Neighborhood Semantics for Modal Logic
Lecture 5

Eric Pacuit

University of Maryland, College Park
pacuit.org
epacuit@umd.edu

August 15, 2014

Course Plan

✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Subset Spaces, Neighborhood Structures, Motivating Non-Normal Modal Logics/Neighborhood Semantics

✓ **Core Theory**: Relationship with Other Semantics for Modal Logic, Model Theory; Completeness, Decidability, Complexity, Incompleteness

1. **Extensions and Applications**: First-Order Modal Logic, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods

- First-Order Modal Logic (Skip)
- Game Logic (Skip)
- Evidence Dynamics (Skip)

Neighborhood Models for First-Order Modal Logic

H. Arlo Costa and E. Pacuit. *First-Order Classical Modal Logic*. Studia Logica, **84**, pgs. 171 - 210 (2006).

Higher-Order Coalition Logic (time permitting)

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 2010.

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \wedge A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)

First-Order Modal Logic

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $V(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution σ' is said to be an x-**variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $V(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in V(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M},\sigma}$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp(\wp(W))$.

A **constant domain Neighborhood model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $V(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in V(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $[\![\varphi]\!]_{\mathcal{M},\sigma} \in N(w)$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

Let **S** be any (classical) propositional modal logic, by **FOL** $+$ **S** we mean the set of formulas closed under the following rules and axioms:

(S) All instances of axioms and rules from **S**.

($\forall$) $\forall x A \rightarrow A^x_t$ (where t is free for x in A)

(Gen) $\dfrac{A \rightarrow B}{A \rightarrow \forall x B}$, where x is not free in A.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Observation 1: *CBF* is provable in **FOL** + **EM**

Observation 2: *BF* and *CBF* both valid on relational frames with constant domains

Observation 3: *BF* is valid in a *varying* domain relational frame iff the frame is anti-monotonic; *CBF* is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the '**lottery paradox**':

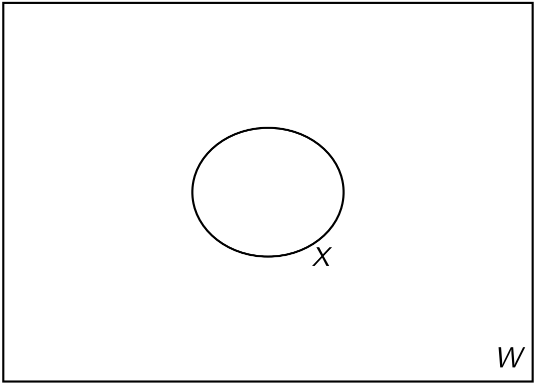
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

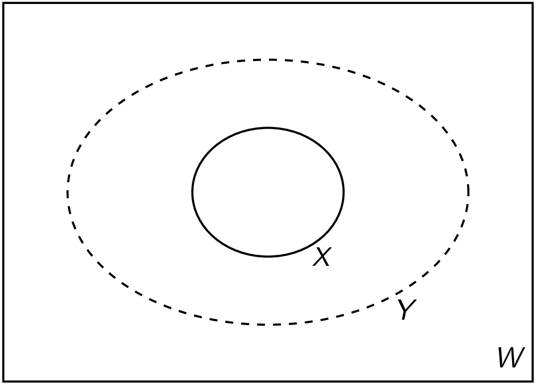
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

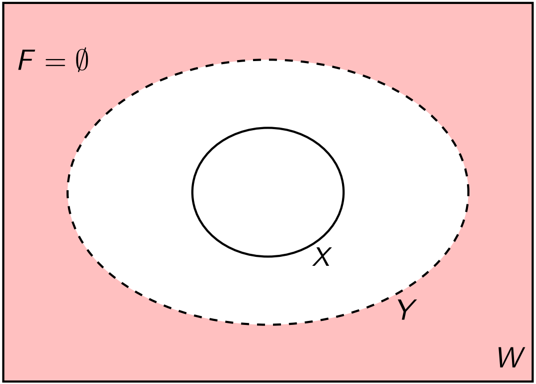
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



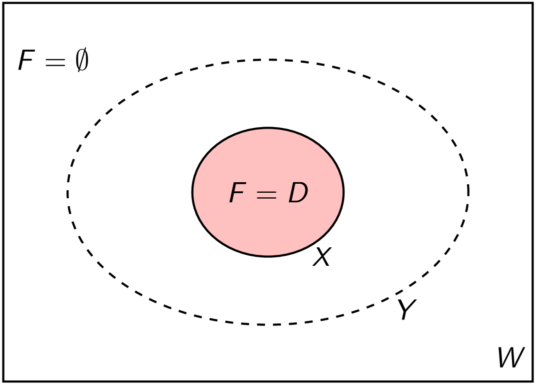
$$X \in N(w)$$



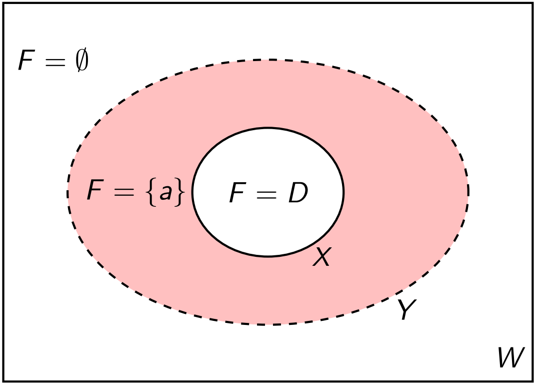
$$Y \notin N(w)$$



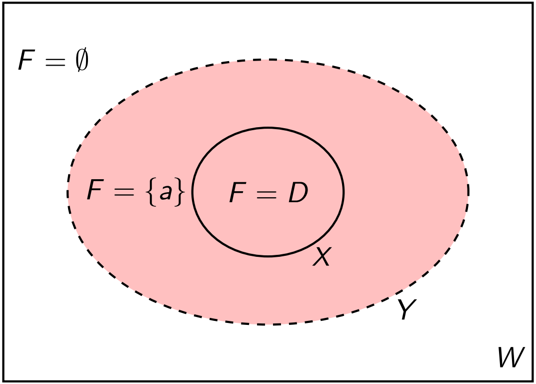
$$\forall v \notin Y, \quad I(F, v) = \emptyset$$



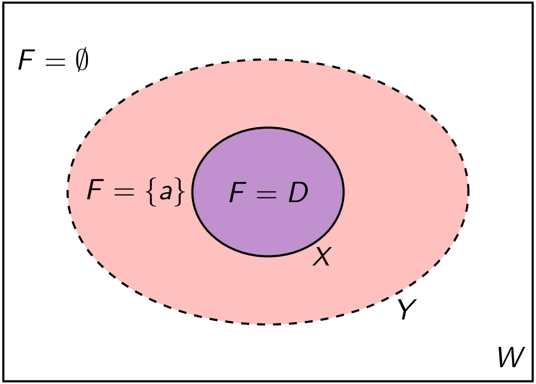
$$\forall v \in X, \quad I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \quad I(F, v) = D = \{a\}$$



$$\color{red}{(F[a])^{\mathcal{M}}} = Y \notin N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

$$\text{hence} \quad w \models \Box \forall x F(x)$$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Completeness Theorems

Theorem FOL $+$ E is sound and strongly complete with respect to the class of **all** frames.

Completeness Theorems

Theorem $\mathsf{FOL} + \mathsf{E}$ is sound and strongly complete with respect to the class of **all** frames.

Theorem $\mathsf{FOL} + \mathsf{EC}$ is sound and strongly complete with respect to the class of frames that are closed under intersections.

Completeness Theorems

Theorem $\mathbf{FOL} + \mathbf{E}$ is sound and strongly complete with respect to the class of **all** frames.

Theorem $\mathbf{FOL} + \mathbf{EC}$ is sound and strongly complete with respect to the class of frames that are closed under intersections.

Theorem $\mathbf{FOL} + \mathbf{EM}$ is sound and strongly complete with respect to the class of supplemented frames.

Completeness Theorems

Theorem $\textbf{FOL} + \textbf{E}$ is sound and strongly complete with respect to the class of **all** frames.

Theorem $\textbf{FOL} + \textbf{EC}$ is sound and strongly complete with respect to the class of frames that are closed under intersections.

Theorem $\textbf{FOL} + \textbf{EM}$ is sound and strongly complete with respect to the class of supplemented frames.

Theorem $\textbf{FOL} + \textbf{E} + CBF$ is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

Observation The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$. In fact, the closure under infinite intersection of the minimal canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$.

$\mathbf{FOL} + \mathbf{K}$ and $\mathbf{FOL} + \mathbf{K} + BF$

Theorem $\mathbf{FOL} + \mathbf{K}$ is sound and strongly complete with respect to the class of filters.

Observation The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$. In fact, the closure under infinite intersection of the minimal canonical model for $\mathbf{FOL} + \mathbf{K}$ is not a canonical model for $\mathbf{FOL} + \mathbf{K}$.

Lemma The augmentation of the smallest canonical model for $\mathbf{FOL} + \mathbf{K} + BF$ is a canonical for $\mathbf{FOL} + \mathbf{K} + BF$.

Theorem $\mathbf{FOL} + \mathbf{K} + BF$ is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

Is the addition of quantifiers straightforward?

Is the addition of quantifiers straightforward?

1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

Is the addition of quantifiers straightforward?

1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

2. **S4.2** is S4 with $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$. This logics is complete for the class of frames that are reflexive, transitive and *convergent*. However, **FOL** + **S4M** + *BF* is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Creswell, pg. 271)

Is the addition of quantifiers straightforward?

1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

2. **S4.2** is S4 with $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$. This logics is complete for the class of frames that are reflexive, transitive and *convergent*. However, **FOL** + **S4M** + *BF* is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Creswell, pg. 271)

3. The quantified extension of **GL** is not complete (with respect to varying domains models).

What is going on?

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press, 2011.

An Application: Coalition Logic

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 19th European Conference on Artificial Intelligence, pgs. 555 - 560, 2010.

▸ Skip

Q. Chen and K. Su. *Higher-Order Epistemic Coalition Logic for Multi-Agent Systems*. 7th Workshop on Logical Aspects of Multi-Agent Systems, 2014.

Coalition Logic: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [C]\varphi$

Coalition Logic: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [C]\varphi$

$\mathcal{M}, w \models [C]\varphi$ iff $(\varphi)^{\mathcal{M}} \in N(w, C)$: "Coalition C has a joint strategy to force the outcome to satisfy φ".

Coalition Logic: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [C]\varphi$

$\mathcal{M}, w \models [C]\varphi$ iff $(\varphi)^{\mathcal{M}} \in N(w, C)$: "Coalition C has a joint strategy to force the outcome to satisfy φ".

Higher-Order Coalition Logic: $\varphi :=$
$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X\varphi \mid \forall x\varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

Coalition Logic: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [C]\varphi$

$\mathcal{M}, w \models [C]\varphi$ iff $(\varphi)^{\mathcal{M}} \in N(w, C)$: "Coalition C has a joint strategy to force the outcome to satisfy φ".

Higher-Order Coalition Logic: $\varphi :=$
$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X\varphi \mid \forall x\varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

- $F(x_1, \ldots, x_n)$ is a first-order atomic formula
- x is a first-order variable
- X is a set variable
- $\{x\}\psi$ is a group operator representing the set of all d such that $\psi[d/x]$ holds

HCL: Expressivity

What does the added expressive power give you?

HCL: Expressivity

What does the added expressive power give you?

- Relationships between coalitions:
  $\forall x(\mathit{super\_user}(x) \rightarrow \mathit{user}(x))$

HCL: Expressivity

What does the added expressive power give you?

- Relationships between coalitions:
  $\forall x(\mathit{super\_user}(x) \rightarrow \mathit{user}(x))$

- General quantification over coalitions:
  $\forall X(\forall x(Xx \rightarrow \mathit{user}(x)) \rightarrow [\{y\}Xy]\varphi)$

  Every coalition such that all of its members are users can achieve φ.

HCL: Expressivity

What does the added expressive power give you?

- Relationships between coalitions:
  $\forall x(\mathit{super\_user}(x) \rightarrow \mathit{user}(x))$
- General quantification over coalitions:
  $\forall X(\forall x(Xx \rightarrow \mathit{user}(x)) \rightarrow [\{y\}Xy]\varphi)$

  Every coalition such that all of its members are users can achieve φ.
- Complex relationships between coalitions and agents:
  $[\{x\}\varphi(x)]\psi \rightarrow [\{y\}\exists x(\varphi(x) \wedge \mathit{collaborates}(y,x))]\psi$

  If the coalition represented by φ can achieve ψ then so can any group that collaborates with at least one member of $\varphi(x)$.

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y\psi(y)$

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \to \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \to [\{x\}\varphi(x)]\forall y\psi(y)$

$[\{x\}x = \mathit{Eric}]\forall y(\mathit{ESSLLI}(y) \to \mathit{happy}(y)) \to$
$\forall y[\{x\}x = \mathit{Eric}](\mathit{ESSLLI}(y) \to \mathit{happy}(y))$

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y\psi(y)$

$[\{x\}x = Eric]\forall y(ESSLLI(y) \rightarrow happy(y)) \rightarrow$
$\forall y[\{x\}x = Eric](ESSLLI(y) \rightarrow happy(y))$

If I can do something to make everyone happy at ESSLLI implies for each person at ESSLLI, I can do something to make them happy.

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y\psi(y)$

$[\{x\}x = \mathit{Eric}]\forall y(\mathit{ESSLLI}(y) \rightarrow \mathit{happy}(y)) \rightarrow$
$\forall y[\{x\}x = \mathit{Eric}](\mathit{ESSLLI}(y) \rightarrow \mathit{happy}(y))$

If I can do something to make everyone happy at ESSLLI implies for each person at ESSLLI, I can do something to make them happy.

$\forall y[\{x\}x = \mathit{Eric}](\mathit{ESSLLI}(y) \rightarrow \mathit{happy}(y)) \not\rightarrow$
$[\{x\}x = \mathit{Eric}]\forall y(\mathit{ESSLLI}(y) \rightarrow \mathit{happy}(y))$

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \to \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \to [\{x\}\varphi(x)]\forall y\psi(y)$

$[\{x\}x = \mathit{Eric}]\forall y(\mathit{ESSLLI}(y) \to \mathit{happy}(y)) \to$
$\forall y[\{x\}x = \mathit{Eric}](\mathit{ESSLLI}(y) \to \mathit{happy}(y))$

If I can do something to make everyone happy at ESSLLI implies for each person at ESSLLI, I can do something to make them happy.

$\forall y[\{x\}x = \mathit{Eric}](\mathit{ESSLLI}(y) \to \mathit{happy}(y)) \not\to$
$[\{x\}x = \mathit{Eric}]\forall y(\mathit{ESSLLI}(y) \to \mathit{happy}(y))$

For each person at ESSLLI, I can make them happy does not imply that I can do something to make everyone at ESSLLI happy.

Higher-Order Coalition Logic

Sound and complete axiomatization combines ideas from coaltion logic, first-order extensions of non-normal modal logics and Henkin-style completeness for second-order logic.

- ✓ First-Order Modal Logic
- Game Logic (Skip)
- Evidence Dynamics (Skip)

Background: Propositional Dynamic Logic

Let P be a set of atomic programs and At a set of atomic propositions.

Formulas of **PDL** have the following syntactic form:

$$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid [\alpha]\varphi$$

$$\alpha := a \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \varphi?$$

where $p \in \mathrm{At}$ and $a \in \mathrm{P}$.

$[\alpha]\varphi$ is intended to mean "after executing the program α, φ is true"

Background: Propositional Dynamic Logic

Semantics: $\mathcal{M} = \langle W, \{R_a \mid a \in \mathsf{P}\}, V\rangle$ where for each $a \in \mathsf{P}$, $R_a \subseteq W \times W$ and $V : \mathsf{At} \rightarrow \wp(W)$

- $R_{\alpha\cup\beta} := R_\alpha \cup R_\beta$
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$
- $R_{\alpha^*} := \cup_{n\geq 0} R_\alpha^n$
- $R_{\varphi?} = \{(w, w) \mid \mathcal{M}, w \models \varphi\}$

$$\mathcal{M}, w \models [\alpha]\varphi \text{ iff for each } v, \text{ if } wR_\alpha v \text{ then } \mathcal{M}, v \models \varphi$$

Background: Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$
7. $\varphi \wedge [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow [\alpha^*]\varphi$
8. Modus Ponens and Necessitation (for each program α)

Background: Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$ (Fixed-Point Axiom)
7. $\varphi \wedge [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow [\alpha^*]\varphi$ (Induction Axiom)
8. Modus Ponens and Necessitation (for each program α)

Background: Propositional Dynamic Logic

Theorem PDL is sound and weakly complete with respect to the Segerberg Axioms.

Theorem The satisfiability problem for **PDL** is decidable ($\mathrm{EXPTIME}$-Complete).

D. Kozen and R. Parikh. *A Completeness proof for Propositional Dynamic Logic*.
.

D. Harel, D. Kozen and Tiuryn. *Dynamic Logic*. 2001.

Concurrent Programs

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

In PDL: $R_\alpha \subseteq W \times W$, where $wR_\alpha v$ means executing α in state w leads to state v.

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

In PDL: $R_\alpha \subseteq W \times W$, where $wR_\alpha v$ means executing α in state w leads to state v.

With Concurrent Programs: $R_\alpha \subseteq W \times \wp(W)$, where $wR_\alpha V$ means executing α in parallel from state w to reach all states in V.

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

In PDL: $R_\alpha \subseteq W \times W$, where $wR_\alpha v$ means executing α in state w leads to state v.

With Concurrent Programs: $R_\alpha \subseteq W \times \wp(W)$, where $wR_\alpha V$ means executing α in parallel from state w to reach all states in V.

$w \models \langle\alpha\rangle\varphi$ iff $\exists U$ such that $(w, U) \in R_\alpha$ and $\forall v \in U,\ v \models \varphi$.

$R_{\alpha\cap\beta} := \{(w, V) \mid \exists U, U',\ (w, U) \in R_\alpha,\ (w, U') \in R_\beta,\ V = U \cup U'\}$

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

Main Idea:
In **PDL**: $w \models \langle\pi\rangle\varphi$: there is a run of the program π starting in state w that ends in a state where φ is true.

The programs in **PDL** can be thought of as *single player games*.

From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

Main Idea:
In **PDL**: $w \models \langle\pi\rangle\varphi$: there is a run of the program π starting in state w that ends in a state where φ is true.

The programs in **PDL** can be thought of as *single player games*.

Game Logic generalized **PDL** by considering two players:

In **GL**: $w \models \langle\gamma\rangle\varphi$: Angel has a **strategy** in the game γ to ensure that the game ends in a state where φ is true.

From **PDL** to Game Logic

Consequences of two players:

From **PDL** to Game Logic

Consequences of two players:

$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

$[\gamma]\varphi$: Demon has a strategy in γ to ensure φ is true

From **PDL** to Game Logic

Consequences of two players:

$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

$[\gamma]\varphi$: Demon has a strategy in γ to ensure φ is true

Either Angel or Demon can win: $\langle\gamma\rangle\varphi \vee [\gamma]\neg\varphi$

From **PDL** to Game Logic

Consequences of two players:

$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

$[\gamma]\varphi$: Demon has a strategy in γ to ensure φ is true

Either Angel or Demon can win: $\langle\gamma\rangle\varphi \vee [\gamma]\neg\varphi$

But not both: $\neg(\langle\gamma\rangle\varphi \wedge [\gamma]\neg\varphi)$

From **PDL** to Game Logic

Consequences of two players:

$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

$[\gamma]\varphi$: Demon has a strategy in γ to ensure φ is true

Either Angel or Demon can win: $\langle\gamma\rangle\varphi \vee [\gamma]\neg\varphi$

But not both: $\neg(\langle\gamma\rangle\varphi \wedge [\gamma]\neg\varphi)$

Thus, $[\gamma]\varphi \leftrightarrow \neg\langle\gamma\rangle\neg\varphi$ is a valid principle

From **PDL** to Game Logic

Consequences of two players:

$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

$[\gamma]\varphi$: Demon has a strategy in γ to ensure φ is true

Either Angel or Demon can win: $\langle\gamma\rangle\varphi \vee [\gamma]\neg\varphi$

But not both: $\neg(\langle\gamma\rangle\varphi \wedge [\gamma]\neg\varphi)$

Thus, $[\gamma]\varphi \leftrightarrow \neg\langle\gamma\rangle\neg\varphi$ is a valid principle

However, $[\gamma]\varphi \wedge [\gamma]\psi \rightarrow [\gamma](\varphi \wedge \psi)$ is **not** a valid principle

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^\times := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^\times := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^\times := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^\times := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1 ; \gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^\times := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^x := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^\times := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^x := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

Syntax

Let Γ_0 be a set of atomic games and At a set of atomic propositions. Then formulas of Game Logic are defined inductively as follows:

$$\gamma \;:=\; g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$
$$\varphi \;:=\; \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi \mid [\gamma]\varphi$$

where $p \in \mathrm{At}, g \in \Gamma_0$.

Game Logic

A neighborhood game model is a tuple $\mathcal{M} = \langle W, \{E_g \mid g \in \Gamma_0\}, V\rangle$ where

W is a nonempty set of states

For each $g \in \Gamma_0$, $E_g : W \to \wp(\wp(W))$ is a monotonic neighborhood function.

$X \in E_g(w)$ means in state s, Angel has a strategy to force the game to end in *some* state in X (we may write wE_gX)

$V : \mathsf{At} \to \wp(W)$ is a valuation function.

Game Logic

Propositional letters and boolean connectives are as usual.

$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E_\gamma(w)$

Game Logic

Propositional letters and boolean connectives are as usual.

$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E_\gamma(w)$

Suppose $E_\gamma(Y) := \{s \mid Y \in E_g(s)\}$

- $E_{\gamma_1;\gamma_2}(Y) \;:=\; E_{\gamma_1}(E_{\gamma_2}(Y))$
- $E_{\gamma_1\cup\gamma_2}(Y) \;:=\; E_{\gamma_1}(Y) \cup E_{\gamma_2}(Y)$
- $E_{\varphi?}(Y) \;:=\; (\varphi)^{\mathcal{M}} \cap Y$
- $E_{\gamma^d}(Y) \;:=\; \overline{E_\gamma(\overline{Y})}$
- $E_{\gamma^*}(Y) \;:=\; \mu X . Y \cup E_\gamma(X)$

Game Logic: Axioms

1. All propositional tautologies
2. $\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$ Composition
3. $\langle\alpha \cup \beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi$ Union
4. $\langle\psi?\rangle\varphi \leftrightarrow (\psi \wedge \varphi)$ Test
5. $\langle\alpha^{d}\rangle\varphi \leftrightarrow \neg\langle\alpha\rangle\neg\varphi$ Dual
6. $(\varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi) \rightarrow \langle\alpha^*\rangle\varphi$ Mix

and the rules,

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad \frac{\varphi \rightarrow \psi}{\langle\alpha\rangle\varphi \rightarrow \langle\alpha\rangle\psi} \qquad \frac{(\varphi \vee \langle\alpha\rangle\psi) \rightarrow \psi}{\langle\alpha^*\rangle\varphi \rightarrow \psi}$$

Game Logic

- Game Logic is more expressive than **PDL**

Game Logic

- Game Logic is more expressive than **PDL**

$$\langle (g^d)^* \rangle \bot$$

Game Logic

Theorem Dual-free game logic is sound and complete with respect to the class of all game models.

Game Logic

Theorem Dual-free game logic is sound and complete with respect to the class of all game models.

Theorem Iteration-free game logic is sound and complete with respect to the class of all game models.

Game Logic

Theorem Dual-free game logic is sound and complete with respect to the class of all game models.

Theorem Iteration-free game logic is sound and complete with respect to the class of all game models.

Open Question Is (full) game logic complete with respect to the class of all game models?

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics, 1985.

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

Game Logic

Theorem Given a game logic formula φ and a finite game model $\mathcal{M}$, model checking can be done in time $O(|\mathcal{M}|^{ad(\varphi)+1} \times |\varphi|)$

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985).

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

D. Berwanger. *Game Logic is Strong Enough for Parity Games*. Studia Logica **75** (2003).

Game Logic

Theorem The satisfiability problem for game logic is in EXPTIME.

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985).

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

D. Berwanger. *Game Logic is Strong Enough for Parity Games*. Studia Logica **75** (2003).

Game Logic

Theorem Game logic can be translated into the modal μ-calculus

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985).

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

D. Berwanger. *Game Logic is Strong Enough for Parity Games*. Studia Logica **75** (2003).

Game Logic

Theorem Game logic can be translated into the modal μ-calculus

Theorem No finite level of the modal μ-calculus hierarchy captures the expressive power of game logic.

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985).

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).

D. Berwanger. *Game Logic is Strong Enough for Parity Games*. Studia Logica **75** (2003).

Game Algebra

Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models

Game Algebra

Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models (iff $\langle\gamma_1\rangle p \leftrightarrow \langle\gamma_2\rangle p$ is valid for a p which occurs neither in γ_1 nor in γ_2.)

Skip

Game Algebra

Game Boards: Given a set of states or positions B, for each game g and each player i there is an associated relation $E^i_g \subseteq B \times 2^B$:

$pE^i_g T$ holds if in position p, i can force that the outcome of g will be a position in T.

- (monotonicity) if $pE^i_g T$ and $T \subseteq U$ then $pE^i_g U$
- (consistency) if $pE^i_g T$ then not $pE^{1-i}_g (B - T)$

Given a game board (a set B with relations E^i_g for each game and player), we say that two games g, h ($g \approx h$) are equivalent if $E^i_g = E^i_h$ for each i.

Game Algebra

Game Algebra

1. Standard Laws of Boolean Algebras

Game Algebra

1. Standard Laws of Boolean Algebras
2. $(x; y); z \approx x; (y; z)$

Game Algebra

1. Standard Laws of Boolean Algebras
2. $(x; y); z \approx x; (y; z)$
3. $(x \vee y); z \approx (x; z) \vee (y; z)$, $(x \wedge y); z \approx (x; z) \wedge (y; z)$

Game Algebra

1. Standard Laws of Boolean Algebras
2. $(x; y); z \approx x; (y; z)$
3. $(x \vee y); z \approx (x; z) \vee (y; z)$, $(x \wedge y); z \approx (x; z) \wedge (y; z)$
4. $-x; -y \approx -(x; y)$

Game Algebra

1. Standard Laws of Boolean Algebras
2. $(x; y); z \approx x; (y; z)$
3. $(x \vee y); z \approx (x; z) \vee (y; z)$, $(x \wedge y); z \approx (x; z) \wedge (y; z)$
4. $-x; -y \approx -(x; y)$
5. $y \preceq z \Rightarrow x; y \preceq x; z$

Theorem Sound and complete axiomatizations of (iteration free) game algebra

Y. Venema. *Representing Game Algebras*. Studia Logica **75** (2003).

V. Goranko. *The Basic Algebra of Game Equivalences*. Studia Logica **75** (2003).

Concurrent Game Logic

$\gamma_1 \cap \gamma_2$ means "play γ_1 and γ_2 in parallel."

Concurrent Game Logic

$\gamma_1 \cap \gamma_2$ means "play γ_1 and γ_2 in parallel."

Need both the disjunctive and conjunctive interpretation of the neighborhoods.

Main Idea: $R_\gamma \subseteq W \times \wp(\wp(\wp(W)))$

J. van Benthem, S. Ghosh and F. Liu. *Modelling Simultaneous Games in Dynamic Logic*. Synthese, 165(2), pgs. 247-268, 2008.

More Information on Game Logic and Algebra

M. Pauly and R. Parikh. *Game Logic — An Overview*. Studia Logica **75**, 2003.

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics, 1985.

J. van Benthem. *Logics and Games*. The MIT Press, 2014.

- ✓ First-Order Modal Logic
- ✓ Game Logic
- ▸ Evidence Dynamics (Skip)

Background: Modeling Informational Changes

- Modeling strategies:

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based;

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \stackrel{\tau}{\Longrightarrow} \mathcal{M}^{\tau}$$

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \overset{\tau}{\Longrightarrow} \mathcal{M}^{\tau}$$

- Given an operation for transforming a model, what are the "recursion axioms" that characterize this operation?

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \stackrel{\tau}{\Longrightarrow} \mathcal{M}^{\tau}$$

- Given an operation for transforming a model, what are the "recursion axioms" that characterize this operation?

  Example: "Public Announcement of φ": $\mathcal{M}^{!\varphi}$ is the submodel of $\mathcal{M}$ where all states satisfy φ

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \stackrel{\tau}{\Longrightarrow} \mathcal{M}^\tau$$

- Given an operation for transforming a model, what are the "recursion axioms" that characterize this operation?

  Example: "Public Announcement of φ": $\mathcal{M}^{!\varphi}$ is the submodel of $\mathcal{M}$ where all states satisfy φ

$$[!\varphi]K\psi \quad \leftrightarrow \quad (\varphi \to K(\varphi \to [!\varphi]\psi))$$

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \stackrel{\tau}{\Longrightarrow} \mathcal{M}^{\tau}$$

- Given an operation for transforming a model, what are the "recursion axioms" that characterize this operation?

  Example: "Public Announcement of φ": $\mathcal{M}^{!\varphi}$ is the submodel of $\mathcal{M}$ where all states satisfy φ

$$\begin{array}{lcl} [!\varphi]K\psi & \leftrightarrow & (\varphi \to K(\varphi \to [!\varphi]\psi)) \\ [!\varphi]B\psi & \leftrightarrow & \end{array}$$

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \stackrel{\tau}{\Longrightarrow} \mathcal{M}^{\tau}$$

- Given an operation for transforming a model, what are the "recursion axioms" that characterize this operation?

  Example: "Public Announcement of φ": $\mathcal{M}^{!\varphi}$ is the submodel of $\mathcal{M}$ where all states satisfy φ

$$\begin{array}{lcl} [!\varphi]K\psi & \leftrightarrow & (\varphi \rightarrow K(\varphi \rightarrow [!\varphi]\psi)) \\ [!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \end{array}$$

Background: Modeling Informational Changes

- Modeling strategies: temporal-based vs. change-based; rich states and algebra/simple operation vs. simples states and algebra/complex or many operation

$$\mathcal{M} \stackrel{\tau}{\Longrightarrow} \mathcal{M}^{\tau}$$

- Given an operation for transforming a model, what are the "recursion axioms" that characterize this operation?

  Example: "Public Announcement of φ": $\mathcal{M}^{!\varphi}$ is the submodel of $\mathcal{M}$ where all states satisfy φ

$$\begin{array}{lcl} [!\varphi]K\psi & \leftrightarrow & (\varphi \rightarrow K(\varphi \rightarrow [!\varphi]\psi)) \\ [!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \\ [!\varphi]B^{\alpha}\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi) \end{array}$$

"Public Announcements"

Accept evidence from an infallible source.

"Public Announcements"

Accept evidence from an infallible source.

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model and $\varphi \in \mathcal{L}$ a formula. The model $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, E^{!\varphi}, V^{!\varphi} \rangle$ is defined as follows: $W^{!\varphi} = [\![\varphi]\!]_{\mathcal{M}}$, for each $p \in \mathsf{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$ and for all $w \in W$,

$$E^{!\varphi}(w) = \{X \mid \emptyset \neq X = Y \cap [\![\varphi]\!]_{\mathcal{M}} \text{ for some } Y \in E(w)\}.$$

"Public Announcements"

Accept evidence from an infallible source.

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model and $\varphi \in \mathcal{L}$ a formula. The model $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, E^{!\varphi}, V^{!\varphi} \rangle$ is defined as follows: $W^{!\varphi} = [\![\varphi]\!]_{\mathcal{M}}$, for each $p \in \mathsf{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$ and for all $w \in W$,

$$E^{!\varphi}(w) = \{X \mid \emptyset \neq X = Y \cap [\![\varphi]\!]_{\mathcal{M}} \text{ for some } Y \in E(w)\}.$$

$[!\varphi]\psi$: "ψ is true after the public announcement of φ"

$\mathcal{M}, w \models [!\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}^{!\varphi}, w \models \psi$

Public Announcements: Recursion Axioms

$$
\begin{array}{lll}
[!\varphi]p & \leftrightarrow & (\varphi \rightarrow p) \qquad (p \in \mathsf{At}) \\
[!\varphi](\psi \wedge \chi) & \leftrightarrow & ([!\varphi]\psi \wedge [!\varphi]\chi) \\
[!\varphi]\neg\psi & \leftrightarrow & (\varphi \rightarrow \neg[!\varphi]\psi) \\
[!\varphi]\Box\psi & \leftrightarrow & (\varphi \rightarrow \Box^{\varphi}[!\varphi]\psi) \\
[!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \\
[!\varphi]\Box^{\alpha}\psi & \leftrightarrow & (\varphi \rightarrow \Box^{\varphi\wedge[!\varphi]\alpha}[!\varphi]\psi) \\
[!\varphi]B^{\alpha}\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi\wedge[!\varphi]\alpha}[!\varphi]\psi) \\
[!\varphi]A\psi & \leftrightarrow & (\varphi \rightarrow A[!\varphi]\psi)
\end{array}
$$

Public Announcements: Recursion Axioms

$$[!\varphi]p \quad\leftrightarrow\quad (\varphi \to p) \qquad (p \in \mathsf{At})$$

$$[!\varphi](\psi \wedge \chi) \quad\leftrightarrow\quad ([!\varphi]\psi \wedge [!\varphi]\chi)$$

$$[!\varphi]\neg\psi \quad\leftrightarrow\quad (\varphi \to \neg[!\varphi]\psi)$$

$$[!\varphi]\Box\psi \quad\leftrightarrow\quad (\varphi \to \Box^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]B\psi \quad\leftrightarrow\quad (\varphi \to B^{\varphi}[!\varphi]\psi)$$

$$[!\varphi]\Box^{\alpha}\psi \quad\leftrightarrow\quad (\varphi \to \Box^{\varphi\wedge[!\varphi]\alpha}[!\varphi]\psi)$$

$$[!\varphi]B^{\alpha}\psi \quad\leftrightarrow\quad (\varphi \to B^{\varphi\wedge[!\varphi]\alpha}[!\varphi]\psi)$$

$$[!\varphi]A\psi \quad\leftrightarrow\quad (\varphi \to A[!\varphi]\psi)$$

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** ($!\varphi$) is a complex combination of three distinct epistemic operations:

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** ($!\varphi$) is a complex combination of three distinct epistemic operations:

1. **Evidence addition**: accepting that φ is a piece of evidence

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** ($!\varphi$) is a complex combination of three distinct epistemic operations:

1. **Evidence addition**: accepting that φ is a piece of evidence
2. **Evidence removal**: remove evidence for $\neg\varphi$

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** ($!\varphi$) is a complex combination of three distinct epistemic operations:

1. **Evidence addition**: accepting that φ is a piece of evidence
2. **Evidence removal**: remove evidence for $\neg\varphi$
3. **Evidence modification**: incorporate φ into each piece of evidence gathered so far

Evidence Addition

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and φ a formula in $\mathcal{L}$. The model $\mathcal{M}^{+\varphi} = \langle W^{+\varphi}, E^{+\varphi}, V^{+\varphi} \rangle$ has $W^{+\varphi} = W$, $V^{+\varphi} = V$ and for all $w \in W$,

$$E^{+\varphi}(w) = E(w) \cup \{[\![\varphi]\!]_{\mathcal{M}}\}$$

$[+\varphi]\psi$: "ψ is true after φ is accepted as an admissible piece of evidence"

$\mathcal{M}, w \models [+\varphi]\psi$ iff $\mathcal{M}, w \models E\varphi$ implies $\mathcal{M}^{+\varphi}, w \models \psi$

Evidence Addition

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and φ a formula in $\mathcal{L}$. The model $\mathcal{M}^{+\varphi} = \langle W^{+\varphi}, E^{+\varphi}, V^{+\varphi} \rangle$ has $W^{+\varphi} = W$, $V^{+\varphi} = V$ and for all $w \in W$,

$$E^{+\varphi}(w) = E(w) \cup \{[\![\varphi]\!]_{\mathcal{M}}\}$$

$[+\varphi]\psi$: "ψ is true after φ is accepted as an admissible piece of evidence"

$\mathcal{M}, w \models [+\varphi]\psi$ iff $\mathcal{M}, w \models E\varphi$ implies $\mathcal{M}^{+\varphi}, w \models \psi$

Evidence Addition: Recursion Axioms

$$
\begin{array}{lcll}
[+\varphi]p & \leftrightarrow & (E\varphi \to p) & (p \in \mathsf{At}) \\
[+\varphi](\psi \wedge \chi) & \leftrightarrow & ([+\varphi]\psi \wedge [+\varphi]\chi) & \\
[+\varphi]\neg\psi & \leftrightarrow & (E\varphi \to \neg[+\varphi]\psi) & \\
[+\varphi]A\psi & \leftrightarrow & (E\varphi \to A[+\varphi]\psi) &
\end{array}
$$

Evidence Addition: Recursion Axioms

$$
\begin{array}{rcl}
[+\varphi]\Box\psi & \leftrightarrow & (E\varphi \rightarrow (\Box[+\varphi]\psi \vee A(\varphi \rightarrow [+\varphi]\psi))) \\
[+\varphi]\Box^{\alpha}\psi & \leftrightarrow & (E\varphi \rightarrow (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge \\
 & & A((\varphi \wedge [+\varphi]\alpha) \rightarrow [+\varphi]\psi))))
\end{array}
$$

Evidence Addition: Recursion Axioms

$$[+\varphi]\Box\psi \quad\leftrightarrow\quad (E\varphi \rightarrow (\Box[+\varphi]\psi \vee A(\varphi \rightarrow [+\varphi]\psi)))$$

$$[+\varphi]\Box^{\alpha}\psi \quad\leftrightarrow\quad (E\varphi \rightarrow (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge A((\varphi \wedge [+\varphi]\alpha) \rightarrow [+\varphi]\psi))))$$

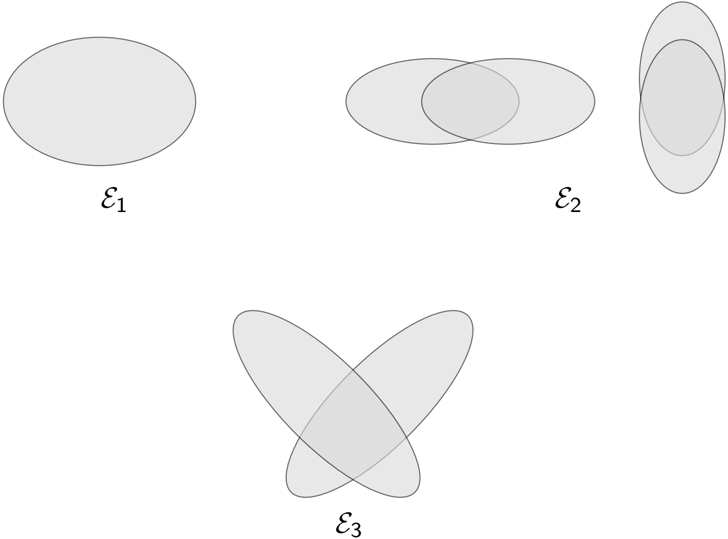
Evidence Addition: Recursion Axioms

$$[+\varphi]B\psi \quad \leftrightarrow \quad ????$$

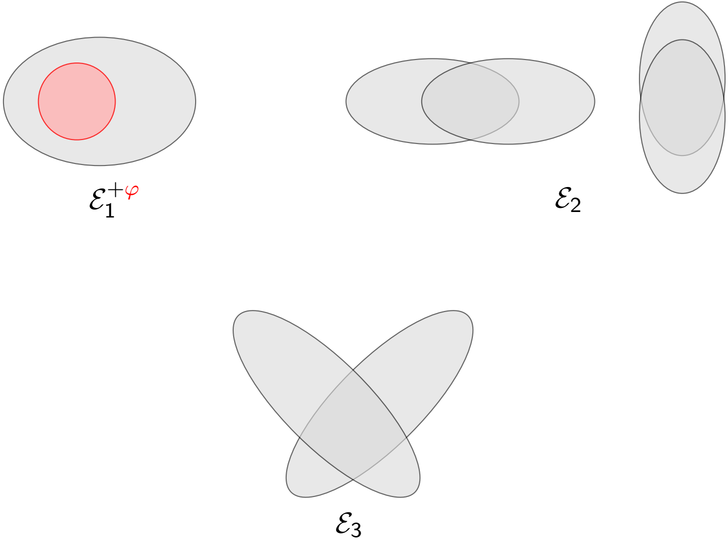
$$[+\varphi]B^{\alpha}\psi \quad \leftrightarrow \quad ????$$

Adding φ

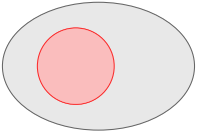
Adding φ



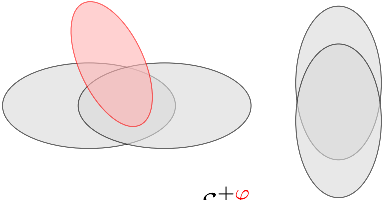
Adding φ



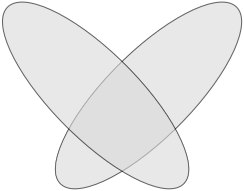
Adding φ



$\mathcal{E}_1^{+\varphi}$

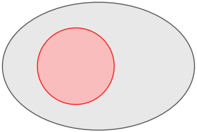


$\mathcal{E}_2^{+\varphi}$

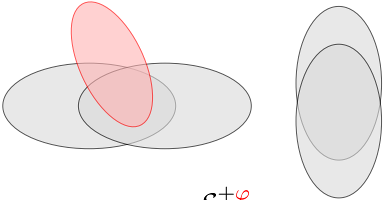


$\mathcal{E}_3$

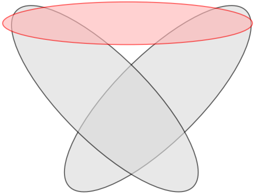
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2^{+\varphi}$



$\mathcal{E}_3^{+\varphi}$

Compatibile vs. Incompatible

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally φ-**compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally **φ-compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and
2. $\mathcal{X}$ is **incompatible** with φ provided there are $X_1, \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots \cap X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$.

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally **φ-compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and

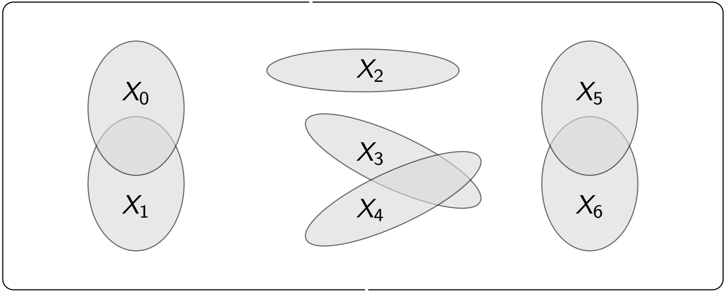
2. $\mathcal{X}$ is **incompatible** with φ provided there are $X_1, \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots \cap X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$.

Conditional belief: $B^{+\varphi}\psi$ iff for each maximally φ-compatible $\mathcal{X} \subseteq E(w)$, $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$

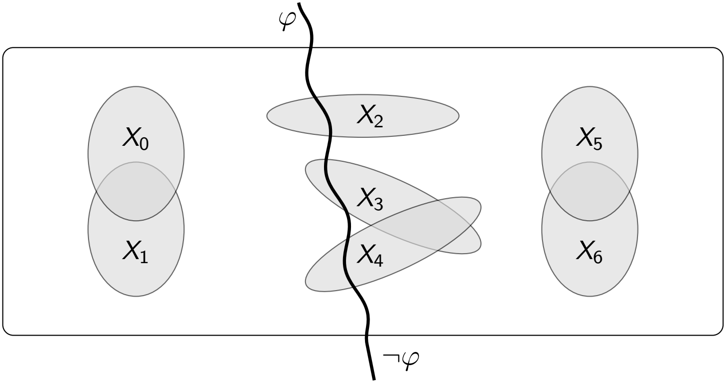
Conditional Beliefs (Incompatibility Version): $\mathcal{M}, w \models B^{-\varphi}\psi$ iff for all maximal f.i.p., if $\mathcal{X}$ is incompatible with φ then $\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

$B^{+\neg\varphi}$ vs. $B^{-\varphi}$

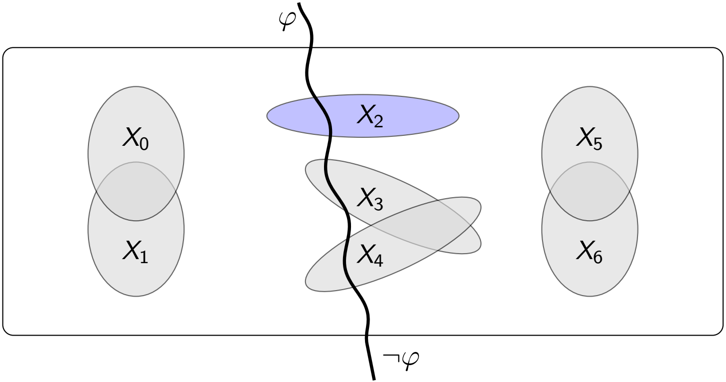
$B^{+\neg\varphi}$ vs. $B^{-\varphi}$



$B^{+\neg\varphi}$ vs. $B^{-\varphi}$



$B^{+\neg\varphi}$ vs. $B^{-\varphi}$



$\{X_2\}$ is (max.) compatible with $\neg\varphi$ but not maximally φ incompatible

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \to (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

Proof Sketch

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

▸ Proof Sketch

But now, we need a recursion axiom for $B^{-\varphi}$.

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

Proof Sketch

But now, we need a recursion axiom for $B^{-\varphi}$.

Language Extension: $\mathcal{M}, w \models B^{\varphi,\psi}\chi$ iff for all maximally φ-compatible sets $\mathcal{X} \subseteq E(w)$, if $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$, then $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\chi]\!]_{\mathcal{M}}$.

$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \to (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

Proof Sketch

But now, we need a recursion axiom for $B^{-\varphi}$.

Language Extension: $\mathcal{M}, w \models B^{\varphi,\psi}\chi$ iff for all maximally φ-compatible sets $\mathcal{X} \subseteq E(w)$, if $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$, then $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\chi]\!]_{\mathcal{M}}$.

$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

Fact. The following is valid:

$$[+\varphi]B^{\psi,\alpha}\chi \leftrightarrow (E\varphi \to (B^{\varphi\wedge[+\varphi]\psi,[+\varphi]\alpha}[+\varphi]\chi \wedge B^{[+\varphi]\psi,\neg\varphi\wedge[+\varphi]\alpha}[+\varphi]\chi))$$

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** $(!\varphi)$ is a complex combination of three distinct epistemic operations:

✓ **Evidence addition**: accepting that φ is a piece of evidence

2. **Evidence removal**: remove evidence for $\neg\varphi$

3. **Evidence modification**: incorporate φ into each piece of evidence gathered so far

Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$ More

Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$ ▸ More

Evidence Modification: $E^{\oplus\varphi}(w) = \{X \cup [\![\varphi]\!]_{\mathcal{M}} \mid X \in E(w)\}$

$\mathcal{M}, w \models [\oplus\varphi]\psi$ iff $\mathcal{M}^{\oplus\varphi}, w \models \psi$

- $[\oplus\varphi]\Box\psi \leftrightarrow (\Box[\oplus\varphi]\psi \wedge A(\varphi \rightarrow [\oplus\varphi]\psi))$

Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$ ▸ More

Evidence Modification: $E^{\oplus\varphi}(w) = \{X \cup [\![\varphi]\!]_{\mathcal{M}} \mid X \in E(w)\}$

$\mathcal{M}, w \models [\oplus\varphi]\psi$ iff $\mathcal{M}^{\oplus\varphi}, w \models \psi$

- $[\oplus\varphi]\Box\psi \leftrightarrow (\Box[\oplus\varphi]\psi \wedge A(\varphi \rightarrow [\oplus\varphi]\psi))$

Evidence Combination: $E^{\#}(w)$ is the smallest set closed under consistent intersection and containing $E(w)$

$\mathcal{M}, w \models [\#]\varphi$ iff $\mathcal{M}^{\#}, w \models \varphi$

- Are $\neg[\#]\Box\neg\varphi \rightarrow B\varphi$ and $[\#]\Box\varphi \rightarrow B\varphi$ valid? ▸ Explain

Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$:	"there is evidence compatible with each of the γ_i for ψ"

Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$:	"there is evidence compatible with each of the γ_i for ψ"
$B\psi$:	"the agent believe χ"

Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$:	"there is evidence compatible with each of the γ_i for ψ"
$B\psi$:	"the agent believe χ"
$B^{\varphi}\psi$:	"the agent believe χ conditional on φ"

Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$:	"there is evidence compatible with each of the γ_i for ψ"
$B\psi$:	"the agent believe χ"
$B^{\varphi}\psi$:	"the agent believe χ conditional on φ"
$B^{\varphi}_{\overline{\gamma}}\psi$:	"the agent believe χ conditional on φ assuming compatibility with each of the γ_i"

Summary: Conditional Belief/Evidence

$\Box\psi$: "there is evidence for ψ"
$\Box^{\varphi}\psi$: "there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$: "there is evidence compatible with each of the γ_i for ψ"
$B\psi$: "the agent believe χ"
$B^{\varphi}\psi$: "the agent believe χ conditional on φ"
$B^{\varphi}_{\overline{\gamma}}\psi$: "the agent believe χ conditional on φ assuming compatibility with each of the γ_i"
$B^{\varphi,\alpha}\psi$: "the agent believe ψ, after having settled on α and conditional on φ"

Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$:	"there is evidence compatible with each of the γ_i for ψ"
$B\psi$:	"the agent believe χ"
$B^{\varphi}\psi$:	"the agent believe χ conditional on φ"
$B^{\varphi}_{\overline{\gamma}}\psi$:	"the agent believe χ conditional on φ assuming compatibility with each of the γ_i"
$B^{\varphi,\alpha}\psi$:	"the agent believe ψ, after having settled on α and conditional on φ"

Complete logical analysis?

$$B^{\varphi}\psi \rightarrow B(\varphi \rightarrow \psi) \quad \text{and} \quad B(\varphi \rightarrow \psi) \rightarrow B^{\top,\varphi}\psi$$

Summary: Evidence Operations

Public announcement:	$[!\varphi]B\psi$	$\leftrightarrow$	$(\varphi \to B^{\varphi}[!\varphi]\psi)$
Evidence addition:	$[+\varphi]B\psi$	$\leftrightarrow$	$(E\varphi \to (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$
Evidence removal:	$[-\varphi]B\psi$	$\leftrightarrow$	$(\neg A\varphi \to B_{\neg\varphi}[-\varphi]\psi)$

Concluding Remarks

Concluding Remarks: Robust Belief, Reliable and Unreliable Evidence

Robust Belief: $\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have $Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Concluding Remarks: Robust Belief, Reliable and Unreliable Evidence

Robust Belief: $\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have $Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Reliable Evidence: $E^C(w) = \{X \in E(w) \mid w \in X\}$

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

Concluding Remarks: Robust Belief, Reliable and Unreliable Evidence

Robust Belief: $\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have $Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

Reliable Evidence: $E^C(w) = \{X \in E(w) \mid w \in X\}$

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

Unreliable Evidence: $E^U(w) = \{X \in E(w) \mid w \notin X\}$.

$\mathcal{M}, w \models \Box^U\varphi$ iff for all $v \in \bigcup E^U(w)$, $\mathcal{M}, v \models \varphi$

Concluding Remarks: Robust Belief, Reliable and Unreliable Evidence

Fact. Let $\mathcal{M}$ be a uniform evidence model, then for all factual formulas φ:

$$\mathcal{M}, w \models \Box^C\varphi \wedge \Box^U\varphi \text{ iff } ORD(\mathcal{M}), w \models B^r\varphi$$

Explain

Fact The operators $\Box^C$ and $\Box^U$ are not definable in evidence belief language $\mathcal{L}$. Proof

Concluding Remarks: Evidence Addition and Plausibility

$$\begin{array}{c} E \\ \Big\downarrow \\ \preceq^E \end{array}$$

Concluding Remarks: Evidence Addition and Plausibility

$$\begin{array}{ccc} E & \xRightarrow{+X} & E^{+} \\ \downarrow & & \\ \preceq^{E} & & \end{array}$$

Concluding Remarks: Evidence Addition and Plausibility

$$\begin{array}{ccc} E & \xrightarrow{+X} & E^+ \\ \downarrow & & \downarrow \\ \preceq^E & \xrightarrow[??]{} & \preceq^{E^+} \end{array}$$

$$\preceq^{E^+} = \preceq^E - \{(w, v) \mid v \in X \text{ and } w \notin X\}.$$

Concluding Remarks: Evidence Addition and Plausibility

$$
\begin{tikzcd}
E \arrow[r, Rightarrow, "??"] \arrow[d, dashed] & E^{+} \arrow[d, dashed] \\
\preceq^{E} \arrow[r, Rightarrow, "!\varphi"'] & \preceq^{E^{+}}
\end{tikzcd}
$$

Concluding Remarks: Many Agents

Social notions: Let $\mathcal{M} = \langle W, \mathcal{E}_i, \mathcal{E}_j, V \rangle$ be a multiagent evidence model. What evidence does the group i, j have?

- $\mathcal{M}, w \models \Box^{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cup \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models \Box_{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cap \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [i \sqcap j]\varphi$ iff there exists $X \in \mathcal{E}_i \sqcap \mathcal{E}_j$ with $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
  $\mathcal{E}_i \sqcap \mathcal{E}_j = \{ Y \mid \emptyset \neq Y = X \cap X' \text{ with } X \in \mathcal{E}_i \text{ and } X' \in \mathcal{E}_j \}$

Concluding Remarks: Some Questions

- What is the right notion of bisimulation for these models?
- What is the complete logic in a language with the conditional belief/evidence operators? ...in a language with the (un)reliable evidence operator?
- We know that the satisfiability problem is decidable, but what is its complexity?
- What happens when the agent notices an inconsistency in her evidence? (eg., Priority structures, represent the sources)
- ...

Course Plan

- ✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Subset Spaces, Neighborhood Structures, Motivating Non-Normal Modal Logics/Neighborhood Semantics
  (Monday, Tuesday)
- ✓ **Core Theory**: Completeness, Decidability, Complexity, Incompleteness, Relationship with Other Semantics for Modal Logic, Model Theory
  (Tuesday, Wednesday, Thursday)
- ✓ **Extensions and Applications**: First-Order Modal Logic, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods
  (Thursday, Friday)

Concluding Remarks (1)

- Why study non-normal modal logics?
- Why study neighborhood semantics for modal logic?

Concluding Remarks (2)

- Proof theory
- Common knowledge/belief in neighborhood structures
- What is the "right" modal language for reasoning about neighborhood structures (hybrid logic, properties of neighborhoods, etc.)
- Model theory (uniform interpolation, Golblatt-Thomason Theorem, ...)
- Alternative semantics for non-normal modal logics
- ...

Thank you!!

`pacuit.org/esslli2014/nbhd`

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$.

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$. Then $\mathcal{X} - \{[\![\varphi]\!]_{\mathcal{M}}\}$ is a maximal f.i.p. in $\mathcal{M}$ that *is compatible with* φ.

$$\bigcap \mathcal{X}^{\varphi} = \bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$. Then $\mathcal{X} - \{[\![\varphi]\!]_{\mathcal{M}}\}$ is a maximal f.i.p. in $\mathcal{M}$ that *is compatible with* φ.

$$\bigcap \mathcal{X}^{\varphi} = \bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

2. $\mathcal{X}$ is incompatible with φ:

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff
for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$. Then $\mathcal{X} - \{[\![\varphi]\!]_{\mathcal{M}}\}$ is a maximal f.i.p. in $\mathcal{M}$ that *is compatible with* φ.

$$\bigcap \mathcal{X}^{\varphi} = \bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

2. $\mathcal{X}$ is incompatible with φ: there exists $X_1 \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$. Then $\mathcal{X}$ is a maximal f.i.p. in $\mathcal{M}$ with

$$\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

Back

Proof Sketch

$\mathcal{M}, w \models [+\varphi]B\psi$ iff $\mathcal{M}, w \models B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi$

for each maximally f.i.p. $\mathcal{X} \subseteq E^{+\varphi}(w)$, $\cap\mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}}$

1. $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{X}$. Then $\mathcal{X} - \{[\![\varphi]\!]_{\mathcal{M}}\}$ is a maximal f.i.p. in $\mathcal{M}$ that *is compatible with* φ.

$$\bigcap \mathcal{X}^{\varphi} = \bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

   $\mathcal{M}, w \models B^{+\varphi}[+\varphi]\psi$

2. $\mathcal{X}$ is incompatible with φ: there exists $X_1 \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$. Then $\mathcal{X}$ is a maximal f.i.p. in $\mathcal{M}$ with

$$\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}^{+\varphi}} = [\![[+\varphi]\psi]\!]_{\mathcal{M}}$$

   $\mathcal{M}, w \models B^{-\varphi}[+\varphi]\psi$

Back

Evidence Removal

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and $\varphi \in \mathcal{L}$. The model $\mathcal{M}^{-\varphi} = \langle W^{-\varphi}, E^{-\varphi}, V^{-\varphi} \rangle$ has $W^{-\varphi} = W$, $V^{-\varphi} = V$ and for all $w \in W$,

$$E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}.$$

Evidence Removal

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and $\varphi \in \mathcal{L}$. The model $\mathcal{M}^{-\varphi} = \langle W^{-\varphi}, E^{-\varphi}, V^{-\varphi} \rangle$ has $W^{-\varphi} = W$, $V^{-\varphi} = V$ and for all $w \in W$,

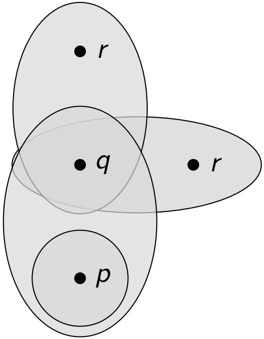
$$E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}.$$

$[-\varphi]\psi$: "after removing the evidence that φ, ψ is true"

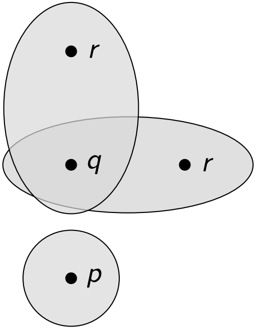
$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$

Fact. Evidence removal *extends* the language.

Fact. Evidence removal *extends* the language.



$\mathcal{E}_1$



$\mathcal{E}_2$

$[-p]\Box(p \vee q)$ is true in $\mathcal{M}_1$ but not in $\mathcal{M}_2$.

Compatible Evidence

$\Box_{\overline{\varphi}}\psi$: "$\psi$ is entailed by some admissible evidence *compatible* with each of $\overline{\varphi}$"

Compatible Evidence

$\Box_{\overline{\varphi}}\psi$: "$\psi$ is entailed by some admissible evidence *compatible* with each of $\overline{\varphi}$"

Let $\mathcal{M} = \langle W, E, V\rangle$ be an evidence model and $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ a finite sequence of formulas. We say that a subset $X \subseteq W$ is **compatible with** $\overline{\varphi}$ provided that, for each formula φ_i, $X \cap [\![\varphi_i]\!]_{\mathcal{M}} \neq \emptyset$.

$\mathcal{M}, w \models \Box_{\overline{\varphi}}\psi$ iff there is some $X \in E(w)$ compatible with $\overline{\varphi}$ where $X \subseteq [\![\psi]\!]_{\mathcal{M}}$

Compatible Evidence

$\Box_{\overline{\varphi}}\psi$: "$\psi$ is entailed by some admissible evidence *compatible* with each of $\overline{\varphi}$"

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model and $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ a finite sequence of formulas. We say that a subset $X \subseteq W$ is **compatible with** $\overline{\varphi}$ provided that, for each formula φ_i, $X \cap [\![\varphi_i]\!]_{\mathcal{M}} \neq \emptyset$.

$\mathcal{M}, w \models \Box_{\overline{\varphi}}\psi$ iff there is some $X \in E(w)$ compatible with $\overline{\varphi}$ where $X \subseteq [\![\psi]\!]_{\mathcal{M}}$

Recursion axiom: $[-\varphi]\Box\psi \leftrightarrow (\neg A\varphi \rightarrow \Box_{\neg\varphi}[-\varphi]\psi)$

Evidence Removal: Recursion Axioms

Langage $\mathcal{L}'$: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\alpha}_{\varphi}\psi \mid \Box^{\alpha}_{\varphi}\psi \mid A\varphi$

Evidence Removal: Recursion Axioms

Langage $\mathcal{L}'$: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\alpha}_{\varphi}\psi \mid \Box^{\alpha}_{\varphi}\psi \mid A\varphi$

- $\mathcal{M}, w \models \Box^{\alpha}_{\varphi}\psi$ iff there is $X \in E(w)$ compatible with $\overline{\varphi}$, α such that $X \cap [\![\alpha]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$.
- $\mathcal{M}, w \models B^{\alpha}_{\varphi}\psi$ iff for each maximal α-f.i.p. $\mathcal{X}$ compatible with $\overline{\varphi}$, $\bigcap \mathcal{X}^{\alpha} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

Back

Evidence Removal: Recursion Axioms

Langage $\mathcal{L}'$: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\alpha}_{\varphi}\psi \mid \Box^{\alpha}_{\varphi}\psi \mid A\varphi$

- $\mathcal{M}, w \models \Box^{\alpha}_{\varphi}\psi$ iff there is $X \in E(w)$ compatible with $\overline{\varphi}$, α such that $X \cap [\![\alpha]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$.
- $\mathcal{M}, w \models B^{\alpha}_{\varphi}\psi$ iff for each maximal α-f.i.p. $\mathcal{X}$ compatible with $\overline{\varphi}$, $\bigcap \mathcal{X}^{\alpha} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

Back

$$
\begin{array}{lcll}
[-\varphi]p & \leftrightarrow & (\neg A\varphi \rightarrow p) & (p \in \mathsf{At}) \\
[-\varphi](\psi \wedge \chi) & \leftrightarrow & ([-\varphi]\psi \wedge [-\varphi]\chi) & \\
[-\varphi]\neg\psi & \leftrightarrow & (\neg A\varphi \rightarrow \neg[-\varphi]\psi) & \\
[-\varphi]\Box^{\alpha}_{\psi}\chi & \leftrightarrow & (\neg A\varphi \rightarrow \Box^{[-\varphi]\alpha}_{[-\varphi]\overline{\psi}, \neg\varphi}[-\varphi]\chi) & \\
[-\varphi]B^{\alpha}_{\psi}\chi & \leftrightarrow & (\neg A\varphi \rightarrow B^{[-\varphi]\alpha}_{[-\varphi]\overline{\psi}, \neg\varphi}[-\varphi]\chi) & \\
[-\varphi]A\psi & \leftrightarrow & (\neg A\varphi \rightarrow A[-\varphi]\psi) &
\end{array}
$$

Evidence Removal: Recursion Axioms

Langage $\mathcal{L}'$: $p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\alpha}_{\varphi}\psi \mid \Box^{\alpha}_{\varphi}\psi \mid A\varphi$

- $\mathcal{M}, w \models \Box^{\alpha}_{\varphi}\psi$ iff there is $X \in E(w)$ compatible with $\overline{\varphi}$, α such that $X \cap [\![\alpha]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$.
- $\mathcal{M}, w \models B^{\alpha}_{\varphi}\psi$ iff for each maximal α-f.i.p. $\mathcal{X}$ compatible with $\overline{\varphi}$, $\bigcap \mathcal{X}^{\alpha} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

Back

$$
\begin{array}{lcl}
[-\varphi]p & \leftrightarrow & (\neg A\varphi \rightarrow p) \quad (p \in \mathsf{At}) \\
[-\varphi](\psi \wedge \chi) & \leftrightarrow & ([-\varphi]\psi \wedge [-\varphi]\chi) \\
[-\varphi]\neg\psi & \leftrightarrow & (\neg A\varphi \rightarrow \neg[-\varphi]\psi) \\
[-\varphi]\Box^{\alpha}_{\psi}\chi & \leftrightarrow & (\neg A\varphi \rightarrow \Box^{[-\varphi]\alpha}_{[-\varphi]\overline{\psi},\neg\varphi}[-\varphi]\chi) \\
[-\varphi]B^{\alpha}_{\psi}\chi & \leftrightarrow & (\neg A\varphi \rightarrow B^{[-\varphi]\alpha}_{[-\varphi]\overline{\psi},\neg\varphi}[-\varphi]\chi) \\
[-\varphi]A\psi & \leftrightarrow & (\neg A\varphi \rightarrow A[-\varphi]\psi)
\end{array}
$$

Evidence Combination (1)

One-round evidence combination:

$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$

Evidence Combination (1)

One-round evidence combination:

$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$

Is $(E(\varphi \wedge \psi) \wedge \Box\varphi \wedge \Box\psi) \rightarrow [\#_1]\Box(\varphi \wedge \psi)$ valid?

Evidence Combination (1)

One-round evidence combination:

$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$

Is $(E(\varphi \wedge \psi) \wedge \Box\varphi \wedge \Box\psi) \rightarrow [\#_1]\Box(\varphi \wedge \psi)$ valid? No!

Evidence Combination (1)

One-round evidence combination:

$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$

Is $(E(\varphi \wedge \psi) \wedge \Box\varphi \wedge \Box\psi) \rightarrow [\#_1]\Box(\varphi \wedge \psi)$ valid? No!

Evidence That Operator $\mathcal{M}, w \models \boxplus\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E(w)$

Evidence Combination (1)

One-round evidence combination:

$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$

Is $(E(\varphi \wedge \psi) \wedge \Box\varphi \wedge \Box\psi) \rightarrow [\#_1]\Box(\varphi \wedge \psi)$ valid? No!

Evidence That Operator $\mathcal{M}, w \models \boxplus\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in E(w)$

Fact. $(E(\varphi \wedge \psi) \wedge \boxplus\varphi \wedge \boxplus\psi) \rightarrow [\#_1] \boxplus (\varphi \wedge \psi)$. is valid.

Evidence Combination (2)

Evidence combination Let $\mathcal{M} = \langle W, E, V\rangle$ be an evidence model. The model $\mathcal{M}^{\#} = \langle W^{\#}, E^{\#}, V^{\#}\rangle$ has $W^{\#} = W$, $V^{\#} = V$ and for all $w \in W$, $E^{\#}(w)$ is the smallest set closed under consistent intersection and containing $E(w)$.

Evidence Combination (2)

Evidence combination Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model. The model $\mathcal{M}^{\#} = \langle W^{\#}, E^{\#}, V^{\#} \rangle$ has $W^{\#} = W$, $V^{\#} = V$ and for all $w \in W$, $E^{\#}(w)$ is the smallest set closed under consistent intersection and containing $E(w)$.

$[\#]\varphi$: "φ is true after the agent (consistently) combines (*all* of) her evidence"

$\mathcal{M}, w \models [\#]\varphi$ iff $\mathcal{M}^{\#}, w \models \varphi$.

Evidence Combination: Some Properties

1. $\Box[\#]\varphi \rightarrow [\#]\Box\varphi$ (combining evidence does not remove any of the original evidence)

Evidence Combination: Some Properties

1. $\Box[\#]\varphi \rightarrow [\#]\Box\varphi$ (combining evidence does not remove any of the original evidence)

2. $B[\#]\varphi \leftrightarrow [\#]B\varphi$ (beliefs are immune to evidence combination)

Evidence Combination: Some Properties

1. $\Box[\#]\varphi \to [\#]\Box\varphi$ (combining evidence does not remove any of the original evidence)

2. $B[\#]\varphi \leftrightarrow [\#]B\varphi$ (beliefs are immune to evidence combination)

3. $B\varphi \to [\#]\Box\varphi$ (beliefs are explicitly supported after consistently combining evidence)

Evidence Combination: Some Properties

1. $\Box[\#]\varphi \to [\#]\Box\varphi$ (combining evidence does not remove any of the original evidence)

2. $B[\#]\varphi \leftrightarrow [\#]B\varphi$ (beliefs are immune to evidence combination)

3. $B\varphi \to [\#]\Box\varphi$ (beliefs are explicitly supported after consistently combining evidence)

4. For factual φ, $B\varphi \to \neg[\#]\Box\neg\varphi$ (if an agent believes φ then the agent cannot combine her evidence so that there is evidence for $\neg\varphi$)

Dynamically Relating Beliefs with Evidence

$B\varphi \to \Box\varphi$ vs. $B\varphi \to [\#]\Box\varphi$

Dynamically Relating Beliefs with Evidence

$B\varphi \to \Box\varphi$ vs. $B\varphi \to [\#]\Box\varphi$

$\Box\varphi \to B\varphi$ vs. $\Box\neg\varphi \to \neg B\varphi$ vs. $B\varphi \to \neg[\#]\Box\neg\varphi$

Dynamically Relating Beliefs with Evidence

$B\varphi \to \Box\varphi$ vs. $B\varphi \to [\#]\Box\varphi$

$\Box\varphi \to B\varphi$ vs. $\Box\neg\varphi \to \neg B\varphi$ vs. $B\varphi \to \neg[\#]\Box\neg\varphi$

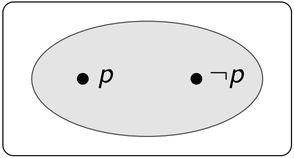
Can we dynamically characterize beliefs in terms of evidence? Are $\neg[\#]\Box\neg\varphi \to B\varphi$ and $[\#]\Box\varphi \to B\varphi$ valid?

Dynamically Relating Beliefs with Evidence

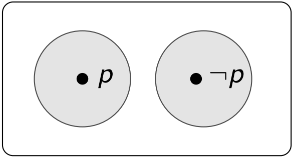
$B\varphi \to \Box\varphi$ vs. $B\varphi \to [\#]\Box\varphi$

$\Box\varphi \to B\varphi$ vs. $\Box\neg\varphi \to \neg B\varphi$ vs. $B\varphi \to \neg[\#]\Box\neg\varphi$

Can we dynamically characterize beliefs in terms of evidence? Are $\neg[\#]\Box\neg\varphi \to B\varphi$ and $[\#]\Box\varphi \to B\varphi$ valid? No!



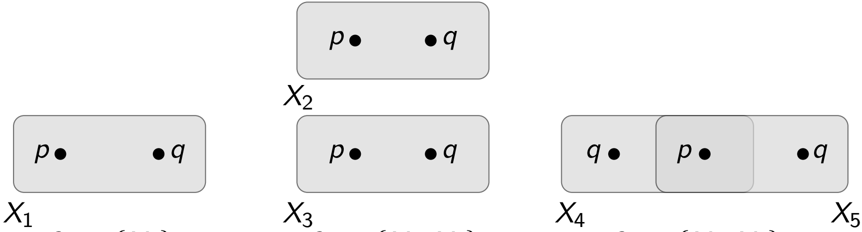
$\mathcal{E}_1$



$\mathcal{E}_2$

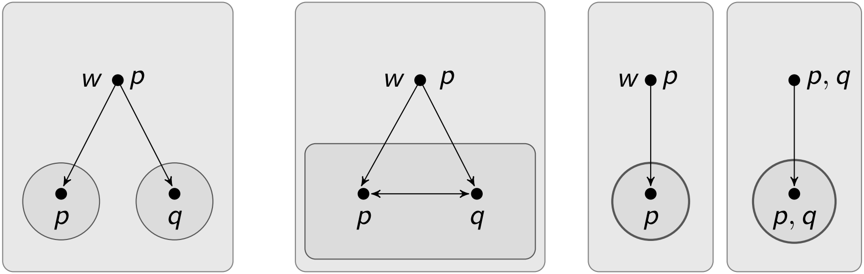
Back

Different Evidential Situations



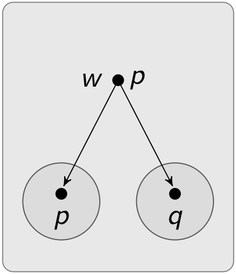
$\mathcal{E}_1 = \{X_1\}$ $\mathcal{E}_2 = \{X_2, X_3\}$ $\mathcal{E}_3 = \{X_4, X_5\}$

Plausibility Models $\hookrightarrow$ Evidence Models (2)

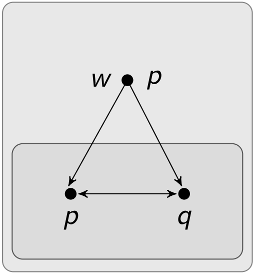


$EV(\mathcal{M}_1)$ $EV(\mathcal{M}_2)$ $EV(\mathcal{M}_3)$

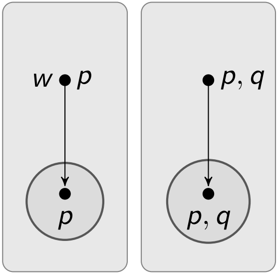
Plausibility Models $\hookrightarrow$ Evidence Models (2)



$EV(\mathcal{M}_1)$



$EV(\mathcal{M}_2)$



$EV(\mathcal{M}_3)$

Fact. The evidence sets of generated models $EV(\mathcal{M})$ are closed under intersections.

Plausibility Models $\hookrightarrow$ Evidence Models (3)

P-**translation**: $(\cdot)^P : \mathcal{L}_E \to \mathcal{L}_{\preceq}$ is defined as follows:

Plausibility Models $\hookrightarrow$ Evidence Models (3)

P-**translation**: $(\cdot)^P : \mathcal{L}_E \to \mathcal{L}_{\preceq}$ is defined as follows:

- $p^P = p$, $(\neg\varphi)^P = \neg\varphi^P$, $(\varphi \wedge \psi)^P = \varphi^P \wedge \psi^P$,
- $(\Box\varphi)^P = E[\preceq]\varphi^P$, $(A\varphi)^P = A\varphi^P$,
- $(\Box^\varphi\psi)^P = E\langle\preceq\rangle(\varphi^P \wedge [\preceq](\varphi^P \to \psi^P))$,
- $(\Box^{\varphi}_{\vec{\gamma}}\psi)^P = E(\bigwedge_i\langle\preceq\rangle\gamma_i^P \wedge \langle\preceq\rangle(\varphi^P \wedge [\preceq](\varphi^P \to \psi^P)))$,
- $(B^\varphi\psi)^P = A(\varphi^P \to (\langle\preceq\rangle\varphi^P \wedge [\preceq](\varphi^P \to \psi^P)))$,
- $(B^{\varphi,\alpha}\psi)^P = A(([\preceq]\alpha^P \wedge [\preceq]\langle\preceq\rangle\varphi^P) \to \langle\preceq\rangle(\varphi^P \wedge [\preceq](\varphi^P \to \psi^P)))$, and
- $(B^{\varphi}_{\vec{\gamma}}\psi)^P = A((\varphi^P \wedge \bigwedge_i\langle\preceq\rangle\gamma_i^P) \to (\langle\preceq\rangle(\varphi^P \wedge \bigwedge_i\langle\preceq\rangle\gamma_i^P) \wedge [\preceq]((\varphi^P \wedge \bigwedge_i\langle\preceq\rangle\gamma_i^P) \to \psi^P)))$.

Plausibility Models $\hookrightarrow$ Evidence Models (3)

P-**translation**: $(\cdot)^P : \mathcal{L}_E \to \mathcal{L}_{\preceq}$ is defined as follows:

- $p^P = p$, $(\neg\varphi)^P = \neg\varphi^P$, $(\varphi \wedge \psi)^P = \varphi^P \wedge \psi^P$,
- $(\Box\varphi)^P = E[\preceq]\varphi^P$, $(A\varphi)^P = A\varphi^P$,
- $(\Box^\varphi \psi)^P = E\langle\preceq\rangle(\varphi^P \wedge [\preceq](\varphi^P \to \psi^P))$,
- $(\Box^{\varphi}_{\gamma}\psi)^P = E(\bigwedge_i \langle\preceq\rangle\gamma_i^P \wedge \langle\preceq\rangle(\varphi^P \wedge [\preceq](\varphi^P \to \psi^P)))$,
- $(B^\varphi \psi)^P = A(\varphi^P \to (\langle\preceq\rangle\varphi^P \wedge [\preceq](\varphi^P \to \psi^P)))$,
- $(B^{\varphi,\alpha}\psi)^P = A(([\preceq]\alpha^P \wedge [\preceq]\langle\preceq\rangle\varphi^P) \to \langle\preceq\rangle(\varphi^P \wedge [\preceq](\varphi^P \to \psi^P)))$, and
- $(B^{\varphi}_{\gamma}\psi)^P = A((\varphi^P \wedge \bigwedge_i\langle\preceq\rangle\gamma_i^P) \to (\langle\preceq\rangle(\varphi^P \wedge \bigwedge_i\langle\preceq\rangle\gamma_i^P) \wedge [\preceq]((\varphi^P \wedge \bigwedge_i\langle\preceq\rangle\gamma_i^P) \to \psi^P)))$.

Lemma. Let $\mathcal{M} = \langle W, \preceq, V\rangle$ be a plausibility model. For any $\varphi \in \mathcal{L}_E$ and world $w \in W$,

$$\mathcal{M}, w \models \varphi^P \text{ iff } EV(\mathcal{M}), w \models \varphi$$

Robust Beliefs

$[\preceq]\varphi$: "the agent **robustly believes** that φ"

Fact. $\mathcal{M}, w \models [\preceq]\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \psi$.

the agent robustly believew φ iff she continues to believe φ given any true "evidence"

Robust Beliefs with Incomparable Worlds

$\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have

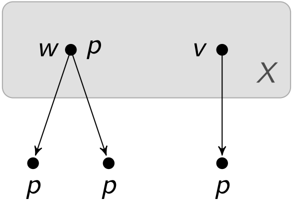
$$Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

Robust Beliefs with Incomparable Worlds

$\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have

$$Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

Fact. On arbitrary pre-orders, $B^r\varphi$ is not equivalent to $[\preceq]\varphi$.

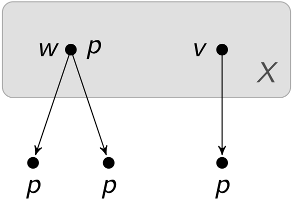


Robust Beliefs with Incomparable Worlds

$\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have

$$Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

Fact. On arbitrary pre-orders, $B^r\varphi$ is not equivalent to $[\preceq]\varphi$.



Fact. On arbitrary pre-orders $\mathcal{M}$, robust belief $B^r\varphi$ holds at w iff φ is true at all worlds $v \not\succ w$ that are not strictly less plausible than w ($[\not\succ]\varphi$).

Reliable Evidence

Suppose that $\mathcal{M} = \langle W, E, V \rangle$ is an evidence model.

Reliable Evidence: $E^C(w) = \{X \in E(w) \mid w \in X\}$
$\Box^C\varphi$: "the agent's reliable evidence entails φ"

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

Reliable Evidence

Suppose that $\mathcal{M} = \langle W, E, V\rangle$ is an evidence model.

Reliable Evidence: $E^C(w) = \{X \in E(w) \mid w \in X\}$
$\Box^C\varphi$: "the agent's reliable evidence entails φ"

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

Fact. Let $\mathcal{M}$ be a uniform evidence model, φ a ground formula, then:

$$\mathcal{M}, w \models [\preceq]\varphi \text{ iff } ORD(\mathcal{M}), w \models \Box^C\varphi$$

Reliable Evidence

Suppose that $\mathcal{M} = \langle W, E, V \rangle$ is an evidence model.

Reliable Evidence: $E^C(w) = \{X \in E(w) \mid w \in X\}$
$\Box^C\varphi$: "the agent's reliable evidence entails φ"

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

Fact. Let $\mathcal{M}$ be a uniform evidence model, φ a ground formula, then:

$$\mathcal{M}, w \models [\preceq]\varphi \text{ iff } ORD(\mathcal{M}), w \models \Box^C\varphi$$

But this is not robust belief on plausibility orders that are not connected!

Reliable and Unreliable Evidence

Suppose that $\mathcal{M} = \langle W, E, V \rangle$ is an evidence model.

Unreliable Evidence: $E^U(w) = \{X \in E(w) \mid w \notin X\}$.
$\Box^U$: "φ follows from the unreliable evidence at w"

$\mathcal{M}, w \models \Box^U\varphi$ iff for all $v \in \bigcup E^U(w)$, $\mathcal{M}, v \models \varphi$

Reliable and Unreliable Evidence

Suppose that $\mathcal{M} = \langle W, E, V \rangle$ is an evidence model.

Unreliable Evidence: $E^U(w) = \{X \in E(w) \mid w \notin X\}$.
$\Box^U$: "φ follows from the unreliable evidence at w"

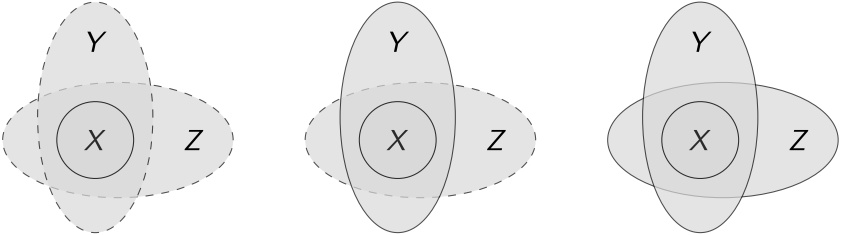
$\mathcal{M}, w \models \Box^U \varphi$ iff for all $v \in \bigcup E^U(w)$, $\mathcal{M}, v \models \varphi$

Fact. Let $\mathcal{M}$ be a uniform evidence model, then for all factual formulas φ:

$$\mathcal{M}, w \models \Box^C \varphi \wedge \Box^U \varphi \text{ iff } ORD(\mathcal{M}), w \models B^r \varphi$$

▸ Back

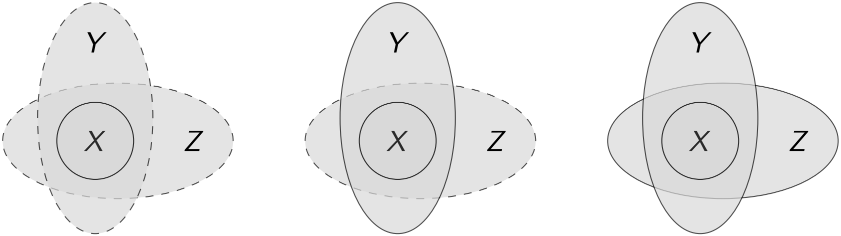
Simple Observations about Expressivity



$\mathcal{E}_1 = \{X\}$ $\mathcal{E}_2 = \{X, Y\}$ $\mathcal{E}_3 = \{X, Y, Z\}$

Fact. Our language is invariant under adding supersets of evidence already contained in an evidence state.

Simple Observations about Expressivity



$\mathcal{E}_1 = \{X\}$ $\mathcal{E}_2 = \{X, Y\}$ $\mathcal{E}_3 = \{X, Y, Z\}$

Fact. Our language is invariant under adding supersets of evidence already contained in an evidence state.

Simple Observations about Expressivity

Fact. The belief modality is not definable in terms of evidence modalities only.

Simple Observations about Expressivity

Definition. Let $\mathcal{M}_1 = \langle W_1, E_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, E_2, V_2 \rangle$ be two evidence models. A non-empty relation $Z \subseteq W_1 \times W_2$ is a (monotonic) **bisimulation** if, for all worlds $w_1 \in W_1$ and $w_2 \in W_2$:

(Prop) If $w_1 Z w_2$, then for all $p \in \mathrm{At}$, $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$.

(Forth) If $w_1 Z w_2$, then for each $X \in E_1^{sup}(w_1)$ there is a $X' \in E_2^{sup}(w_2)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

(Back) If $w_1 Z w_2$, then for each $X \in E_2^{sup}(w_2)$ there is a $X' \in E_1^{sup}(w_1)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

Simple Observations about Expressivity

Definition. Let $\mathcal{M}_1 = \langle W_1, E_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, E_2, V_2 \rangle$ be two evidence models. A non-empty relation $Z \subseteq W_1 \times W_2$ is a (monotonic) **bisimulation** if, for all worlds $w_1 \in W_1$ and $w_2 \in W_2$:

(Prop) If $w_1 Z w_2$, then for all $p \in \mathsf{At}$, $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$.

(Forth) If $w_1 Z w_2$, then for each $X \in E_1^{sup}(w_1)$ there is a $X' \in E_2^{sup}(w_2)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

(Back) If $w_1 Z w_2$, then for each $X \in E_2^{sup}(w_2)$ there is a $X' \in E_1^{sup}(w_1)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

Simple Observations about Expressivity

Definition. Let $\mathcal{M}_1 = \langle W_1, E_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, E_2, V_2 \rangle$ be two evidence models. A non-empty relation $Z \subseteq W_1 \times W_2$ is a (monotonic) **bisimulation** if, for all worlds $w_1 \in W_1$ and $w_2 \in W_2$:

(Prop) If $w_1 Z w_2$, then for all $p \in \mathsf{At}$, $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$.

(Forth) If $w_1 Z w_2$, then for each $X \in E_1^{sup}(w_1)$ there is a $X' \in E_2^{sup}(w_2)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

(Back) If $w_1 Z w_2$, then for each $X \in E_2^{sup}(w_2)$ there is a $X' \in E_1^{sup}(w_1)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

Simple Observations about Expressivity

Definition. Let $\mathcal{M}_1 = \langle W_1, E_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, E_2, V_2 \rangle$ be two evidence models. A non-empty relation $Z \subseteq W_1 \times W_2$ is a (monotonic) **bisimulation** if, for all worlds $w_1 \in W_1$ and $w_2 \in W_2$:

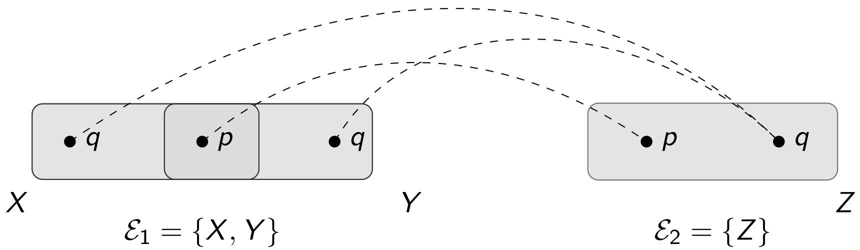
(Prop) If $w_1 Z w_2$, then for all $p \in \mathrm{At}$, $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$.

(Forth) If $w_1 Z w_2$, then for each $X \in E_1^{sup}(w_1)$ there is a $X' \in E_2^{sup}(w_2)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

(Back) If $w_1 Z w_2$, then for each $X \in E_2^{sup}(w_2)$ there is a $X' \in E_1^{sup}(w_1)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

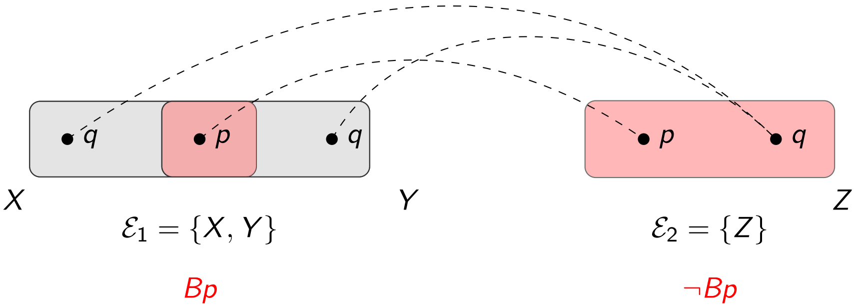
Simple Observations about Expressivity

Fact. The belief modality is not definable in terms of evidence modalities only.



Simple Observations about Expressivity

Fact. The belief modality is not definable in terms of evidence modalities only.



Conditional Beliefs: Valid Principles

1. $B^{\varphi}\varphi$
2. $B^{\varphi}\psi \rightarrow B^{\varphi}(\psi \vee \chi)$
3. $(B^{\varphi}\psi_1 \wedge B^{\varphi}\psi_2) \rightarrow B^{\varphi}(\psi_1 \wedge \psi_2)$
4. $(B^{\varphi_1}\psi \wedge B^{\varphi_2}\psi) \rightarrow B^{\varphi_1 \vee \varphi_2}\psi$
5. $(B^{\varphi}\psi \wedge B^{\psi}\varphi) \rightarrow (B^{\varphi}\chi \leftrightarrow B^{\psi}\chi)$

Back

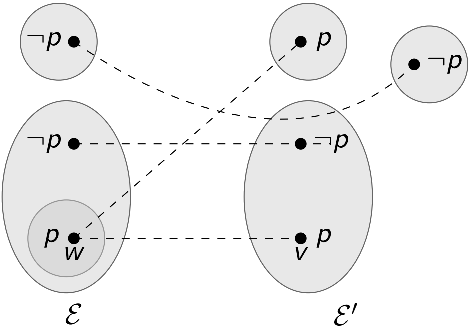
Monotonic Bisimulation

Let $\mathcal{M}_1 = \langle W_1, E_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, E_2, V_2 \rangle$ be two evidence models. A non-empty relation $Z \subseteq W_1 \times W_2$ is a **bisimulation** if, for all worlds $w_1 \in W_1$ and $w_2 \in W_2$:

(Prop) If $w_1 Z w_2$, then for all $p \in \mathsf{At}$, $p \in V_1(w_1)$ iff $p \in V_2(w_2)$.

(Forth) If $w_1 Z w_2$, then for each $X \in E_1^{sup}(w_1)$ there is a $X' \in E_2^{sup}(w_2)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.

(Back) If $w_1 Z w_2$, then for each $X \in E_2^{sup}(w_2)$ there is a $X' \in E_1^{sup}(w_1)$ such that for all $x' \in X'$, there is a $x \in X$ such that xZx'.



Back

- Proof theory
- Common knowledge/belief in neighborhood structures
- What is the "right" modal language for reasoning about neighborhood structures (hybrid logic, properties of neighborhoods, etc.)
- Model theory (uniform interpolation, Golblatt-Thomason Theorem, ...)
- Alternative semantics for non-normal modal logics
- ...

Course Plan

- ✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Subset Spaces, Neighborhood Structures, Motivating Non-Normal Modal Logics/Neighborhood Semantics
  (Monday, Tuesday)
- ✓ **Core Theory**: Completeness, Decidability, Complexity, Incompleteness, Relationship with Other Semantics for Modal Logic, Model Theory
  (Tuesday, Wednesday, Thursday)
- ✓ **Extensions and Applications**: First-Order Modal Logic, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods
  (Thursday, Friday)

Thank you!!

pacuit.org/esslli2014/nbhd