Reasoning, Games, Action and Rationality

Lecture 5

ESSLLI'08, Hamburg

Eric Pacuit Olivier Roy

Stanford University University of Groningen

ai.stanford.edu/~epacuit philos.rug.nl/~oroy

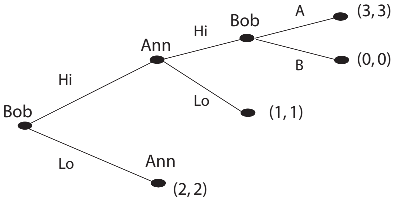
August 15, 2008

Plan for Today

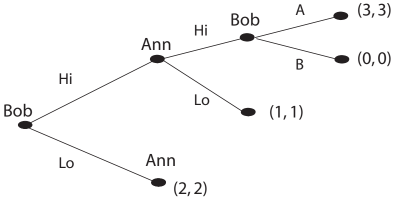
- Rationality and belief revision in extensive games.
- General discussion: recap, logical characterizations of solution concepts.

Extensive games

- An extensive game is a game where the players move *sequentially*.



Extensive games

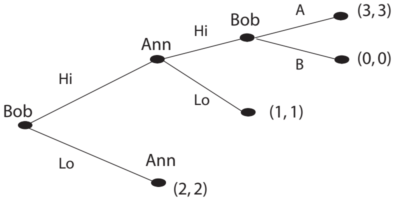


Definition

Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I} \rangle$ such that:

-

Extensive games

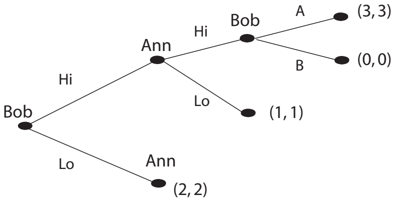


Definition

Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I}\rangle$ such that:

- I is a finite set of players.

Extensive games

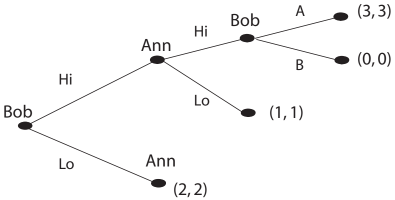


Definition

Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I} \rangle$ such that:

- T is finite set of finite sequences of *actions*, called *histories*, such that:
  - The empty sequence $\emptyset$, the *root* of the tree, is in T.
  - T is prefix-closed: if $(a_1, \ldots, a_n, a_{n+1}) \in T$ then $(a_1, \ldots, a_n) \in T$.

Extensive games

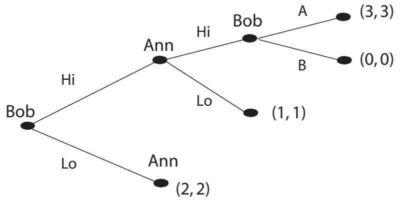


Definition

Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I}\rangle$ such that:

- Given a history $h = (a_1, \ldots, a_n)$, the history $(a_1, \ldots, a_n, a)$, h followed by the action a, is denoted ha.

Extensive games

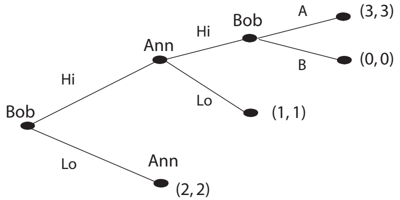


Definition

Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I} \rangle$ such that:

- A history h is *terminal in* T whenever it is the sub-sequence of no other history $h' \in T$. Z denotes the set of terminal histories in T.

Extensive games

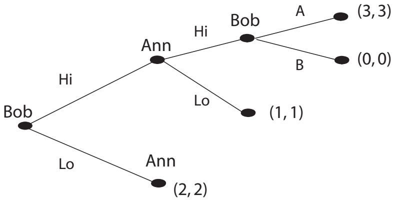


Definition

Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I} \rangle$ such that:

- $\tau : (T - Z) \longrightarrow I$ is a *turn function* which assigns to every non-terminal history h the player whose turn it is to play at h.

Extensive games

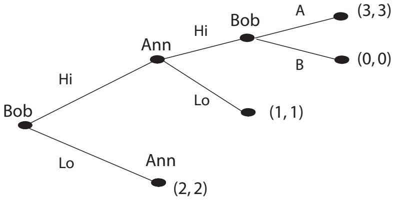


Definition

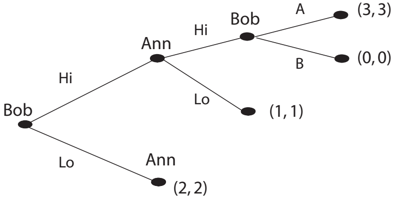
Extensive form games - basic definition A *game in extensive form* $\mathcal{T}$ is a tuple $\langle I, T, \tau, \{v_i\}_{i \in I} \rangle$ such that:

- $v_i : Z \longrightarrow \mathbb{R}$ is a *payoff function* for player i which assigns i's payoff at each terminal history.

Extensive form games - strategies and reachability



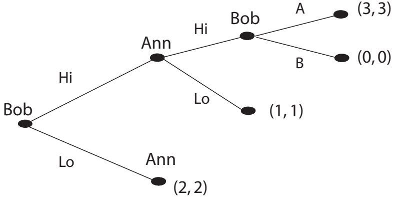
Extensive form games - strategies and reachability



Definition

A *strategy* s_i for agent i is a function that gives, for every history h such that $i = \tau(h)$, an action $a \in A(h)$. S_i is the set of strategies for agent i.

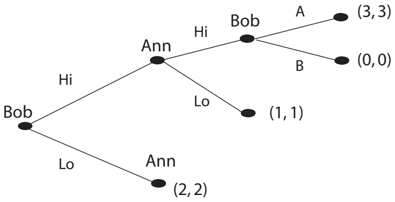
Extensive form games - strategies and reachability



Definition

A *strategy profile* $\sigma \in \Pi_{i \in I} S_i$ is a combination of strategies, one for each agent, and $\sigma(h)$ is a shorthand for the action a such that $a = \sigma_i(h)$ for the agent i whose turn it is at h.

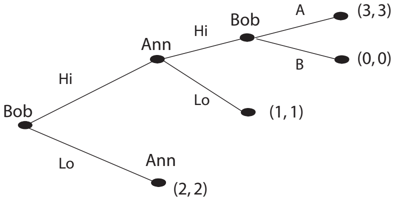
Extensive form games - strategies and reachability



Definition

A history h' is *reachable* or *not excluded* by the profile σ from h if $h' = (h, \sigma(h), \sigma(h, \sigma(h)), \ldots)$ for some finite number of application of σ.

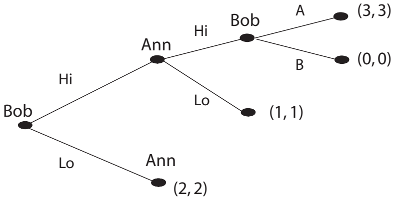
Extensive form games - strategies and reachability



Definition

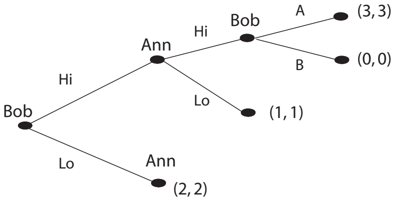
A history h' is *reachable* from h by playing the strategy s_i, for $i = \tau(h)$ if there is a combination of strategies for the other players $\sigma_{j \neq i}$ such that h' is reachable from h by the profile $(s_i, \sigma_{j \neq i})$. We denote $v_i^h(\sigma)$ the value of $util_i$ at the unique terminal history reachable from h by the profile σ.

Extensive form games - strategies and reachability



Definition

Extensive form games - strategies and reachability



Definition

- Perfect information games. No random moves.

Models of information in extensive games

- Type spaces and epistemic plausibility models are directly usable.

Models of information in extensive games

- Type spaces and epistemic plausibility models are directly usable. For now we only look at hard information, however.

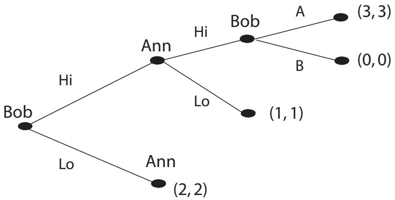
Models of information in extensive games

- Type spaces and epistemic plausibility models are directly usable. For now we only look at hard information, however.

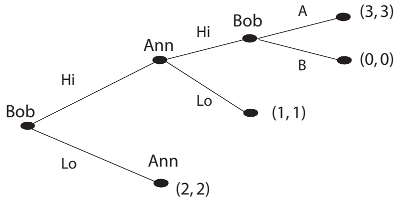
Definition

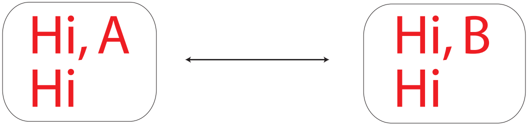
Given an extensive game $\mathcal{T}$, an *epistemic model* is a tuple $\langle W, f, \{\sim_i\}_{i \in I} \rangle$, where W is a finite set of states, f is a function which assigns to each state a strategy profile σ of $\mathcal{T}$, and $\sim_i$ is an equivalence relation such that if $w \sim_i w'$ then $\sigma_i(w) = \sigma_i(w')$.

Epistemic models: simple example



Epistemic models: simple example





Rationality in epistemic models

- Two loci of rationality:

Rationality in epistemic models

- Two loci of rationality:
  1. Rationality at *a state*:

Definition

A player is *rational* at a state w whenever he is rational at all histories h given that the game playing situation is w.

Rationality in epistemic models

- Two loci of rationality:
  1. Rationality at *a state*:

Definition

A player is *rational* at a state w whenever he is rational at all histories h given that the game playing situation is w.

  2. Rationality at a state-history pair:

Rationality in epistemic models

- Two loci of rationality:
    1. Rationality at *a state*:

Definition

A player is *rational* at a state w whenever he is rational at all histories h given that the game playing situation is w.

2. Rationality at a state-history pair:
    - Many proposals.

Rationality in epistemic models: Aumann 1994

- Rationality at a state-history pair:

R.J. Aumann. *Backward induction and common knowledge of rationality*. *Games and Economic Behavior*, 8:121-133, 1994.

Rationality in epistemic models: Aumann 1994

- Rationality at a state-history pair:

Definition

Player i is *rational$_A$* at history h given that the game playing situation is state w whenever for all strategy $s_i' \neq \sigma_i(w)$, there is a $w' \sim_i w$ such that $v_i^h(f(w')) \geq v_i^h(s_i', f_{-i}(w'))$.

R.J. Aumann. *Backward induction and common knowledge of rationality*. *Games and Economic Behavior*, 8:121-133, 1994.

Rationality in epistemic models: Aumann 1994

- Rationality at a state-history pair:

Definition

Player i is *rational*$_A$ at history h given that the game playing situation is state w whenever for all strategy $s_i' \neq \sigma_i(w)$, there is a $w' \sim_i w$ such that $v_i^h(f(w')) \geq v_i^h(s_i', f_{-i}(w'))$.

- Intuition: a strategy is rational$_A$ at a given history and state if it is not strictly dominated.

R.J. Aumann. *Backward induction and common knowledge of rationality. Games and Economic Behavior*, 8:121-133, 1994.

Common knowledge of rationality$_A$ and backward induction

R.J. Aumann. *Backward induction and common knowledge of rationality*. *Games and Economic Behavior*, 8:121-133, 1994.

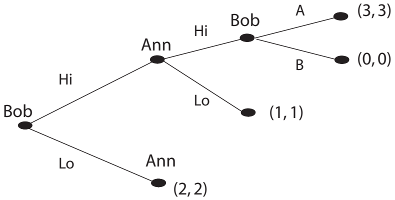
Common knowledge of rationality$_A$ and backward induction

Theorem

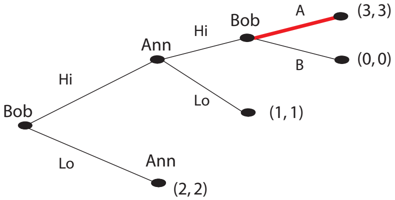
Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

R.J. Aumann. *Backward induction and common knowledge of rationality*. *Games and Economic Behavior*, 8:121-133, 1994.

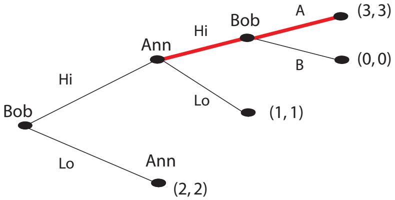
Backward induction and sub-game perfect equilibrium



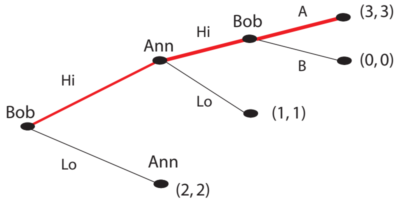
Backward induction and sub-game perfect equilibrium



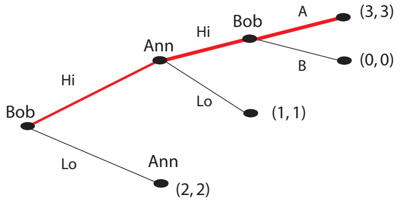
Backward induction and sub-game perfect equilibrium



Backward induction and sub-game perfect equilibrium



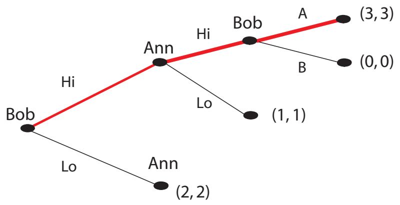
Backward induction and sub-game perfect equilibrium



- In extensive games (* with no ties), the backward induction algorithm computes the unique *sub-game perfect equilibrium*.

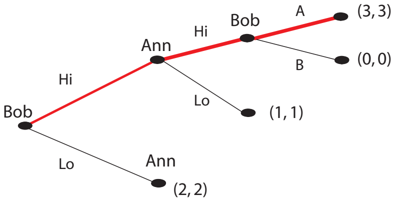
R. Selten. *Reexamination of the perfectness concept for equilibrium points in extensive games*. *International Journal of Game Theory*, 4 (1):25-55, 1975.

Backward induction and sub-game perfect equilibrium



	Hi, A	Hi, B	Lo, A	Lo, B
Hi	3, 3	0, 0	2, 2	2, 2
Lo	1, 1	1, 1	2, 2	2, 2

Backward induction and sub-game perfect equilibrium



	Hi, A	Hi, B	Lo, A	Lo, B
Hi	3, 3	0, 0	2, 2	2, 2
Lo	1, 1	1, 1	2, 2	2, 2

- Some Nash equilibria of the strategic form or this extensive game seem to involve incredible threats.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

Theorem

Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

R.J. Aumann. *Backward induction and common knowledge of rationality. Games and Economic Behavior*, 8:121-133, 1994.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

Theorem

Given an extensive games of perfect information $\mathcal{T}$ $(^*)$ and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

- Confirms intuitions about non-sub-game perfect equilibria.

R.J. Aumann. *Backward induction and common knowledge of rationality*. *Games and Economic Behavior*, 8:121-133, 1994.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

Theorem

Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

- Confirms intuitions about non-sub-game perfect equilibria. (Observe the contrast with strategic games. The sequential structure *is* important.)

R.J. Aumann. *Backward induction and common knowledge of rationality. Games and Economic Behavior*, 8:121-133, 1994.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

Theorem

Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

- Confirms intuitions about non-sub-game perfect equilibria. (Observe the contrast with strategic games. The sequential structure *is* important.)
- Is this all that there is to be said?

R.J. Aumann. *Backward induction and common knowledge of rationality. Games and Economic Behavior*, 8:121-133, 1994.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

Theorem

Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

- Confirms intuitions about non-sub-game perfect equilibria. (Observe the contrast with strategic games. The sequential structure *is* important.)
- Is this all that there is to be said?
  - Knowledge of rationality at off-BI histories?

R.J. Aumann. *Backward induction and common knowledge of rationality*. *Games and Economic Behavior*, 8:121-133, 1994.

Backward induction and sub-game perfect equilibrium

- Some Nash equilibria of the strategic form or an extensive game can involve incredible threats.

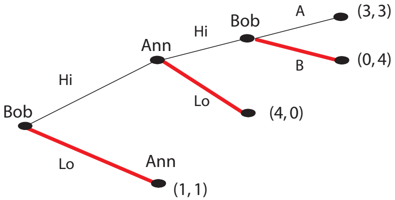
Theorem

Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

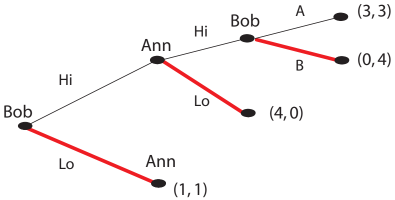
- Confirms intuitions about non-sub-game perfect equilibria. (Observe the contrast with strategic games. The sequential structure *is* important.)
- Is this all that there is to be said?
  - Knowledge of rationality at off-BI histories?
  - "Paradoxes" of backward induction.

R.J. Aumann. *Backward induction and common knowledge of rationality. Games and Economic Behavior*, 8:121-133, 1994.

"Paradoxes" of backward induction.



“Paradoxes” of backward induction.



- Common knowledge of rationality implies that Bob goes down at the first node.

Alternative notion of rationality & belief revision.

- Players might *revise* their beliefs in case someone does something unexpected.

R. C. Stalnaker. *Knowledge, belief and counterfactual reasoning in games. Economics and Philosophy*, 12:133-163, 1996.

Alternative notion of rationality & belief revision.

- Players might *revise* their beliefs in case someone does something unexpected.

R. C. Stalnaker. *Knowledge, belief and counterfactual reasoning in games. Economics and Philosophy*, 12:133-163, 1996.

J. Y. Halpern. *Substantive rationality and backward induction. Games and Economic Behavior*, 37(2):425-435, 2001.

Belief revision in extensive games.

Definition

(Halpern, 2001) Given an extensive game $\mathcal{T}$, an *epistemic-doxastic model* is a tuple $\langle W, f, \{\sim_i, \rho_i\}_{i \in I}\rangle$, where $\langle W, f, \{\sim_i\}_{i \in I}\rangle$ is an epistemic model for $\mathcal{T}$ and $\rho_i : W \times H \longrightarrow W$ is a *selection function* such that, for all state w, history h and agent i :

Belief revision in extensive games.

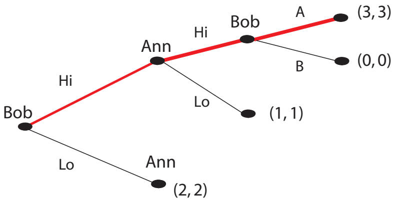
Definition

(Halpern, 2001) Given an extensive game $\mathcal{T}$, an *epistemic-doxastic model* is a tuple $\langle W, f, \{\sim_i, \rho_i\}_{i \in I} \rangle$, where $\langle W, f, \{\sim_i\}_{i \in I} \rangle$ is an epistemic model for $\mathcal{T}$ and $\rho_i : W \times H \longrightarrow W$ is a *selection function* such that, for all state w, history h and agent i :

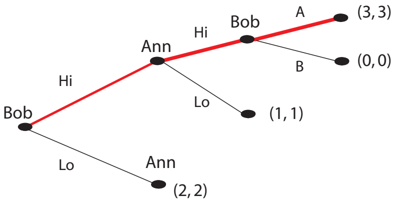
1. h is reachable from the root by playing $f(\rho_i(w, h))$.
2. If h is reachable from the root by playing in $f(w)$ then $w = \rho_i(w, h)$.
3. $f(\rho(w, h)) = f(w)$ on the sub-tree below h.

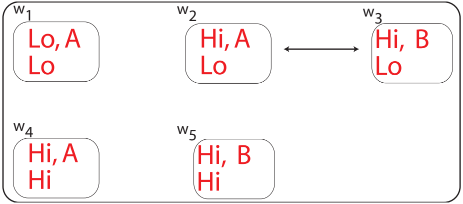
- Intuition: ρ_i assigns to each pair (w, h) the state w' where h is played and which is the most similar to w according to i.

An example

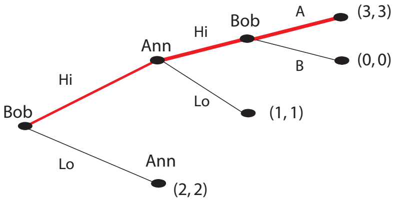


An example

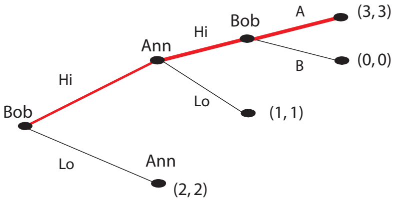

Bob
Hi
Lo
Ann
Hi
Lo
Bob
A
B
(3, 3)
(0, 0)
(1, 1)
Ann
(2, 2)


w_1
Lo, A
Lo
w_2
Hi, A
Lo
w_3
Hi, B
Lo
w_4
Hi, A
Hi
w_5
Hi, B
Hi

An example

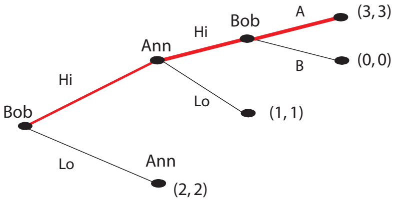


An example



- Take the following selection function (which satisfies conditions 1, 2 and 3):

An example



- Take the following selection function (which satisfies conditions 1, 2 and 3):

	Lo	Hi	Hi, Lo	Hi, Hi	Hi, Hi, A	Hi, Hi, B
w_1	w_1	w_2	w_2	w_4	w_4	w_5
w_2	w_1	w_2	w_2	w_4	w_4	w_5
w_3	w_1	w_3	w_3	w_5	w_4	w_5
w_4	w_1	w_4	w_2	w_4	w_4	w_5
w_5	w_1	w_5	w_3	w_5	w_4	w_5

Rationality in epistemic models: Stalnaker

- Rationality at a state-history pair:

Rationality in epistemic models: Stalnaker

- Rationality at a state-history pair:

Definition

Player i is *rational$_S$* at history h given that the game playing situation is state w whenever he is rational$_A$ at $\rho_i(w, h)$.

Rationality in epistemic models: Stalnaker

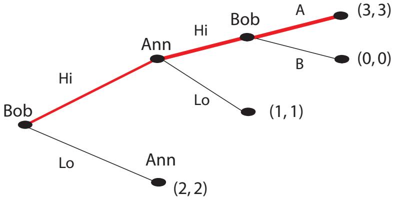
- Rationality at a state-history pair:

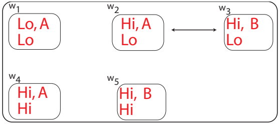
Definition

Player i is *rational*$_S$ at history h given that the game playing situation is state w whenever he is rational$_A$ at $\rho_i(w, h)$.

- Intuition: a strategy is rational at a given history and state if, in the closest or most plausible scenario, it is not strictly dominated.

An example

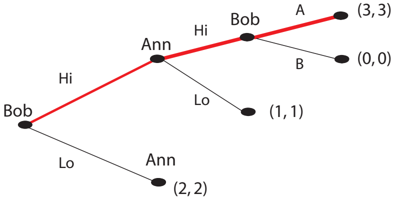


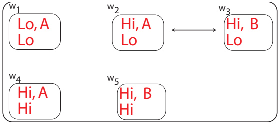


$$\rho(w_1, Hi) = w_2 \text{ and } \rho(w_1, Hi, Hi) = w_4$$

- Ann is not rational$_A$ at the pair (w_1, Hi) and, indeed, the BI-profile is not played here.

An example





$$\rho(w_1, Hi) = w_2 \text{ and } \rho(w_1, Hi, Hi) = w_4$$

- Ann and Bob *are* rational$_S$ at w_1, and this fact is common knowledge.

Rationality$_S$ and backward induction

Theorem

(Aumann, 1994) Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

Theorem

(Stalnaker, 1996) Common knowledge of rationality$_B$ at a state w of an epistemic-doxastic model for an extensive game $\mathcal{T}$ does not imply that $f(w)$ is the backward induction solution of that game.

- Remarks :

Rationality$_S$ and backward induction

Theorem

(Aumann, 1994) Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

Theorem

(Stalnaker, 1996) Common knowledge of rationality$_B$ at a state w of an epistemic-doxastic model for an extensive game $\mathcal{T}$ does not imply that $f(w)$ is the backward induction solution of that game.

- Remarks :
  - Rationality in extensive games is sensitive to *counterfactual* reasoning.

Rationality$_S$ and backward induction

Theorem

(Aumann, 1994) Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

Theorem

(Stalnaker, 1996) Common knowledge of rationality$_B$ at a state w of an epistemic-doxastic model for an extensive game $\mathcal{T}$ does not imply that $f(w)$ is the backward induction solution of that game.

- Remarks :
  - Rationality in extensive games is sensitive to *counterfactual* reasoning.
  - Rationality$_S$ does not avoid the "paradox" of BI.

Theorem

(Aumann, 1994) Given an extensive games of perfect information $\mathcal{T}$ (*) and an epistemic model $\mathbb{M}$ for it, if at a state w rationality$_A$ is common knowledge then $f(w)$ is the backward induction solution of that game.

Theorem

(Stalnaker, 1996) Common knowledge of rationality$_B$ at a state w of an epistemic-doxastic model for an extensive game $\mathcal{T}$ does not imply that $f(w)$ is the backward induction solution of that game.

- Remarks :
  - Rationality in extensive games is sensitive to *counterfactual* reasoning.
  - Rationality$_S$ does not avoid the "paradox" of BI.
  - Recent developments on this issue: Baltag, Smets, Zvesper (2008).

Epistemic Program in Game Theory

Fundamental Problem: What does it mean to say that the players in a game are **rational**, each **thinks** each other is rational, each thinks each other thinks the others are rational, etc.?

Epistemic Program in Game Theory

Fundamental Problem: What does it mean to say that the players in a game are **rational**, each **thinks** each other is rational, each thinks each other thinks the others are rational, etc.?

Epistemic Program in Game Theory: An explicit description of the players' beliefs is part of the basic description of a game (even games with complete information).

Epistemic Program in Game Theory

Fundamental Problem: What does it mean to say that the players in a game are **rational**, each **thinks** each other is rational, each thinks each other thinks the others are rational, etc.?

Epistemic Program in Game Theory: An explicit description of the players' beliefs is part of the basic description of a game (even games with complete information).

Identify for any game the strategies that are chosen by rational and intelligent players who know the strucutre of the game, the preference of the other players and recognize each others rationality and beliefs.

Literature See, for example,

R. Aumann. *Interactive Epistemology I & II*. International Journal of Game Theory (1999).

Literature See, for example,

R. Aumann. *Interactive Epistemology I & II*. International Journal of Game Theory (1999).

P. Battigalli and G. Bonanno. *Recent results on belief, knowledge and the epistemic foundations of game theory*. Research in Economics (1999).

Literature See, for example,

R. Aumann. *Interactive Epistemology I & II*. International Journal of Game Theory (1999).

P. Battigalli and G. Bonanno. *Recent results on belief, knowledge and the epistemic foundations of game theory*. Research in Economics (1999).

B. de Bruin. *Explaining Games*. Ph.D. Thesis, ILLC (2004).

Literature See, for example,

R. Aumann. *Interactive Epistemology I & II*. International Journal of Game Theory (1999).

P. Battigalli and G. Bonanno. *Recent results on belief, knowledge and the epistemic foundations of game theory*. Research in Economics (1999).

B. de Bruin. *Explaining Games*. Ph.D. Thesis, ILLC (2004).

A. Brandenburger. *The Power of Paradox: Some Recent Developments in Interactive Epistemology*. International Journal of Game Theory (2007).

Epistemic Characterizations of Solutions Concepts

If the players all satisfy some **epistemic condition** involving some form of **rationality** (eg., common knowledge of rationality) then the players will play according to some solution concept (eg., Nash equilibrium, iterated removal of strongly dominated strategies, . . .).

Epistemic Characterizations of Solutions Concepts

If the players all satisfy some **epistemic condition** involving some form of **rationality** (eg., common knowledge of rationality) then the players will play according to some solution concept (eg., Nash equilibrium, iterated removal of strongly dominated strategies, ...).

The key "axioms" and assumptions:

1. Players know there own strategies (and types)
2. Players are expected utility maximizers
3. The above facts are common *knowledge*
4. Players do not completely rule out choices of the other players
5. The players do not have any (soft) information about the other players

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given.

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given. Neither is it meant as a description of what human beings actually do in interactive situations.

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given. Neither is it meant as a description of what human beings actually do in interactive situations. The most appropriate term is perhaps "analytic"; it asks, what are the implications *of rationality in interactive situations? Where does it lead?*

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given. Neither is it meant as a description of what human beings actually do in interactive situations. The most appropriate term is perhaps "analytic"; it asks, what are the implications *of rationality in interactive situations? Where does it lead? This question may be as important as, or even more important than, more direct "tests" of the relevance of the rationality hypothesis.*

R. Aumann. *Irrationality in Game Theory*. 1992.

Dynamic Analysis of Solution Concepts

Dynamic Analysis of Solution Concepts

From Games to Logic: Given some algorithmic algorithm defining a solution concept, try to find epistemic actions driving its dynamics

Dynamic Analysis of Solution Concepts

From Games to Logic: Given some algorithmic algorithm defining a solution concept, try to find epistemic actions driving its dynamics

From Logic to Games: Any type of epistemic assertion defines and iterated solution process.

J. van Benthem. *Rational Dynamics and Epistemic Logic in Games*. IJGT, 2007.

Logics for Games

Recognize that (extensive, strategic) games form a class of modal models.

G. Bonanno. *Modal logic and game theory: Two alternative approaches*. Risk Decision and Policy 7 (2002).

J. van Benthem. *Extensive Games as Process Models*. JOLLI 11 (2002).

Logics for Games

Recognize that (extensive, strategic) games form a class of modal models.

G. Bonanno. *Modal logic and game theory: Two alternative approaches*. Risk Decision and Policy 7 (2002).

J. van Benthem. *Extensive Games as Process Models*. JOLLI 11 (2002).

What is a "good language" for expressing properties of these structures?

Example

- $[i]\varphi$: "φ holds in all states at least as preferable to the present one"
- $[\sigma]\varphi$: "if from here all players adhere to σ, then play will eventually end in a state in which φ holds"
- $[i, \sigma]\varphi$: "φ holds in all states that will be reached if all the players except possibly i play the strategy σ"

Example

- $[i]\varphi$: "φ holds in all states at least as preferable to the present one"
- $[\sigma]\varphi$: "if from here all players adhere to σ, then play will eventually end in a state in which φ holds"
- $[i,\sigma]\varphi$: "φ holds in all states that will be reached if all the players except possibly i play the strategy σ"

Fact: σ is a subgame perfect Nash equilibrium iff

$$\mathcal{F} \models \bigwedge_{i\in\mathcal{A}} (\langle i,\sigma\rangle[i]\varphi \rightarrow [\sigma]\varphi)$$

P. Harrenstein, W. van der Hoek, J-J. Meyer and C. Witteveen. *A Modal Characterization of Nash Equilibrium*. Fundamenta Informaticae 57 (2003).

More Examples

Nash Equilibrium:

More Examples

Nash Equilibrium:

Backwards Induction:

$Win_i := \mu P.(\mathsf{end} \wedge \mathsf{win_i}) \vee (\mathsf{turn}_i \wedge \langle any(i)\rangle P) \vee (\mathsf{turn}_j \wedge [any(j)]P)$

J. van Benthem. *Extensive Games as Process Models*. JOLLI 11 (2002).

More Examples

- (WR') $s_i^k \rightarrow \neg B_i(s_i^l \prec_i s_i^k)$
- (SR') $s_i^k \rightarrow \neg(B_i(s_i^l \preceq_i s_i^k) \wedge \neg B_i \neg(s_i^l \prec_i s_i^k)$

More Examples

- (WR') $s_i^k \rightarrow \neg B_i(s_i^l \prec_i s_i^k)$
- (SR') $s_i^k \rightarrow \neg(B_i(s_i^l \preceq_i s_i^k) \wedge \neg B_i \neg(s_i^l \prec_i s_i^k)$

Facts:

1. WR' characterizes Iterated Removal of Strongly Dominated Strategies (on $KD45$ frames).
2. SR' characterizes Iterated Removal of Weakly Dominated Strategies (on $S5$ frames).

with specific interpretations of the propositional variables

G. Bonanno. *A Syntactic Approach to Rationality in Games with Ordinal Payoffs*. 2007.

W. van der Hoek and M. Pauly. *Modal Logic for Games and Information*. in Handbook of Modal Logic (2007).

It is as if every time we think we finally get a hold on what rational behaviour means, we find ourselves having grasped only a shadow. Maybe this means that rationality is something belonging to the gods themselves, and that should not be stolen from them. Maybe it is the tree of knowledge itself that we should not touch?

J-F. Mertens. *Stable equilibria a reformulation. Part 1. Definition and basic properties*. Math Operations Research (1989).

Thank You and Merci!