Social Choice Theory for Logicians
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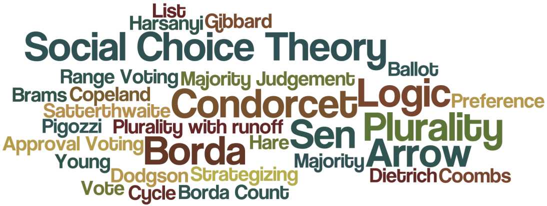

Plan

- Introduction, Background, Voting Theory, May's Theorem, Arrow's Theorem
- Social Choice Theory: Variants of Arrow's Theorem, Weakening Arrow's Conditions (Domain Conditions), Harsanyi's Theorem, Characterizing Voting Methods
- Strategizing (Gibbard-Satterthwaite Theorem) and Iterative Voting/ Introduction to Judgement Aggregation
- Aggregating Judgements (linear pooling, wisdom of the crowds, prediction markets), Probabilistic Social Choice.
- Logics for Social Choice Theory (Modal Logic, Dependence/Independence Logic, First Order Logic)

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Pragmatic considerations: Is the procedure easy to use? Is it legal? The importance of ease of use should not be underestimated: Despite its many flaws, plurality rule is, by far, the most commonly used method.

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- **Unanimity (Pareto)**: If *everyone* ranks A above B, then B should not win the election.
- **Anonymity**: The names of the voters do not matter (if two voters swap votes, then the outcome is unaffected).

Monotonicity

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More-is-Less Paradox: If a candidate C is elected under a given a profile of rankings of the competing candidates, it is possible that, *ceteris paribus*, C may not be elected if some voter(s) raise C in their rankings.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine (1983).

More-is-Less Paradox: Plurality with Runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

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	B	A	C	B
	C	B	A	C

Winner: C

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	B	A	C	**A**
	C	B	A	**C**

Winner: *A*

# voters	6	5	4	2
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- **Truncation Paradox**: A voter may obtain a more preferable outcome if, *ceteris paribus*, he only reveals part of his ranking of the candidates.

No-Show Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

No-Show Paradox: Plurality with Runoff

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Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: B

Twin Paradox: Plurality with Runoff

# voters	4	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: C

# voters	2	3	1	3
	A	B	C	C
	B	C	A	B
	C	A	B	A

Winner: B

Failures of Monotonicity

Example: Burlington, VT 2009 Mayoral Race
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Theorem (Moulin). If there are four or more candidates, then every Condorcet consistent voting methods is susceptible to the No-Show paradox.

H. Moulin. *Condorcet's Principle Implies the No Show Paradox*. Journal of Economic Theory, 45, pgs. 53 - 64, 1988.

Spoiler Candidates: Plurality Rule

# voters	49	48	3
	A	B	C
	B	A	B
	C	C	A

Winner: A

Spoiler Candidates: Plurality Rule

# voters	49	48	3
	A	B	C
	B	A	B
	C	C	A

Winner: B

IIA

Independence of Irrelevant Alternatives: If the voters in two different electorates rank A and B in exactly the same way, then A and B should be ranked the same way in both elections.

Failure of IIA: Borda Count

# voters	3	2	2
3	A	B	C
2	B	C	A
1	C	A	B
0	X	X	X

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

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A (15) $>_{BC}$ B (14) $>_{BC}$ C (13) $>_{BC}$ X (0)

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# voters	3	2	2
3	A	B	C
2	B	C	X
1	C	X	A
0	X	A	B

C (13) $>_{BC}$ B (12) $>_{BC}$ A (11) $>_{BC}$ X (6)

Voting Methods

Positional Scoring Rules: Given the rankings of the candidates provided by the voters, each candidate is assigned a score. The candidate(s) with the highest score is(are) declared the winner(s).

Examples: Borda, Plurality

Generalized Scoring Rules: Voters assign scores, or "grades", to the candidates. The candidate(s) with the "best" aggregate score is(are) declared the winner(s).

Examples: Approval Voting, Majority Judgement, Range Voting

Voting Methods

Staged Procedures: The winner(s) is(are) determined in stages. At each stage, one or more candidates are eliminated. The candidate or candidates that are never eliminated are declared the winner(s).

Examples: Plurality with Runoff, Hare, Coombs

Condorcet Consistent Methods: Voting methods that guarantee that the Condorcet winner is elected.

Examples: Copeland, Dodgson, Young

Principles

Condorcet: Elect the Condorcet winner whenever it exists.

Monotonicity: More support should never hurt a candidate.

Participation: It should never be in a voter's best interests not to vote.

Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

Independence: The group's ranking of A and B should only depend on the voter's rankings of A and B.

More Principles

Pareto: Never elect a candidate if another candidate is strictly preferred by all voters.

Anonymity: The outcome does not depend on the names of the voters.

Neutrality: The outcome does not depend on the names of the candidates.

Universal Domain: The voters are free to rank the candidates (or grade the candidates) in any way they want.

What are the relationships between these principles? Is there a procedure that satisfies *all* of them?

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A few observations:

- Condorcet winners may not exist.
- No positional scoring method satisfies the Condorcet Principle.
- The Condorcet and Participation principles cannot be jointly satisfied.

Different Perspectives

Axiomatics: Characterize the voting procedures in terms of the principles that they satisfy.

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Finding a Compromise: Which voting method produces a ranking that comes "closest" to the "consensus" ranking?

Finding the Optimal Choice: Which voting method is most likely to yield the "correct" choice?

Proceduralist View

"[W]e could identify a set of ideals with which any collective decision-making procedure ought to comply. We might think of these as procedural ideals, and a process of collective decision making would be more or less justifiable depending on the extent to which it satisfies them...

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J. Coleman and J. Ferejohn. *Democracy and social choice*. Ethics, 97(1): 6-25, 1986..

Epistemic View

"Condorcet begins with the premise that the object of government is to make decisions that are in the best interest of society. This leads naturally to the question: what voting rules are most likely to yield good outcomes?

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(pg. 60)

H. P. Young. *Optimal Voting Rules*. The Journal of Economic Perspectives, 9:1, pgs. 51 - 64, 1995.

Axiomatics

"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose.

A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

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A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

The Social Choice Model

Notation

- N is a finite set of voters (assume that $N = \{1, 2, 3, \ldots, n\}$)
- X is a (typically finite) set of alternatives, or candidates
- A relation on X is a linear order if it is transitive, irreflexive, and complete (hence, acyclic)
- $L(X)$ is the set of all linear orders over the set X
- $O(X)$ is the set of all reflexive and transitive relations over the set X

Notation

- A **profile** for the set of voters N is a sequence of (linear) orders over X, denoted $\mathbf{R} = (R_1, \ldots, R_n)$.
- $L(X)^n$ is the set of all **profiles** for n voters (similarly for $O(X)^n$)
- For a profile $\mathbf{R} = (R_1, \ldots, R_n) \in O(X)^n$, let $\mathbf{N}_{\mathbf{R}}(A\ P\ B) = \{i \mid A\ P_i\ B\}$ be the set of voters that rank A above B (similarly for $\mathbf{N}_{\mathbf{R}}(A\ I\ B)$ and $\mathbf{N}_{\mathbf{R}}(B\ P\ A)$)

Preference Aggregation Methods

Social Welfare Function: $F : \mathcal{D} \to L(X)$, where $\mathcal{D} \subseteq L(X)^n$

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Comments

- $\mathcal{D}$ is the *domain* of the function: it is the set of all possible profiles
- Aggregation methods are *decisive*: every profile $\mathbf{R}$ in the domain is associated with exactly one ordering over the candidates
- The range of the function is $L(X)$: the social ordering is assumed to be a linear order
- Tie-breaking rules are built into the definition of a preference aggregation function

Preference Aggregation Methods

Social Welfare Function: $F : \mathcal{D} \to L(X)$, where $\mathcal{D} \subseteq L(X)^n$

Variants

- Social Choice Function: $F : \mathcal{D} \to \wp(X) - \emptyset$, where $\mathcal{D} \subseteq L(X)^n$ and $\wp(X)$ is the set of all subsets of X.
- Allow Ties: $F : \mathcal{D} \to O(X)$ where $O(X)$ is the set of orderings (reflexive and transitive) over X
- Allow Indifference and Ties: $F : \mathcal{D} \to O(X)$ where $O(X)$ is the set of orderings (reflexive and transitive) over X and $\mathcal{D} \subseteq O(X)^n$

Examples

$Maj(\mathbf{R}) = \ >_M$ where $A >_M B$ iff $|\mathbf{N_R}(A\ P\ B)| > |\mathbf{N_R}(B\ P\ A)|$

(*the problem is that $>_M$ may not be transitive (or complete)*)

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(*the problem is that* $>_M$ *may not be transitive (or complete)*)

$Borda(\mathbf{R}) \ = \geq_{BC}$ where $A \geq_{BC} B$ iff the Borda score of A is greater than the Borda score for B.

(*the problem is that* $\geq_{BC}$ *may not be a linear order*)

Characterizing Majority Rule

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Majority Rule: A is ranked above (below) B if more (fewer) voters rank A above B than B above A, otherwise A and B are tied.

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Majority Rule: A is ranked above (below) B if more (fewer) voters rank A above B than B above A, otherwise A and B are tied.

When there are only two options, can we argue that majority rule is the "best" procedure?

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : O(X)^n \to O(X)$, where $O(X)$ is the set of orderings over X
(*there are only three possibilities: $A\ P\ B$, $A\ I\ B$, or $B\ P\ A$*)

$$F_{Maj}(\mathbf{R}) = \begin{cases} A\ P\ B & \text{if } |\mathbf{N_R}(A\ P\ B)| > |\mathbf{N_R}(B\ P\ A)| \\ A\ I\ B & \text{if } |\mathbf{N_R}(A\ P\ B)| = |\mathbf{N_R}(B\ P\ A)| \\ B\ P\ A & \text{if } |\mathbf{N_R}(B\ P\ A)| > |\mathbf{N_R}(A\ P\ B)| \end{cases}$$

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Social Welfare Function: $F : \{1, 0, -1\}^n \to \{1, 0, -1\}$,

where 1 means $A \; P \; B$, 0 means $A \; I \; B$, and -1 means $B \; P \; A$

$$F_{Maj}(\mathbf{v}) = \begin{cases} 1 & \text{if } |\mathbf{N_v}(1)| > |\mathbf{N_v}(-1)| \\ 0 & \text{if } |\mathbf{N_v}(1)| = |\mathbf{N_v}(-1)| \\ -1 & \text{if } |\mathbf{N_v}(-1)| > |\mathbf{N_v}(1)| \end{cases}$$

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there?

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- There are three possible rankings for 2 candidates.
- When there are two voters there are $3^2 = 9$ possible profiles:

$$\{(1,1),(1,0),(1,-1),(0,1),(0,0),(0,-1),(-1,1),(-1,0),(-1,-1)\}$$

- Since there are 9 profiles and 3 rankings, there are $3^9 = 19,683$ possible preference aggregation functions.

May's Theorem: Details

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- **Anonymity**: all voters should be treated equally.

  $F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where $v_i \in \{1, 0, -1\}$ and π is a permutation of the voters.

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- **Neutrality**: all candidates should be treated equally.

  $F(-\mathbf{v}) = -F(\mathbf{v})$ where $-\mathbf{v} = (-v_1, \ldots, -v_n)$.

May's Theorem: Details

- **Positive Responsiveness** (Monotonicity): unidirectional shift in the voters' opinions should help the alternative toward which this shift occurs

If $F(\mathbf{v}) = 0$ or $F(\mathbf{v}) = 1$ and $\mathbf{v} \prec \mathbf{v}'$, then $F(\mathbf{v}') = 1$
where $\mathbf{v} \prec \mathbf{v}'$ means for all $i \in N$ $v_i \leq v_i'$ and there is some $i \in N$ with $v_i < v_i'$.

Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there that satisfy anonymity?

Anonymity: all voters should be treated equally.

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Warm-up Exercise

Suppose that there are two voters and two candidates. How many social choice functions are there that satisfy anonymity? 729

Anonymity: all voters should be treated equally.

$F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- Imposing anonymity reduces the number of preference aggregation functions.
- If F satisfies anonymity, then $F(1, 0) = F(0, 1)$, $F(1, -1) = F(-1, 1)$ and $F(-1, 0) = F(0, -1)$.
- This means that there are essentially 6 elements of the domain. So, there are $3^6 = 729$ preference aggregation functions.

May’s Theorem: Details

May’s Theorem (1952) A social decision method F satisfies unanimity, neutrality, anonymity and positive responsiveness iff F is majority rule.

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  Contradiction.

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Contradiction.

Other characterizations

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Can May's Theorem be generalized to more than 2 candidates?

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Arrow's Theorem

K. Arrow. *Social Choice and Individual Values*. John Wiley & Sons, 1951.

Arrow's Theorem

Let X be a finite set with *at least three elements* and N a finite set of n voters.

Social Welfare Function: $F : \mathcal{D} \to O(X)$ where $\mathcal{D} \subseteq O(X)^n$

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Reminders:

- $O(X)$ is the set of transitive and complete relations on X
- For $R \in O(X)$, let P_R denote the strict subrelation and I_R the indifference subrelation:
  - $A\ P_R\ B$ iff $A\ R\ B$ and not $B\ R\ A$
  - $A\ I_R\ B$ iff $A\ R\ B$ and $B\ R\ A$

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For all profiles $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{D}$:

$$\text{If for each } i \in N,\ A\ P_i\ B \text{ then } A\ P_{F(\mathbf{R})}\ B$$

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The domain of F is the set of *all* profiles, i.e., $\mathcal{D} = O(X)^n$.

Independence of Irrelevant Alternatives

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The social ranking (higher, lower, or indifferent) of two alternatives A and B depends only the relative rankings of A and B for each voter.

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For all profiles $\mathbf{R} = (R_1, \ldots, R_n)$ and $\mathbf{R}' = (R'_1, \ldots, R'_n)$:

$$\text{If } R_{i\{A,B\}} = R'_{i\{A,B\}} \text{ for all } i \in N, \text{ then } F(\mathbf{R})_{\{A,B\}} \text{ iff } F(\mathbf{R}')_{\{A,B\}}.$$

where $R_{\{X,Y\}} = R \cap \{X, Y\} \times \{X, Y\}$

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Dictatorship

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There is a $d \in N$ such that for each profile $\mathbf{R} = (R_1, \ldots, R_d, \ldots, R_n)$, if $A\ P_d\ B$, then $A\ P_{F(\mathbf{R})}\ B$

M. Morreau. *Arrow's Theorem*. Stanford Encyclopedia of Philosophy, 2014.

Arrow’s Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.