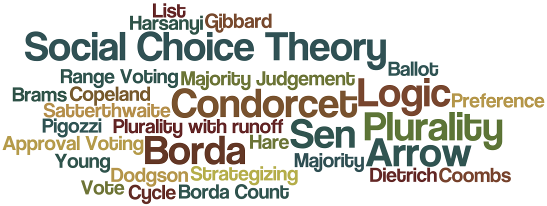
Social Choice Theory for Logicians
ESSLLI 2016

Eric Pacuit

University of Maryland, College Park

pacuit.org



Plan

- Introduction, Background, Voting Theory, May's Theorem, Arrow's Theorem
- Social Choice Theory: Variants of Arrow's Theorem, Weakening Arrow's Conditions (Domain Conditions), Harsanyi's Theorem, Characterizing Voting Methods
- Strategizing (Gibbard-Satterthwaite Theorem) and Iterative Voting/ Introduction to Judgement Aggregation
- Aggregating Judgements (linear pooling, wisdom of the crowds, prediction markets), Probabilistic Social Choice.
- Logics for Social Choice Theory (Modal Logic, Dependence/Independence Logic, First Order Logic)

Voting Situations

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- 21 voters and 4 candidates: Ann (A), Bob (B), Charles (C) and Dora (D)

Voting Situations

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- 21 voters and 4 candidates: Ann (A), Bob (B), Charles (C) and Dora (D)
- Each voter ranks the candidates from best (at the top of the list) to worst (at the bottom of the list) resulting in the 4 voting blocks given in the above table

Voting Situations

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

Who *should* win the election?

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** *A*: More people (8) rank *A* first than any other candidate

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** A: More people rank A first than any other candidate
- **Candidate** A **should *not* win**: more than half rank A last

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** A: More people rank A first than any other candidate
- **Candidate** D **should** ***not*** **win**

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** A: More people rank A first than any other candidate
- **Candidate** D **should *not* win**: *everyone* ranks B higher than D

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Which of B or C should win?

Which candidate *should* be chosen?

VS.

Marquis de Condorcet (1743 - 1794)

Jean-Charles de Borda (1733 -1799)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win:** *C* beats every other candidate in head-to-head elections (*C* is the *Condorcet winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win:** C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** C **should win:** *C* beats every other candidate in head-to-head elections (*C* is the *Condorcet winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	**B**	C
	B	C	D	**B**
	C	**B**	C	D
worst	D	D	A	A

- **Candidate** C **should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate** B **should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** C **should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate** B **should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)
- B gets 13 (vs. A)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** C **should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate** B **should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)
- B gets 13 (vs. A) + 10 (vs. C)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win**: *C* beats every other candidate in head-to-head elections (*C* is the *Condorcet winner*)
- **Candidate *B* should win**: Taking into account the *entire* ordering, *B* has the most "support" (*B* is the *Borda winner*)

- *B* gets 13 (vs. *A*) + 10 (vs. *C*) + 21 (vs. *D*) = 44 points

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate** C **should win**: *C* beats every other candidate in head-to-head elections (*C* is the *Condorcet winner*)
- **Candidate** *B* **should win**: Taking into account the *entire* ordering, *B* has the most "support" (*B* is the *Borda winner*)

- *C* get 13 (vs. *A*) + 11 (vs. *B*) + 14 (vs. *D*) = 38 points

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate B should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)

- C get 13 (vs. A) + 11 (vs. B) + 14 (vs. D) = 38 points

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- **Candidate C should win**: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- **Candidate B should win**: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)
- C get 13 (vs. A) + 11 (vs. B) + 14 (vs. D) = 38 points

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
↑	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate A should *not* win: more than half rank A last
- Candidate D should *not* win: *everyone* ranks B higher than D
- Candidate C: C beats every other candidate in head-to-head elections (C is the *Condorcet winner*)
- Candidate B: Taking into account the *entire* ordering, B has the most "support" (B is the *Borda winner*)

Which candidate *should* be chosen?

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

Conclusion: *there are many ways to answer the above question!*

- …ons (*C* is the *Condorcet winner*)
- **Candidate *B***: Taking into account the *entire* ordering, *B* has the most "support" (*B* is the *Borda winner*)

Preference

Preferences

Preferring or choosing x is different that "liking" x or "having a taste for x": one can prefer x to y but *dislike* both options

In utility theory, preferences are always understood as comparative: "preference" is more like "bigger" than "big"

Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

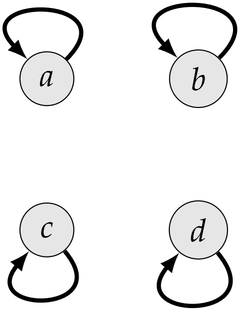
Reflexive relation: for all $x \in X$, $x \mathrel{R} x$

Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Reflexive relation: for all $x \in X$, $x \; R \; x$

E.g., $X = \{a, b, c, d\}$



Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

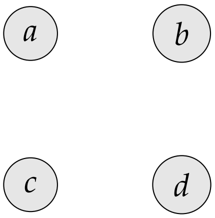
Irreflexive relation: for all $x \in X$, $x \not{R}\, x$ (i.e., $(x, x) \notin R$)

Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Irreflexive relation: for all $x \in X$, $x \not R x$ (i.e., $(x, x) \notin R$)

E.g., $X = \{a, b, c, d\}$



Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

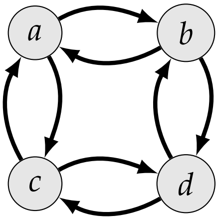
Symmetric relation: for all $x, y \in X$, if $x \; R \; y$, then $y \; R \; x$

Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Symmetric relation: for all $x, y \in X$, if $x \mathrel{R} y$, then $y \mathrel{R} x$

E.g., $X = \{a, b, c, d\}$



Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

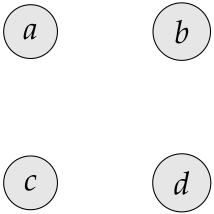
Complete relation: for all $x, y \in X$, either $x \; R \; y$ or $y \; R \; x$

Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Complete relation: for all $x, y \in X$, either $x\ R\ y$ or $y\ R\ x$

E.g., $X = \{a, b, c, d\}$

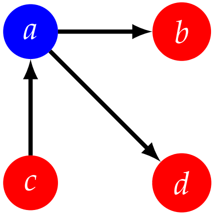


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Complete relation: for all $x, y \in X$, either $x \; R \; y$ or $y \; R \; x$

E.g., $X = \{a, b, c, d\}$

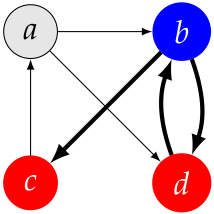


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Complete relation: for all $x, y \in X$, either $x \; R \; y$ or $y \; R \; x$

E.g., $X = \{a, b, c, d\}$

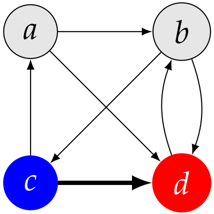


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Complete relation: for all $x, y \in X$, either $x \; R \; y$ or $y \; R \; x$

E.g., $X = \{a, b, c, d\}$

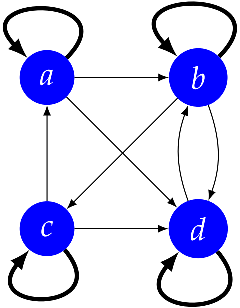


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Complete relation: for all $x, y \in X$, either $x\ R\ y$ or $y\ R\ x$

E.g., $X = \{a, b, c, d\}$



Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

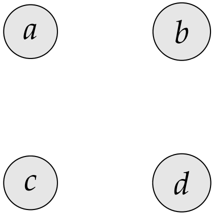
Transitive relation: for all $x, y, z \in X$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$

Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Transitive relation: for all $x, y, z \in X$, if $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z$

E.g., $X = \{a, b, c, d\}$

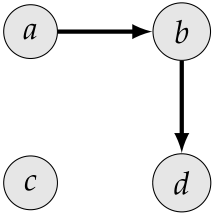


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Transitive relation: for all $x, y, z \in X$, if $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z$

E.g., $X = \{a, b, c, d\}$

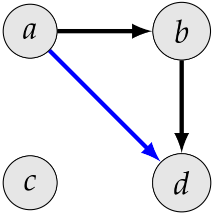


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Transitive relation: for all $x, y, z \in X$, if $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z$

E.g., $X = \{a, b, c, d\}$

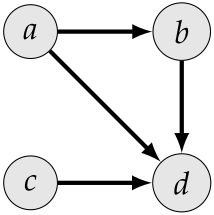


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Transitive relation: for all $x, y, z \in X$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$

E.g., $X = \{a, b, c, d\}$

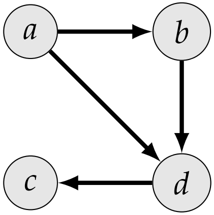


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Transitive relation: for all $x, y, z \in X$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$

E.g., $X = \{a, b, c, d\}$

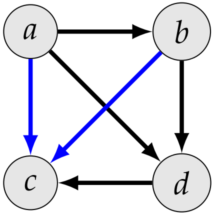


Mathematical background: Relations

Suppose that X is a set and $R \subseteq X \times X$ is a relation.

Transitive relation: for all $x, y, z \in X$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$

E.g., $X = \{a, b, c, d\}$



Maximal elements, Cycles

Suppose that $R \subseteq X \times X$ is a relation.

$x \in X$ is **maximal** with respect to R provided there is no $y \in X$ such that $y \; R \; x$.

For $Y \subseteq X$, let $\max_R(Y) = \{x \in Y \mid \text{ there is no } y \in Y \text{ such that } y \; R \; x\}$

Maximal elements, Cycles

Suppose that $R \subseteq X \times X$ is a relation.

$x \in X$ is **maximal** with respect to R provided there is no $y \in X$ such that $y \mathrel{R} x$.

For $Y \subseteq X$, let $\max_R(Y) = \{x \in Y \mid \text{ there is no } y \in Y \text{ such that } y \mathrel{R} x\}$

A **cycle** is a set of distinct elements $x_1, \ldots, x_n$ such that

$$x_1 \mathrel{R} x_2 \cdots x_{n-1} \mathrel{R} x_n \mathrel{R} x_1$$

R is **acyclic** if it does not contain any cycles.

Representing Preferences

Let X be a set of options/outcomes. A decision maker's *preference* over X is represented by a *relation* $\succeq \subseteq X \times X$.

Representing Preferences

Given $x, y \in X$, there are four possibilities:

Representing Preferences

Given $x, y \in X$, there are four possibilities:

1. $x \succeq y$ and $y \not\succeq x$: *The decision maker ranks x above y* (the decision maker strictly prefers x to y).

Representing Preferences

Given $x, y \in X$, there are four possibilities:

1. $x \succeq y$ and $y \not\succeq x$: *The decision maker ranks x above y* (the decision maker strictly prefers x to y).

2. $y \succeq x$ and $x \not\succeq y$: *The decision maker ranks y above x* (the decision maker strictly prefers y to x).

Representing Preferences

Given $x, y \in X$, there are four possibilities:

1. $x \succeq y$ and $y \not\succeq x$: *The decision maker ranks x above y* (the decision maker strictly prefers x to y).

2. $y \succeq x$ and $x \not\succeq y$: *The decision maker ranks y above x* (the decision maker strictly prefers y to x).

3. $x \succeq y$ and $y \succeq x$: The agent is *indifferent* between x and y.

Representing Preferences

Given $x, y \in X$, there are four possibilities:

1. $x \succeq y$ and $y \not\succeq x$: *The decision maker ranks x above y* (the decision maker strictly prefers x to y).

2. $y \succeq x$ and $x \not\succeq y$: *The decision maker ranks y above x* (the decision maker strictly prefers y to x).

3. $x \succeq y$ and $y \succeq x$: The agent is *indifferent* between x and y.

4. $x \not\succeq y$ and $y \not\succeq x$: The agent *cannot compare* x and y

Representing Preferences

Given $x, y \in X$, there are four possibilities:

1. $x \succeq y$ and $y \not\succeq x$: *The decision maker ranks x above y* (the decision maker strictly prefers x to y).

2. $y \succeq x$ and $x \not\succeq y$: *The decision maker ranks y above x* (the decision maker strictly prefers y to x).

3. $x \succeq y$ and $y \succeq x$: The agent is ***indifferent*** between x and y.

4. $x \not\succeq y$ and $y \not\succeq x$: The agent ***cannot compare*** x and y

Representing Preferences

A relation $\succeq \subseteq X \times X$ is a **(rational) preference relation** (for a decision maker) provided

1. $\succeq$ is complete (and hence reflexive)
2. $\succeq$ is transitive

Representing Preferences

A relation $\succeq \subseteq X \times X$ is a **(rational) preference relation** (for a decision maker) provided

1. $\succeq$ is complete (and hence reflexive)
2. $\succeq$ is transitive

Suppose that $\succeq$ is a preference relation. Then,

- **Strict preference**: $x \succ y$ iff $x \succeq y$ and $y \not\succeq x$
- **Indifference**: $x \sim y$ iff $x \succeq y$ and $y \succeq x$

- What is the relationship between choice and preference?
- Why *should* preferences be complete and transitive?
- *Are* people's preferences complete and transitive?

Invoking someone's preferences will suffice to explain why some choices were not made (i.e. in terms of rational impermissibility) but not typically why some particular choice was made. To take up the slack, explanations must draw on factors other than preference: psychological one such as the framing of the choice problem or the saliency of particular options, or sociological ones such as the existence of norms or conventions governing choices of the relevant kind.

Utility Function

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

Utility Function

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

A preference ordering is **represented** by a utility function iff x is (weakly) preferred to y provided $u(x) \geq u(y)$

Utility Function

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

A preference ordering is **represented** by a utility function iff x is (weakly) preferred to y provided $u(x) \geq u(y)$

What properties does such a preference ordering have?

Ordinal Utility Theory

Fact. Suppose that X is finite and $\succeq$ is a complete and transitive ordering over X, then there is a utility function $u : X \to \Re$ that represents $\succeq$

(i.e., $x \succeq y$ iff $u(x) \geq u(y)$)

Ordinal Utility Theory

Fact. Suppose that X is finite and $\succeq$ is a complete and transitive ordering over X, then there is a utility function $u : X \rightarrow \Re$ that represents $\succeq$

(i.e., $x \succeq y$ iff $u(x) \geq u(y)$)

Utility is *defined* in terms of preference (so it is an error to say that the agent prefers x to y *because* she assigns a higher utility to x than to y).

Important

All three of the utility functions represent the preference $x \succ y \succ z$

Item	u_1	u_2	u_3
x	3	10	1000
y	2	5	99
z	1	0	1

$x \succ y \succ z$ is represented by both $(3, 2, 1)$ and $(1000, 999, 1)$, so one cannot say that y is "closer" to x than to z.

$$X = \{M, C, P, L\}$$

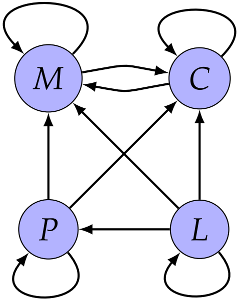
$$X = \{M, C, P, L\}$$

$M\ C\ P\ L$	$M\ C$	$P\ L$
$M\ P\ L$	$M\ P$	M
$C\ P\ L$	$M\ L$	C
$M\ C\ P$	$C\ P$	P
$M\ C\ L$	$C\ L$	L

$$X = \{M, C, P, L\}$$

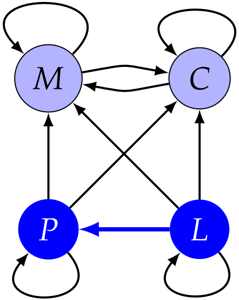
$M\ C\ P\ L$	$M\ C$	$P\ L$
$M\ P\ L$	$M\ P$	M
$C\ P\ L$	$M\ L$	C
$M\ C\ P$	$C\ P$	P
$M\ C\ L$	$C\ L$	L

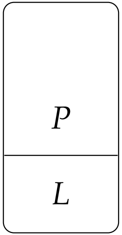
$$X = \{M, C, P, L\}$$



$$\succeq = \{(M, C), (C, M), (M, P), (M, L), (C, P), (C, L), (P, L),$$
$$(M, M), (P, P), (C, C), (L, L)\}$$

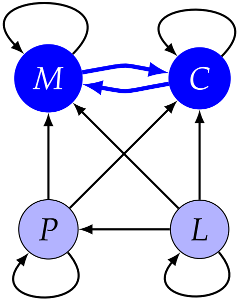
$$X = \{M, C, P, L\}$$

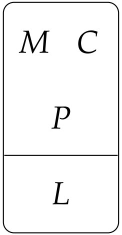




$$\succeq = \{(M, C), (C, M), (M, P), (M, L), (C, P), (C, L), (P, L),$$
$$(M, M), (P, P), (C, C), (L, L)\}$$

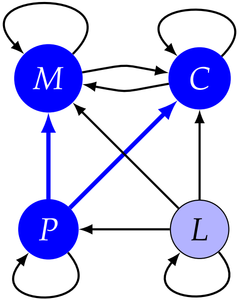
$$X = \{M, C, P, L\}$$

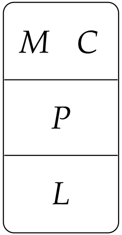




$$\succeq = \{(M,C),(C,M),(M,P),(M,L),(C,P),(C,L),(P,L),$$
$$(M,M),(P,P),(C,C),(L,L)\}$$

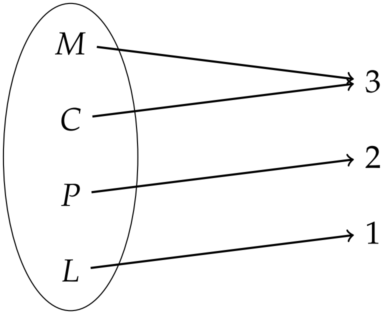
$$X = \{M, C, P, L\}$$



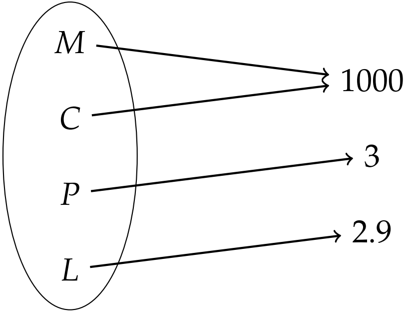


$$\succeq = \{(M,C),(C,M),(M,P),(M,L),(C,P),(C,L),(P,L),$$
$$(M,M),(P,P),(C,C),(L,L)\}$$

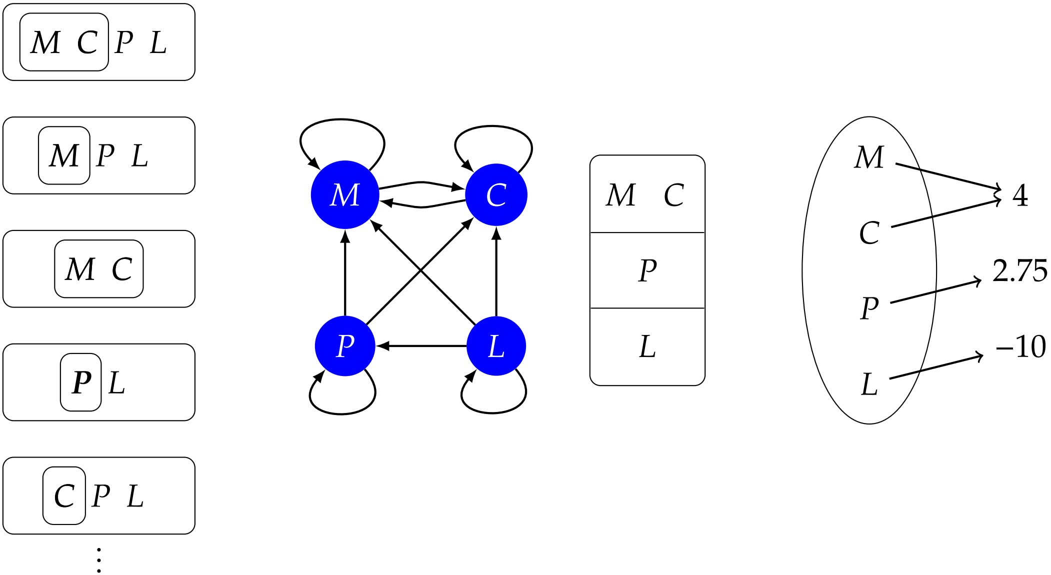
$$X = \{M, C, P, L\}$$



$$X = \{M, C, P, L\}$$

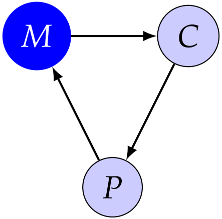


$$X = \{M, C, P, L\}$$

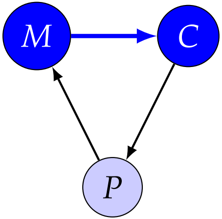


- What is the relationship between choice and preference?
- Why *should* preferences be complete and transitive?
- *Are* people's preferences complete and transitive?

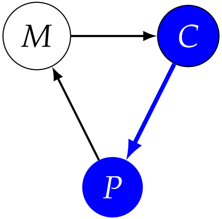
- Transitivity: Money-pump argument
- Completeness: Incommensurable options



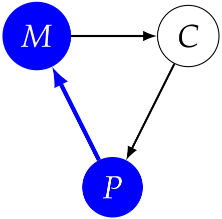
(M)



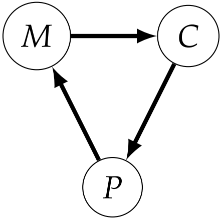
$$(M) \implies (C, -1)$$



$$(M) \implies (C, -1) \implies (P, -2)$$



$$(M) \implies (C, -1) \implies (P, -2) \implies (M, -3)$$



$$(M) \implies (C, -1) \implies (P, -2) \implies (M, -3) \implies (C, -4) \implies \cdots$$

[O]f all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others, it is inaccurate as a description of real life; but unlike them we find it hard to accept even from the normative viewpoint.

(Aumann, 1962)

Rather than trying to provide instrumental or pragmatic justifications for the axioms of ordinal utility, it is better...to see them as constitutive of our conception of a fully rational agent....those disposed to blatantly ignore transitivity are unintelligible to us: we can’t understand their pattern of actions as sensible. [Gaus], pg. 39

The Condorcet Paradox

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections.

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections.

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections.

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections.

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	**B**	C
	B	C	D	**B**
	C	**B**	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections. B is ranked second

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	**B**	C
	B	C	D	**B**
	C	**B**	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections. B is ranked second

Recall Condorcet's Idea

# voters	3	5	7	6
best	A	A	B	C
	B	C	D	B
	C	B	C	D
worst	D	D	A	A

- Candidate C should win since C beats every other candidate in head-to-head elections. B is ranked second, D is ranked third, and A is ranked last.

$$C >_M B >_M D >_M A$$

The Majority Relation

Suppose that X and Y are candidates and P_i represents voter i's *strict preference*.

$\mathbf{N}(X\ P\ Y) = |\{i \mid X\ P_i\ Y\}|$
"the number of voters that rank X strictly above Y"

The Majority Relation

Suppose that X and Y are candidates and P_i represents voter i's *strict preference*.

$\mathbf{N}(X\ P\ Y) = |\{i \mid X\ P_i\ Y\}|$
"the number of voters that rank X strictly above Y"

$X \geq_M Y$ iff $\mathbf{N}(X\ P\ Y) \geq \mathbf{N}(Y\ P\ X)$
"a majority prefers candidate X over candidate Y"

The Majority Relation

Suppose that X and Y are candidates and P_i represents voter i's *strict preference*.

$\mathbf{N}(X\ P\ Y) = |\{i \mid X\ P_i\ Y\}|$
"the number of voters that rank X strictly above Y"

$X \geq_M Y$ iff $\mathbf{N}(X\ P\ Y) \geq \mathbf{N}(Y\ P\ X)$
"a majority prefers candidate X over candidate Y"

X is a **Condorcet winner** if X beats every other candidate in an head-to-head election: there is no candidate Y such that $Y >_M X$

The Majority Relation

Suppose that X and Y are candidates and P_i represents voter i's *strict preference*.

$\mathbf{N}(X\ P\ Y) = |\{i \mid X\ P_i\ Y\}|$
"the number of voters that rank X strictly above Y"

$X \geq_M Y$ iff $\mathbf{N}(X\ P\ Y) \geq \mathbf{N}(Y\ P\ X)$
"a majority prefers candidate X over candidate Y"

X is a **Condorcet winner** if X beats every other candidate in an head-to-head election: there is no candidate Y such that $Y >_M X$

X is a **Condorcet loser** if X loses to every other candidate in an head-to-head elections: there is no candidate Y such that, $X >_M Y$

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B?

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

- Does the group prefer *A* over *B*? Yes
- Does the group prefer *B* over *C*? Yes
- Does the group prefer *A* over *C*? No

The Problem

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

The majority relation $>_M$ is **not** transitive!

There is a **Condorcet cycle**: $A >_M B >_M C >_M A$

How bad is this?

- Final decisions are extremely sensitive to institutional features such as who can set the agenda, arbitrary time limits place on deliberation, who is permitted to make motions, etc.

How bad is this?

- Final decisions are extremely sensitive to institutional features such as who can set the agenda, arbitrary time limits place on deliberation, who is permitted to make motions, etc.
- Is there *empirical evidence* that Condorcet cycles have shown up in real elections?

W. Riker. *Liberalism against Populism*. Waveland Press, 1982.

G. Mackie. *Democracy Defended*. Cambridge University Press, 2003.

How bad is this?

- Final decisions are extremely sensitive to institutional features such as who can set the agenda, arbitrary time limits place on deliberation, who is permitted to make motions, etc.
- Is there *empirical evidence* that Condorcet cycles have shown up in real elections?

W. Riker. *Liberalism against Populism*. Waveland Press, 1982.

G. Mackie. *Democracy Defended*. Cambridge University Press, 2003.

- How *likely* is a Condorcet cycle?

The probability of a Condorcet cycle

Give a set of n voters and m candidates, what is the probability that there is a Condorcet cycle?

The probability of a Condorcet cycle

Give a set of n voters and m candidates, what is the probability that there is a Condorcet cycle?

$$Pr(m, n) = \frac{\text{"the number of preference profiles that generate a Condorcet cycle"}}{\text{"the total number of preference profiles"}}$$

(A **preference profile** is a list of preferences, one for each voter.)

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles?

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$
For $m = n = 3$, there are $(3 \cdot 2 \cdot 1)^3 = 6^3 = 216$ different preference profiles.

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$
For $m = n = 3$, there are $(3 \cdot 2 \cdot 1)^3 = 6^3 = 216$ different preference profiles.
How many of these generate intransitive group preferences?

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$
For $m = n = 3$, there are $(3 \cdot 2 \cdot 1)^3 = 6^3 = 216$ different preference profiles.
How many of these generate intransitive group preferences?
$A >_M B >_M C >_M A$ and $C >_M B >_M A >_M C$

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$
For $m = n = 3$, there are $(3 \cdot 2 \cdot 1)^3 = 6^3 = 216$ different preference profiles.
How many of these generate intransitive group preferences?
$A >_M B >_M C >_M A$ and $C >_M B >_M A >_M C$

G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
C	B	A
B	A	C
A	C	B

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$
For $m = n = 3$, there are $(3 \cdot 2 \cdot 1)^3 = 6^3 = 216$ different preference profiles.
How many of these generate intransitive group preferences? 12
$A >_M B >_M C >_M A$ and $C >_M B >_M A >_M C$

G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
C	B	A
B	A	C
A	C	B

$$G_1 + G_3 > G_2$$
$$G_1 + G_2 > G_3$$
$$G_2 + G_3 > G_1$$

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives
(*assume that the voter's preferences are linear orders*)

What is the total number of preference profiles? $(m!)^n$
For $m = n = 3$, there are $(3 \cdot 2 \cdot 1)^3 = 6^3 = 216$ different preference profiles.
How many of these generate intransitive group preferences? 12
$A >_M B >_M C >_M A$ and $C >_M B >_M A >_M C$

G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
C	B	A
B	A	C
A	C	B

$G_1 + G_3 > G_2$
$G_1 + G_2 > G_3$
$G_2 + G_3 > G_1$

$$Pr(3,3) = \frac{12}{216} = 0.0555555\ldots$$

The probability of a Condorcet paradox, II

	Number of voters						
Number of candidates	3	5	7	9	11	$\cdots$	∞
3	.056	.069	.075	.078	.080		.088
4	.111	.139	.150	.156	.160		.176
5	.160	.200	.215				.251
6	.212						.315
$\vdots$							$\vdots$
∞	1.00	1.00	1.00	1.00	1.00		1.00

(Source: W. Riker, *Liberalism Against Populism*, pg. 122)

Key assumptions

- voters are free to adopt any preference they want (there are no *domain restrictions*)
- all of the linear orderings are equally likely to be submitted by an individual (the voter's preferences are *independent* and drawn from an *impartial culture* over the set of all linear orderings)

Key assumptions

"...changing the distribution in *any fashion* (whether we call it 'realistic' or not) away from an impartial culture over linear orders will automatically have the effect of reducing the probability of majority cycles in infinite samples..." (pg., 28, 29)

This means that assuming an impartial culture is a *worst case analysis*.

M. Regenwetter, B. Gromfan, A. Marley, and I. Tsetlin. *Behavioral Social Choice*. Cambridge University Press, 2006.

See, also,

W. Gehrlein. *Condorcet's Paradox*. Springer, 2006.

Should we select a Condorcet winner (when one exists)?

Is the Condorcet winner the “best” choice?

# voters	47	47	3	3
	A	B	C	C
	C	C	A	B
	B	A	B	A

C is the Condorcet winner

Is the Condorcet winner the “best” choice?

# voters	47	47	3	3
	A	B	C	C
	C	C	A	B
	B	A	B	A

C is the Condorcet winner; however, it seems that supporters of the main rivals A and B would rather see C win than their candidate’s principal opponent, but this does not mean that there is “positive support” for C.

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$$
$$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$$
$$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$$

$$B >_{BC} A >_{BC} C$$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\quad\quad$ $A >_{M} B >_{M} C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$B >_{BC} A >_{BC} C$ $\qquad\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...
$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$
$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$B >_{BC} A >_{BC} C$ $\quad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$

$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$Score(A) > Score(B) \Rightarrow 31s_2 + 39s_1 > 39s_2 + 31s_1 \Rightarrow s_1 > s_2$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Theorem (Fishburn 1974). For all $m \geq 3$, there is some voting situation with a Condorcet winner such that every scoring rule will have at least $m - 2$ candidates with a greater score than the Condorcet winner.

P. Fishburn. *Paradoxes of Voting*. The American Political Science Review, 68:2, pgs. 537 - 546, 1974.

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$$
$$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$$
$$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$$

$$B >_{BC} A >_{BC} C$$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_{M} B >_{M} C$

Condorcet Triples

G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
A	C	B
C	B	A
B	A	C

If $G_1 = G_2 = G_3$, then this group of voters “cancel out” each other’s votes

Saari's argument

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Saari's argument

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

Saari's argument

# voters	20	1	29	0	0	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

Saari's argument

# voters	20	0	28	0	0	0
	A		B			
	B		A			
	C		C			

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

There are many different voting methods

Many different electoral methods: Plurality, Borda Count, Antiplurality/Veto, and k-approval; Plurality with Runoff; Single Transferable Vote (STV)/Hare; Approval Voting; Cup Rule/Voting Trees; Copeland; Banks; Slater Rule; Schwartz Rule; the Condorcet rule; Maximin/Simpson, Kemeny; Ranked Pairs/Tideman; Bucklin Method; Dodgson Method; Young's Method; Majority Judgment; Cumulative Voting; Range/Score Voting; …

Voting Methods

Positional Scoring Rules: Given the rankings of the candidates provided by the voters, each candidate is assigned a score. The candidate(s) with the highest score is(are) declared the winner(s).

Examples: Borda, Plurality

Generalized Scoring Rules: Voters assign scores, or "grades", to the candidates. The candidate(s) with the "best" aggregate score is(are) declared the winner(s).

Examples: Approval Voting, Majority Judgement, Range Voting

Voting Methods

Staged Procedures: The winner(s) is(are) determined in stages. At each stage, one or more candidates are eliminated. The candidate or candidates that are never eliminated are declared the winner(s).

Examples: Plurality with Runoff, Hare, Coombs

Condorcet Consistent Methods: Voting methods that guarantee that the Condorcet winner is elected.

Examples: Copeland, Dodgson, Young

Voting Methods Tutorial