function.

- Utility is unique only *up to linear transformations*. So, it still does not make sense to add two different agents cardinal utility functions.
- Issue with continuity: 1EUR $\succ$ 1 cent $\succ$ death, but who would accept a lottery which is p for 1EUR and $(1-p)$ for death??
- Important issues about how to identify correct descriptions of the outcomes and options.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

Individual and Social Rationality Each citizen and the Planner have a ranking $\succeq_1, \succeq_2, \ldots, \succeq_n, \succeq$ over $\mathcal{L}(X)$ (the set of lotteries over the social states X) satisfying the Von Neumann-Morgenstern axioms.

- Each citizen's preference is represented by a linear utility function u_i
- The Planner's preference is represented by a linear utility function u
- Assume that all the citizens use 0 to 1 utility scales.
- Assume that 0 is the lowest utility scale for the Planner.

Strong Pareto

(P1) For each L, L' if $L \sim_i L'$ for all $i \in N$, then $L \sim L'$

(P2) For each L, L' if $L \succeq_i L'$ for all $i \in N$ and $L \succ_j L'$ for some $j \in N$, then $L \succ L'$

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \Re^n$. That is, the sequence of utility values of L for each agent.

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \Re^n$. That is, the sequence of utility values of L for each agent.

Defined the following two sets:

$$\mathcal{R}^n = \{(r_1, \ldots, r_n) \in \Re^n \mid \text{ there is a } L \in \mathcal{L} \text{ such that for all } i = 1, \ldots, n, u_i(L) = r_i\}$$

and

$$\mathcal{R} = \{r \in \Re \mid \text{there is a } L \in \mathcal{L} \text{ such that } u(L) = r\}$$

Each lottery L is associated with a vector of real numbers, $(u_i(L), \ldots, u_n(L)) \in \Re^n$. That is, the sequence of utility values of L for each agent.

Defined the following two sets:

$$\mathcal{R}^n = \{(r_1, \ldots, r_n) \in \Re^n \mid \text{ there is a } L \in \mathcal{L} \text{ such that for all } i = 1, \ldots, n, u_i(L) = r_i\}$$

and

$$\mathcal{R} = \{r \in \Re \mid \text{there is a } L \in \mathcal{L} \text{ such that } u(L) = r\}$$

Define a function $f : \mathcal{R}^n \to \mathcal{R}$ as follows: for all $(r_1, \ldots, r_n)$, let $f(r_1, \ldots, r_n) = r$ where $r = u(L)$ with L a lottery such that $(u_1(L), \ldots, u_n(L)) = (r_1, \ldots, r_n)$.

Observation. The function f is well-defined.

Observation. The function f is well-defined.

Proof. Suppose that $L, L' \in \mathcal{L}$ such that $(u_1(L), \ldots, u_n(L)) = (u_1(L'), \ldots, u_n(L'))$. Then, for all $i \in N$, i is indifferent between L and L' (i.e., $L \sim_i L'$). Then, by axiom $P1$, we have $L \sim L'$. Thus, $u(L) = u(L')$; and so, f is well-defined.

Equity

(E) All agents should be treated equally by the Planner. Formally, this means that $f(r_1, \ldots, r_n) = f(r'_1, \ldots, r'_n)$ when there is a permutation $\pi : N \to N$ such that for each $i = 1, \ldots, n$, $r'_i = r_{\pi(i)}$.

Harsanyi's Theorem For all $(r_1, \ldots, r_n) \in \mathcal{R}^n$, $f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$.

For each $i \in N$ and $L \in \mathcal{L}$, we have $0 \leq u_i(L) \leq 1$.

For each $i \in N$, let $e_i = (0, 0, \ldots, 1, \ldots, 0)$ (where there is a 1 in the ith position and 0 everywhere else).

This corresponds to a situation in which a single agent gets her most preferred outcome while all the other agents get their least-preferred outcome.

Lemma. For each $i, j \in N$, $f(e_i) = f(e_j)$

Lemma. For all $a \in \mathfrak{R}$, $af(r_1, \ldots, r_n) = f(ar_1, \ldots, ar_n)$.

Let L be the lottery such that for each $i \in N$, $u_i(L) = r_i$. Consider the lottery $L' = [L : a, \mathbf{0} : (1 - a)]$, where $\mathbf{0}$ is the lottery in which everyone gets their lowest-ranked outcome.

Then, for each $i \in N$, $u_i(\mathbf{0}) = 0$. Furthermore, by the Pareto principle $P1$, we must have $u(\mathbf{0}) = 0$.

Then, for all $i \in N$, we have

1. $u_i(L') = au_i(L) + (1-a)u_i(\mathbf{0}) = au_i(L) = ar_i$; and
2. $u(L') = au(L) + (1-a)u(\mathbf{0}) = au(L)$

$$af(r_1, \ldots, r_n) \quad = \quad au(L) \qquad\qquad \text{(definition of } f\text{)}$$

Then, for all $i \in N$, we have

1. $u_i(L') = au_i(L) + (1-a)u_i(\mathbf{0}) = au_i(L) = ar_i$; and
2. $u(L') = au(L) + (1-a)u(\mathbf{0}) = au(L)$

$$\begin{array}{rcll} af(r_1,\ldots,r_n) & = & au(L) & \text{(definition of } f\text{)} \\ & = & u(L') & \text{(item 2.)} \end{array}$$

Then, for all $i \in N$, we have

1. $u_i(L') = au_i(L) + (1 - a)u_i(\mathbf{0}) = au_i(L) = ar_i$; and
2. $u(L') = au(L) + (1 - a)u(\mathbf{0}) = au(L)$

$$\begin{aligned} af(r_1, \ldots, r_n) &= au(L) && \text{(definition of } f\text{)} \\ &= u(L') && \text{(item 2.)} \\ &= f(u_1(L'), \ldots, u_n(L')) && \text{(definition of } f\text{)} \end{aligned}$$

Then, for all $i \in N$, we have

1. $u_i(L') = au_i(L) + (1-a)u_i(\mathbf{0}) = au_i(L) = ar_i$; and
2. $u(L') = au(L) + (1-a)u(\mathbf{0}) = au(L)$

$$
\begin{aligned}
af(r_1, \dots, r_n) &= au(L) && \text{(definition of } f\text{)} \\
&= u(L') && \text{(item 2.)} \\
&= f(u_1(L'), \dots, u_n(L')) && \text{(definition of } f\text{)} \\
&= f(ar_1, \dots ar_n) && \text{(item 1.)}
\end{aligned}
$$

Theorem. For all $(r_1, \ldots, r_n) \in \mathcal{R}^n, f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$.

Consider a lottery L such that for all $i \in N$, $u_i(L) = r_i$. Consider lotteries L_i such that $u_i(L_i) = r_i$ and for all $j \neq i$, $u_j(L_i) = 0$. Consider the lottery $L' = [L_1 : 1/n, \ldots, L_n : 1/n]$.

Consider a lottery L such that for all $i \in N$, $u_i(L) = r_i$. Consider lotteries L_i such that $u_i(L_i) = r_i$ and for all $j \neq i$, $u_j(L_i) = 0$. Consider the lottery $L' = [L_1 : 1/n, \ldots, L_n : 1/n]$.

- $u_i(L') = \sum_{k=1}^{n} \frac{1}{n} u_i(L_k) = \frac{1}{n} u_i(L_i) = \frac{1}{n} r_i$.
- $f(0, \ldots, r_k, \ldots, 0) = r_k f(0, \ldots, 1, \ldots, 0) = r_k$

Consider a lottery L such that for all $i \in N$, $u_i(L) = r_i$. Consider lotteries L_i such that $u_i(L_i) = r_i$ and for all $j \neq i$, $u_j(L_i) = 0$. Consider the lottery $L' = [L_1 : 1/n, \ldots, L_n : 1/n]$.

	1	2	P
L_1	r_1	0	$f(r_1, 0) = r_1 f(1, 0)$
L_2	0	r_2	$f(0, r_2) = r_2 f(0, 1)$
L'	$\frac{1}{2}u(L_1) + \frac{1}{2}u(L_2) = \frac{1}{2}r_1$	$\frac{1}{2}u(L_1) + \frac{1}{2}u(L_2) = \frac{1}{2}r_1$	$f(\frac{1}{2}r_1, \frac{1}{2}r_2)$

$$\frac{1}{2} f(r_1, r_2) = f(\frac{1}{2} r_1, \frac{1}{2} r_2) = u(L') = \frac{1}{2} u(L_1) + \frac{1}{2} u(L_2) = \frac{1}{2} r_1 f(1, 0) + \frac{1}{2} r_2 f(0, 1)$$

$$u(L') = \sum_{k=1}^{n} \frac{1}{n} u(L_k)$$

$$
\begin{aligned}
u(L') &= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(u_1(L_k), \ldots, u_k(L_k), \ldots, u_n(L_k))
\end{aligned}
$$

$$
\begin{aligned}
u(L') &= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(u_1(L_k), \ldots, u_k(L_k), \ldots, u_n(L_k)) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(0, \ldots, r_k, \ldots, 0)
\end{aligned}
$$

$$
\begin{aligned}
u(L') &= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(u_1(L_k), \ldots, u_k(L_k), \ldots, u_n(L_k)) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(0, \ldots, r_k, \ldots, 0) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, r_k f(0, \ldots, 1, \ldots, 0)
\end{aligned}
$$

$$
\begin{aligned}
u(L') &= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, u(L_k) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(u_1(L_k), \ldots, u_k(L_k), \ldots, u_n(L_k)) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n} f(0, \ldots, r_k, \ldots, 0) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, r_k f(0, \ldots, 1, \ldots, 0) \\
&= \textstyle\sum_{k=1}^{n} \frac{1}{n}\, r_k
\end{aligned}
$$

$$u(L') = f(u_1(L'), \ldots, u_n(L'))$$

$$
\begin{aligned}
u(L') &= f(u_1(L'), \ldots, u_n(L')) \\
&= f(\tfrac{1}{n}\, r_1, \ldots, \tfrac{1}{n} r_n)
\end{aligned}
$$

$$
\begin{aligned}
u(L') &= f(u_1(L'), \ldots, u_n(L')) \\
&= f(\tfrac{1}{n}\, r_1, \ldots, \tfrac{1}{n} r_n) \\
&= \tfrac{1}{n} f(r_1, \ldots, r_n)
\end{aligned}
$$

Thus,

$$\frac{1}{n} f(r_1, \ldots, r_k) = u(L') = \sum_{k=1}^{n} \frac{1}{n}\, r_k = \frac{1}{n}\, \sum_{k=1}^{n} r_k$$

Hence, $f(r_1, \ldots, r_n) = r_1 + \cdots + r_n$, as desired.

For 2 citizens, Harsanyi's Theorem require the existence of the following vectors of utilities:

$$(0,0) \quad (0,1) \quad (1,0) \quad (u_1,0) \quad (0,u_2) \quad (u_1,u_2)$$

For 2 citizens, Harsanyi's Theorem require the existence of the following vectors of utilities:

$$(0,0) \quad (0,1) \quad (1,0) \quad (u_1,0) \quad (0,u_2) \quad (u_1,u_2)$$

Problem. None of Harsanyi's conditions guarantee the existence of this social outcomes.

Suppose the problem is to give a scholarship to *exactly* one of the citizens.

- $(1, 0)$: give the scholarship to citizen 1
- $(0, 1)$: give the scholarship to citizen 2

Suppose the problem is to give a scholarship to *exactly* one of the citizens.

- $(1, 0)$: give the scholarship to citizen 1
- $(0, 1)$: give the scholarship to citizen 2

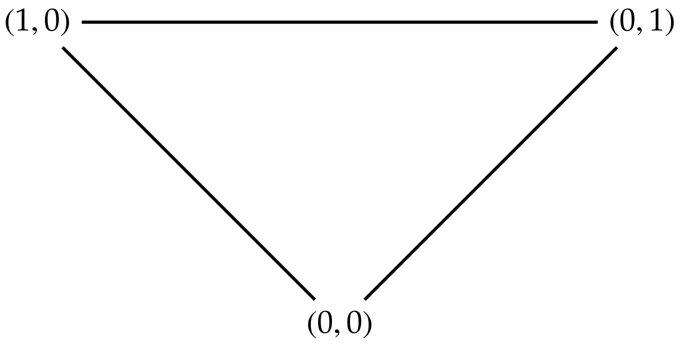
- What is the outcome $(0, 0)$?

Distributable Goods Assumption

For every vector of numbers $(u_1, \ldots, u_n)$ with $0 \leq u_i \leq 1$, there is at least one social option for which the distribution of citizens' utilities equals the vector in question.

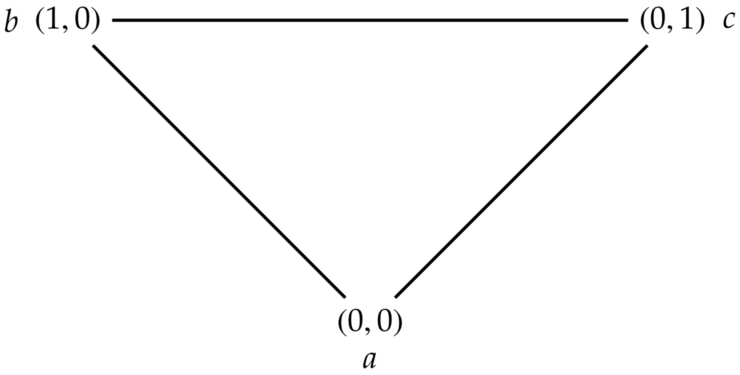
A distributable good is one, such as food, health, education, talent, friendship, for which all distributions throughout society are at least logically possible.

Problem: Philosophers also look to social choice theory for help in resolving problems in which interests conflict-situations, for example, in which citizens gain only at the expense of others, or ones in which the citizens envy each other, or prefer to sacrifice for each other. These are situations in which we cannot count on the distributable goods assumption to hold.



- $(1, 0)$ is the best for citizen 1 and worst for citizen 2
- $(0, 1)$ is the best for citizen 2 and worst for citizen 1
- $(0, 0)$ is the worst for both citizens

Special Prospects Assumption. There are three social options a, b and c such that (1) the first citizen prefers b to a and is indifferent between a and c, (2) the second citizen prefers c to a and is indifferent between a and$b$.



- 1 prefers b to a and is indifferent between a and c
- 2 prefers c to a and is indifferent between a and b.
- a, b, and c can be very similar or "close" to each other

- A defense of the theorem must argue either that a "true" representation of the citizens' preferences will give rise to the appropriate vectors or that there is a set of "background" options sufficiently rich to support the same vectors, or that certain profiles, such as those in which considerations of envy or altruism are operative, should not be considered.

Mary seashore $\succ_M$ museums $\succ_M$ camping

Sam camping $\succ_S$ museums $\succ_S$ seashore

- The seashore is the only alternative that Mary finds bearable, although she feels more negative about going to the mountains than to the museums.
- Each choice is fine with Sam, although he would much prefer going to the mountains.

	Mary	Sam	Total
Seashore	20		
Museums	10		
Mountains	9		

	Mary	Sam	Total
Seashore	20	86	
Museums	10	93	
Mountains	9	100	

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

For Mary, the difference between the seashore and the mountains crosses the threshold between the bearable and the intolerable. She feels that her "right to an emotionally recuperative vacation will be violated by following a utilitarian scheme.

	Mary	Sam	Total
Seashore	200	86	286
Museums	100	93	190
Mountains	90	100	190

Mary: My preferences are so intense in comparison with yours that my scale should range between 0 and 1,000, if yours range between 0 and 100.

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

Sam: You think that my preferences are rather weak, but the fact is I feel things quite deeply. I have been brought up in a culture very different from yours and have been trained to avoid emotional outbursts...But I have strong feelings all the same.

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

Sam: I do not think that extra weight should be given in a utilitarian calculation to those who are capable of more intense preferences. , the difference between the seashore and the mountains crosses the threshold between the bearable and the intolerable. She feels that her “right to an emotionally recuperative vacation will be violated by following a utilitarian scheme.

- Is Mary's preference for the seashore *really* stronger than Sam's for the mountains? Or, is Mary just a more vocal person?
- If some people's preferences are in fact stronger than others', how could we *know* this?
- Does it make any more sense to compare Sam's preferences with Mary's than it does to compare a dog's preference for steak bones with a horse's preference for oats?
- Even if we answer all these questions affirmatively, is it morally proper to respond to such information in making social choices?

Can’t we just wait for psychologists to develop an adequate theory of emotions?

Don’t we make interpersonal comparisons all the time?

Can’t we just wait for psychologists to develop an adequate theory of emotions?

Don’t we make interpersonal comparisons all the time?

Is there more to emotions than our display of them?