Reasoning, Games, Action and Rationality

Lecture 3

ESSLLI'08, Hamburg

Eric Pacuit — Stanford University — ai.stanford.edu/~epacuit

Olivier Roy — University of Groningen — philos.rug.nl/~oroy

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Plan for Today

- Hard knowledge and Nash equilibrium.
- Prior beliefs, mixed strategies and equilibrium of beliefs.

Generic and Specific Expectations

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- Common beliefs of rationality is a *generic* kind of expectation: Independent of the game structure.
- In many games these expectations do not exclude any strategy.
- What about more specific expectations?

More Specific Expectations

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- What happens if the players have *correct beliefs* about each others' choices?

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- If Ann believes that Bob plays **A**, the only rational choice for her is **a**.
- The same hold for Bob.
- If, furthermore, these beliefs are *true*, then **aA** is played.

Nash Equilibrium

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- Intuitions behind Nash equilibrium:
  - Best response given the choices of others.
  - No regret.

Knowledge of Strategies and Nash Equilibrium

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- In general:

Theorem

(Aumann and Brandenburger, 1995) For any two-players strategic game and model for that game, if at state w both players are rational and "know" the other's strategy choice, then $\sigma(w)$ is a Nash equilibrium.

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  - 2 players (more on this in the notes).
  - Hard knowledge, or even correct beliefs, are very *specific*: Ann knows that Bob is playing **A**. How can the agents have such information? Is it something we can expect to happen?

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1. *Player 2 remains rational after the announcement of player 1's choice.*
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What remains to be said when they have so much information?

break

Equilibrium play

- Question: can we understand equilibrium play as resulting from more *generic* information or expectations?
  - Yes: as equilibrium *of posterior beliefs* given common *prior* beliefs and common knowledge of rationality.

Prior and posterior beliefs

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- At the *ex ante* stage the players do not have any specific information about which profile will be played. In particular, they didn't make up their mind. *Prior beliefs*.
- At the *ex interim* stage they know more, *at least* they know what they have chosen. *Posterior beliefs*.

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A type space $\mathbb{T}$ is *generated* by the set of priors $\{p_i\}_{i \in I}$ whenever, for every state (σ, t) and set of states E:

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Where E_i is defined as the set of pairs (σ'_{-i}, t'_{-i}) such that $(\sigma', t') \in E$.

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Intuition: each player's beliefs at a state (σ, t) are generated by conditioning the prior on him choosing σ_i and being of type t_i.

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- Think of a card game.

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- *Harsanyi doctrine* to justify *common* prior assumption.

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- Common but not uncontroversial assumption.

S. Morris. *The Common Prior Assumption in Economic Theory. Economics and Philosophy*, 11(2):227253, 1995.

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- Assume that $\mathbb{T}_{Ann} = \{t_{Ann}\}$ and $\mathbb{T}_{Bob} = \{t_{Bob}\}$.
- At state $(aA, t_{Bob}t_{Ann})$ Bob is certain about his strategy choice:

$$\lambda_{Bob}(t_{Bob})(A_{Bob}) = \frac{p(A)}{p(A)} = 1$$

but Ann is not certain about Bob's choice:

$$\lambda_{Ann}(t_{Ann})(A_{Ann}) = \frac{p(A)}{p(a)} = 1/2$$

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  - As *objects of choice*.
  - As *beliefs* of the others about what one will choose.
    - In particular, in two-players games, first-order beliefs can be naturally read as mixed strategies.

Mixed strategies as beliefs

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- Vice-versa for the mixed strategy ρ_{Bob}.

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Theorem

(Aumann 1987, Aumann and Dreze 2005) At any state (σ, t) in a type structure $\mathbb{T}$ for a game $\mathbb{G}$, if λ_i is generated by a non-correlated common prior p for all player i, and all players are rational at all states (σ, t), then the profile ρ induced by $\mathbb{T}$ at (σ, t) is a Nash equilibrium of $\mathbb{G}$.

- Remarks:
  - *Non-correlated* common prior.

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  - Equilibrium play as equilibrium of beliefs.
  - For Aumann, CPA and CKR are inherent to the notion of interactive rationality.

Tomorrow

- Not excluding any eventualities and “admissible” strategies.