Reasoning, Games, Action and Rationality

Lecture 2

ESSLLI'08, Hamburg

Eric Pacuit Olivier Roy

Stanford University University of Groningen

ai.stanford.edu/~epacuit philos.rug.nl/~epacuit

August 11, 2008

Plan for Today

- Logic for Knowledge, beliefs and preferences in games.
- Expectation 1: Common knowledge of rationality

Logics, logic and logic

- Logic
- is
- great.

break

Expectation 1: Rationality and common belief of rationality

- What happens if all players are rational, believe that all players are rational, believe that all players believe that (...)?

Expectation 1: Rationality and common belief of rationality

- What happens if all players are rational, believe that all players are rational, believe that all players believe that (...)?
- "Classical" assumption about game-theoretic analysis. See e.g. Myerson (1991).

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- B is a bad strategy for Bob.

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- B is a bad strategy for Bob.
- It is *never* rational for him to choose B.

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

$$EV_{t_B}(B) \geq EV_{t_B}(A)$$

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

$$v_{Bob}(aB)\lambda_{Bob}(t_{Bob})(a) + v_{Bob}(bB)\lambda_{Bob}(t_{Bob})(b) \geq v_{Bob}(aA)\lambda_{Bob}(t_{Bob})(a) + v_{Bob}(bA)\lambda_{Bob}(t_{Bob})(b)$$

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

$$1\lambda_{Bob}(t_{Bob})(a)+0\lambda_{Bob}(t_{Bob})(b) \geq 2\lambda_{Bob}(t_{Bob})(a)+1\lambda_{Bob}(t_{Bob})(b)$$

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

$$\lambda_{Bob}(t_{Bob})(a) \geq 2\lambda_{Bob}(t_b)(a) + \lambda_{Bob}(t_{Bob})(b)$$

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

$$0 \geq \lambda_{Bob}(t_{Bob})(a) + \lambda_{Bob}(t_{Bob})(b)$$

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- A type t_B of Bob would be rational in choosing B iff:

$$0 \geq \lambda_{Bob}(t_{Bob})(a) + \lambda_{Bob}(t_{Bob})(b)$$

$$\text{But } \lambda_{Bob}(t_{Bob})(a) + \lambda_{Bob}(t_{Bob})(b) = 0!$$

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- Bob never plays **B** at state (σ, t) if he is rational at that state.

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- Bob never plays **B** at state (σ, t) if he is rational at that state.
- But then if Ann's type at that state believes that Bob is rational,

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- Bob never plays **B** at state (σ, t) if he is rational at that state.
- But then if Ann's type at that state believes that Bob is rational, that type must assign probability 1 to Bob playing **A**.

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- Bob never plays **B** at state (σ, t) if he is rational at that state.
- But then if Ann's type at that state believes that Bob is rational, that type must assign probability 1 to Bob playing **A**.
- Given this belief, **a** is her only rational strategy.

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- If Ann and Bob are rational, and Ann believes that Bob is rational at state (σ, t),

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- If Ann and Bob are rational, and Ann believes that Bob is rational at state (σ, t), then $\sigma =$ aA.

Example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

- If Ann and Bob are rational, and Ann believes that Bob is rational at state (σ, t), then $\sigma =$ aA.
- This strategy profile is the only one that survives *iterated elimination of strictly dominated strategies*.

Strictly dominated strategies

Definition

A strategy s_i is *strictly dominated* by another strategy s_i' iff for all combinations of choices of the other players σ_{-i} :

$$v_i(s_i, \sigma_{-i}) < v_i(s_i', \sigma_{-i})$$

Iterated elimination of strictly dominated strategies

1. Start with a game;

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;
3. Look at the reduced game;

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;
3. Look at the reduced game;
4. Eliminate all strictly dominated strategies here;

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;
3. Look at the reduced game;
4. Eliminate all strictly dominated strategies here;
5. Repeat 3 and 4 until you don't eliminate anything.

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;
3. Look at the reduced game;
4. Eliminate all strictly dominated strategies here;
5. Repeat 3 and 4 until you don't eliminate anything.

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;
3. Look at the reduced game;
4. Eliminate all strictly dominated strategies here;
5. Repeat 3 and 4 until you don't eliminate anything.

	A
a	1, 2
b	0, 1

Iterated elimination of strictly dominated strategies

1. Start with a game;
2. Eliminate all strictly dominated strategies;
3. Look at the reduced game;
4. Eliminate all strictly dominated strategies here;
5. Repeat 3 and 4 until you don't eliminate anything.

	A
a	1, 2

Common knowledge of rational and elimination of strictly dominated strategies

- If Ann and Bob are rational, and Ann believes that Bob is rational at state (σ, t), then $\sigma = \text{aA}$.

Common knowledge of rational and elimination of strictly dominated strategies

- If Ann and Bob are rational, and Ann believes that Bob is rational at state (σ, t), then $\sigma = \text{aA}$.
- For this game we need rationality and only one level of higher-order information to conclude that aA will be played.

Common knowledge of rational and elimination of strictly dominated strategies

- If Ann and Bob are rational, and Ann believes that Bob is rational at state (σ, t), then $\sigma =$ aA.
- For this game we need rationality and only one level of higher-order information to conclude that aA will be played. But in the general case:

Theorem

For any state (σ, t) of a type structure for an arbitrary finite game $\mathbb{G}$, if all players are rational and it is common belief that all players are rational at (σ, t), then σ is a iteratively non-dominated strategy profile.

A. Brandenburger and E. Denkel. *Rationalizability and correlated equilibria*. *Econometrica*, 55:13911402, 1987.

Comments on characterization results

- If [such and such expectations] at state (σ, t), then [such and such *solution concept*] is played at that state.

Comments on characterization results

- If [such and such expectations] at state (σ, t), then [such and such *solution concept*] is played at that state.
- *Solution concepts*: proposal as to what is *rational* in a game. (traditionally)

Comments on characterization results

- If [such and such expectations] at state (σ, t), then [such and such *solution concept*] is played at that state.
- *Solution concepts*: proposal as to what is *rational* in a game. (traditionally)
- What about the converse?

Comments on characterization results

- If [such and such expectations] at state (σ, t), then [such and such *solution concept*] is played at that state.
- *Solution concepts*: proposal as to what is *rational* in a game. (traditionally)
- What about the converse?
  - If [such and such *solution concept*] then one can build a state in a model such that [such and such expectations] hold.

Alternative (Dynamic) Characterization

J. van Benthem. Rational. *Rational dynamic and epistemic logic in games*. In S. Vannucci, editor, Logic, *Game Theory and Social Choice III*, 1923. University of Siena, department of political economy, 2003..

Alternative (Dynamic) Characterization

J. van Benthem. Rational. *Rational dynamic and epistemic logic in games*. In S. Vannucci, editor, Logic, *Game Theory and Social Choice III*, 1923. University of Siena, department of political economy, 2003..

- Recover the *dynamic* aspect of elimination of strictly dominated strategies.
- Understand how common knowledge of rationality *arises*.

Alternative (Dynamic) Characterization

- Some restrictions:

Alternative (Dynamic) Characterization

- Some restrictions:
  1. Consider only hard information (knowledge).

Alternative (Dynamic) Characterization

- Some restrictions:
  1. Consider only hard information (knowledge).
  2. Work with *full models*:
     - 1 to 1 correspondence between the profiles and the states.

Alternative (Dynamic) Characterization

- Some restrictions:
  1. Consider only hard information (knowledge).
  2. Work with *full models*:
     - 1 to 1 correspondence between the profiles and the states.
     - $\sigma_i(w) = \sigma_i(w')$ if *and only if* $w \sim_i w'$.

Full model: an example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

Full model: an example

	A	B
a	1, 2	0, 1
b	0, 1	1, 0

$$\begin{tikzcd}
aA \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dashed] & aB \arrow[d, leftrightarrow, dashed] \\
bA \arrow[r, leftrightarrow] & bB
\end{tikzcd}$$

Announcing rationality

- van Benthem's proposal: look at what happens when the players start to publicly announce that they are rational.

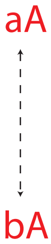
Announcing rationality

- van Benthem's proposal: look at what happens when the players start to publicly announce that they are rational.

$$
\begin{tikzcd}
aA \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dashed] & aB \arrow[d, leftrightarrow, dashed] \\
bA \arrow[r, leftrightarrow] & bB
\end{tikzcd}
$$

Announcing rationality

- van Benthem's proposal: look at what happens when the players start to publicly announce that they are rational.



After announcing that everybody is rational

Announcing rationality

- van Benthem's proposal: look at what happens when the players start to publicly announce that they are rational.

aA

After announcing once more that everybody is rational.

Announcing rationality

- Again, here we only needed two announcements to reach the state where aA is played.

Announcing rationality

- Again, here we only needed two announcements to reach the state where aA is played. In general:

Theorem

(van Benthem, 2003) The following are equivalent for states w in full game models:

1. $\sigma_i(w)$ *is a iteratively non-dominated strategy for all* i.
2. w *is in the submodel which remains after any number of successive announcement of rationality.*

Remarks and open questions

Theorem

(van Benthem, 2003) The following are equivalent for states w in full game models:

1. $\sigma_i(w)$ *is a iteratively non-dominated strategy for all* i.
2. w *is in the submodel which remains after any number of successive announcement of rationality.*

- How to reach common knowledge.

Remarks and open questions

Theorem

(van Benthem, 2003) The following are equivalent for states w in full game models:

1. *$\sigma_i(w)$ is a iteratively non-dominated strategy for all i.*
2. *w is in the submodel which remains after any number of successive announcement of rationality.*

- How to reach common knowledge.
- This is an *if and only if*.

Remarks and open questions

Theorem

(van Benthem, 2003) The following are equivalent for states w in full game models:

1. $\sigma_i(w)$ *is a iteratively non-dominated strategy for all* i.
2. w *is in the submodel which remains after any number of successive announcement of rationality.*

- How to reach common knowledge.
- This is an *if and only if*. Why?
  - Full models?

Remarks and open questions

Theorem

(van Benthem, 2003) The following are equivalent for states w in full game models:

1. $\sigma_i(w)$ *is a iteratively non-dominated strategy for all* i.
2. w *is in the submodel which remains after any number of successive announcement of rationality.*

- How to reach common knowledge.
- This is an *if and only if*. Why?
  - Full models?
- What happens in general?

Tomorrow

- Hard knowledge and Nash equilibrium.
- Prior beliefs, correlated beliefs and Nash equilibrium.