Many Agent States of Knowledge

Ann is sitting on a chair in front of which there is a vase with a dozen roses in it. Bob can see both Ann and the roses. Charlie can see Ann and Bob and the roses.

We could now ask:

- *Does Ann know p?* where $p =$ There are roses in front of her. I.e., $K_a(p)$?

- Does Bob know that she knows? $(K_b K_a(p)\,?)$

- Does Charlie know that Bob knows that Ann knows $(K_c K_b K_a(p))$?

Both common sense and the corresponding Kripke structure tell us that the answer to all three questions is *yes*. Indeed if they can see each other then p is *common knowledge* among them.

Let us now change the meaning of p. In this new example, Ann, Bob and Charlie are all as before, but what is in front of Ann is not a vase of roses, but a blackboard with the number 1243 written on it. Let p now denote the fact that the number on the blackboard is composite.

Logically the situation is not changed. Since 1243 is composite (113 times 11), this is a necessary truth, Ann knows it, Bob knows that Ann knows it, and Charlie knows that Bob knows that Ann knows it.

But are *we* sure that this is the case? It could be that Ann finds numbers greater than 100 to be a mystery. Or perhaps she *is* actually a number theorist but sexist Bob thinks that she is number-challenged. Or perhaps Bob knows her quite well, but Charlie thinks that Bob is a chauvinist who has a poor opinion of the mathematical abilities of women.

So we are no longer sure that $K_a(p)$, $K_bK_a(p)$ and $K_cK_bK_a(p)$ are all true.

Consider now the following game. Ann is sitting (again) in a chair in front of a blackboard on which the number n is written. In front of her are three buttons, 1, 2, 3. Bob can see her and the blackboard, and Charlie can see both Ann and Bob and the blackboard. Bob and Charlie also have buttons. No one can see the buttons of the other people.

The Game:
Ann should push button 1 if she thinks n is prime, button 2 if she thinks it is composite, and button 3 if she does not know.
If she presses the right button she gets one Euro. If she guesses wrong, she *pays* $\mathcal{E}10$. And if she presses 3, there is no gain or loss.

Bob has four buttons, and he should press a button corresponding to Ann's if he knows which button it is, and he presses button 4 if he does not know. If he guesses right, he gets $\mathcal{E}1$, if he guesses wrong, he pays $\mathcal{E}10$, and if he presses 4, no gain or loss.

Charlie has 5 buttons, buttons 1-4 to indicate what he thinks Bob pressed, and button 5 if he does not know. His payments are similar to Bob's.

If p denotes the fact that n is composite, then we ought to have $K_a(p)$, $K_bK_a(p)$ and $K_cK_bK_a(p)$. Thus all three should press button 2, all of them getting $\mathcal{E}1$.

Will this happen? Not necessarily! As we saw, Ann may not realize that the number is composite, or if she does, Bob might think the number is too big for her to factorize etc. Thus in fact we do not have a definite map from physical situations to Kripke structures. The physical set up leaves out the mental facts, and there are many interpretations (not all of which are Kripke structures) for the *same* physical situation.

So how will the game be played? It depends, even if *some* of the three payers are logically omniscient. But note that Ann's best strategy is to press button 2 regardless of what Bob and Charlie press. *Given that she presses button 2*, Bob's best strategy is to press 2 also, and *given* that they are both pressing 2, Charlie should also play 2.

The *standard* Kripke structure that we get out of the physical situation does not necessarily represent the mental situation, but *it does represent the unique Nash equilibrium.*

A Formalism:

Suppose we have a finite n-agent Kripke structure $\mathcal{M}$. The set of states is W with cardinality m. We use this structure to construct a game $\mathcal{G}$. Each agent is told what $\mathcal{M}$ loooks like. Moreover, each agent has a set of symbols corresponding to the (finitely many) equivalence classes of that agent. I.e. the space W splits into finitely many pieces which are the equivalence classes of the agent's accessibility relation and the agent has a symbol for each such class. Thus each agent has his own alphabet. Let $[s]_i$ be i's symbol (equivalence class) for $s \in W$. Thus $s \sim_i t$ iff $[s]_i = [t]_i$. When the agent sees the symbol, he knows which equivalence class he is in, but not *where* he is in that class.

At any moment of time, some state $s \in W$ is picked with probability $1/m$. Then each agent i is given the symbol $[s]_i$. i is also given a finite set X_i of formulas with the following properties. Only atoms are negated in any formula – there are no other negations in any formula. The only connectives are $\wedge$, $\vee$, K_j, $L_j = \neg K_j \neg$. Every knowledge formula (without common knowledge) can be written in this way with all negations driven in using de Morgan's laws, etc.

If $A \wedge B$ $(A \vee B)$ is in X_i, then so are A, B.

If $K_j(A)$ or $L_j(A)$ is in X_i, then A is in X_j.

At time t, each agent i is asked to mark each formula in X_i of level $t - 1$ with a *yes*, or a *no*, or a *don't know*. The process goes on until all formulas have been marked. (We could have made this a one shot game, but the extended form is a bit prettier.)

After this, each agent gets $\mathcal{E}1$ for each formula **correctly marked**, $\mathcal{E}0$ for each **don't know**, and is fined $\mathcal{E}(m \times k)$, for each **incorrectly marked** formula, where m is the cardinality of W, and k is the cardinality of the finite set $\bigcup X_i$. A formula marked with *don't know* is not considered marked.

A literal (atomic formula or its negation) is considered correctly marked by i iff it is true and marked *yes*, or false and marked *no*. ("true, false" are relative to $\mathcal{M}, s$.)

Formulas $A \wedge B$ and $A \vee B$ are considered correctly marked by i if the yes/no corresponds to the truth value (of $K_i(A \wedge B)$ and $K_i(A \vee B)$) at state s.

A formula $K_j(A)$ is considered to be correctly marked by i if
either $K_j(A)$ is true and A marked *yes* by j or
$K_j(A)$ is marked *no* and either A is false, or A is not marked *yes* by j.

A formula $L_j(A)$ is considered correctly marked by i if either $L_j(A)$ is marked *yes*, it is true, and A is not marked *no* by j or $L_j(A)$ is false, marked *no*, and A is marked *no* by j.

Each agent may have a strategy for playing this game given by the Kripke structure and the sets X_i. We will say that an n-tuple $S = (s_1, ..., s_n)$ of strategies is **safe** for i if i does not have a negative expected value. It is **safe** if no agent makes an expected loss.

Clearly the strategy where some agent says *don't know* for all formulas, is safe for him. Indeed the *don't know* strategy is safe regardless of how the other agents play.

On the other hand, a strategy where an agent says *yes* for a formula A when he does not know A (i.e., where $s \models \neg K_i(A)$), can never be safe, because if he does not know A,, then there is probability at least $1/m$ that he is wrong once, and his loss of $m \times k$ will make up for all possible gains from other cases where he is accidentally right.

Definition: A **knowledge state** for n-agents is a set of *safe* strategies for them. A **belief state** is a set of not necessarily safe strategies.

A state where **every** agent marks formulas according to their knowledge value is safe.

Bob could have a false belief that Ann does not know that 1243 is composite. That is not (on the face of it) a false belief about the world, *but it is a false belief nonetheless. And if Bob has such a false belief, he will make a bad move and pay for it in our game.*

Theorem: The only Nash equilibrium is where each agent marks each formula correctly according to its value at s, where A is considered to be correctly marked by A if it is marked *yes* and $s \models K_i(A)$ or it is marked *no* and $s \models K_i(\neg A)$. (Formulas A where the agent does not know whether A should be marked with a *don't know*).

Proof: Straightforward by induction on formula complexity.

Definition: Let the **knowledge depth** $d(A)$ be the maximum length of a chain of embedded knowledge operators (K or L) in formula A. We will say that a strategy s of some agent is l-complete if the agent correctly marks all formulas of knowledge depth at most l.

Lemma: Suppose all agents other than i are ℓ-complete. Then agent i can safely be $(\ell + 1)$-complete.

Thus for agent i to infer to level $\ell + 1$ it is sufficient that other agents do infer to level ℓ. In the Ann, Bob, Charlie example, if Ann correctly infers facts (that p is true) then Bob can safely infer one level higher, and if he does, then Charlie can safely infer two levels higher.

Thus there can be evolution towards the Nash equilibrium as follows. Each agent can safely start by marking true all knowledge-free formulas which the Kripke structure says they know, and marking false all knowledge-free formulas which the Kripke structure says they know to be false. They are not dependent on other players being intelligent.

Suppose now that all the agents proceed from some level ℓ to $\ell + 1$. They are still safe since all agents were ℓ-complete. In a finite number of steps, they will arrive at a stage where all formulas A where agent i knows *whether A* according to the Kripke structure, have been marked. Now the agents have earned the maximum they possibly could and the Nash equilibrium has been reached.

We can make a stronger assertion. Starting with the strategy where all agents say *don't know* all the time, there is a sequence of changes where at each stage, *only one* agent changes his valuation of *one formula*, and which ends up with the Nash equilibrium. Moreover, no agent is unsafe at any stage of these transformations.

What about **common knowledge**? We could extend the game by saying that Ann and Bob can mark the formula $C_{a,b}(p)$ *yes*, provided it is true in the conventional sense and they *both* mark it *yes*. But now there is no individually safe way to proceed to this situation! They must do it together.

However, if the Kripke structure $\mathcal{M}$ does satisfy $C_{a,b}(p)$, then for each formula A of the form $K_a K_b K_a \ldots K_b(p)$ (for example) there is a way for the two agents to proceed to a stage where both agents mark A with *yes*.

We can now consider the case where some agents are – or are believed to be, logically deficient by other agents. Thus suppose that of agents 1,2,3, agents 1 and 2 are logically adequate, but they know that agent 3 has no notion that other people even have minds (perhaps he is autistic). All three are looking at a vase of flowers. Let p stand for *There is a vase of flowers.* Then p will be common knowledge among 1 and 2, and in fact, *that 3 knows p* will be common knowledge among 1 and 2. But p cannot be common knowledge among $\{1,2,3\}$, for 3 has no notion of what 1 and 2 are thinking! For example, 1 cannot mark $K_3K_1(p))$ *yes*, because he cannot count on 3 marking $K_1(p)$ *yes.* Thus the formula $K_1K_3K_1(p)$ fails to be true not because of a deficiency in 1, but because of a deficiency in 3.

Thus there will be a sort of Nash equlibrium where agents 1, 2 are doing their best *given* 3's deficiency!

This is an important fact. When we talk about Nash equlibria, we are implicitly attributing equal logical abilities to all agents, but if some agents are logically deficient, and known to be so, there can still be a 'Nash' equilibrium for the other players, which depends on their knowledge of how the deficient players have played. Spelling out 'W', 'A', 'L', 'K' in the presence of a dog is a common ploy to prevent the dog from getting too excited on hearing the word 'walk'.

Conclusion: We have defined a more general set of knowledge states than those provided by Kripke structures. Hopefully, this more flexible notion will allow us to address various puzzles like that of the **No Trade theorem**, or the issue of mathematical knowledge.

Single Agent Omniscience

Socrates: Do you see, Meno, what advances he has made in his power of recollection? He did not know at first, and he does not know now, what is the side of a figure of eight feet: but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows.

Meno: True.

Socrates: Is he not better off in knowing his ignorance?

Meno: I think that he is.

Socrates: If we have made him doubt, and given him the "torpedo's shock," have we done him any harm?

Meno: I think not.

Socrates: We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double space should have a double side.

Socrates: And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal?

Boy: Certainly, Socrates.

Socrates: What do you say of him, Meno? Were not all these answers given out of his own head?

Meno: Yes, they were all his own.

Socrates: And yet, as we were just now saying, he did not know?

Meno: True.

Socrates: But still he had in him those notions of his-had he not?

Meno: Yes.

Socrates: Then he who does not know may still have true notions of that which he does not know?

Meno: He has.

The soul, then, as being immortal, and having been born again many times, and having seen all things that exist, whether in this world or in the world below, has knowledge of them all; and it is no wonder that she should be able to call to remembrance all that she ever knew about virtue, and about everything.

A conclusion similar to that of Socrates also follows from the Kripke Semantics. The semantics for the logic of knowledge uses Kripke structures with an accessibility relation R, typically assumed to be reflexive, symmetric, and transitive. If we are talking about belief rather than knowledge, then R would be serial, transitive, and euclidean.

Then some formula ϕ is said to be believed (known) at state s iff ϕ is true at all states R-accessible from s. Formally,

$$s \models B(\phi) \quad \text{iff} \quad (\forall t)(sRt \rightarrow t \models \phi)$$

If follows immediately that if a formula is logically valid then it is true at all states and hence it is both known and believed. Moreover, if ϕ and $\phi \rightarrow \psi$ are believed at s then both are true at all t such that sRt, and hence ψ is true at all such t. Thus ψ is also believed at s. Moreover, a logically inconsistent formula can be neither known nor believed.

There have been suggestions, [8, 21] that these 'problems' can be dealt with by using tools like **awareness**, **impossible possible worlds**, or by referring to **computational complexity**. In our view, these methods do not address the fundamental problem, namely **what is it that we are trying to model?** In order to have a theory of knowledge, we need to have some criterion for "measuring" what someone knows.

The following two examples are from Daniel Kahneman's Nobel lecture, 2002.

> *A bat and a ball cost €1.10 in total. The bat costs €1 more than the ball. How much does the ball cost? Almost everyone reports an initial tendency to answer 10 cents because the sum €1.10 separates naturally into €1 and 10 cents, and 10 cents is about the right magnitude. Frederick found that many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students, and 56% (164/293) of students at the University of Michigan gave the wrong answer. Clearly, these respondents offered a response without checking it.*

Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in antinuclear demonstrations.

#6 Linda is a bank teller

#8 Linda is a bank teller and active in the feminist movement

89% of respondents rated item #8 higher in probability than item #6.

But the set of bank tellers who are active in the feminist movement is a proper subset (perhaps even a rather small subset) of the set of all bank tellers, so #8 *cannot* have higher probability than #6.

Imagine that Carol assigns probabilities of .3, .3, and .8 respectively to events $X, Y, X \cup Y$. One could say that these probabilities are **inconsistent**. But in fact nothing prevents Carol from *accepting* bets based on these probabilities. What makes them incoherent is that we can make Dutch book against Carol – i.e., place bets in such a way that no matter what happens, she will end up losing money.

For instance we can bet £3 on X, £3 on Y, and £2 against $X \cup Y$. If either X or Y happens, we earn £7 (at least), and lose (at most) £5, thus gaining £2. If neither happens, we gain £8 and lose £6, so that we again make a profit – and Carol makes a loss.

Thus incoherent beliefs, on this account, are **possible**, (and hence consistent) but unwise. It will be important to keep in mind this distinction between inconsistency and incoherence.

Hayek [12] considers an isolated person acting over a period according to a preconceived plan. The plan

> *may, of course, be based on wrong assumptions concerning external facts and on this account may have to be changed. But there will always be a conceivable set of external events which would make it possible to execute the plan as originally conceived.*

Beliefs are related here to an agent's plans, and it is implicit in Hayek's view that the agent believes that the world is such that his plans will work out (or have a good chance of doing so).

The belief states which are implicit in plans are more general than the belief states which correspond to Kripke structures,

Having a plan does not require that the plan be formulated explicitly in language, or even that the planner *has* a language. It is perfectly possible for an animal to have a plan and to a smaller extent, it is also possible for a pre-lingual child to engage in deliberate behaviour which is plan-like.

Animals Can also have Plans!

It wasn't until an ape saved a member of our own species that there was public awakening to the possibility of nonhuman kindness. This happened on August 16, 1996 when an eight-year old female gorilla named Binti Jua helped a three-year-old boy who had fallen eighteen feet into the primate exhibit at Chicago's Brookfield Zoo. Reacting immediately, Binti scooped up the boy and carried him to safety.

Frans de Waal

Next to the ridicule of denying an evident truth is that of taking much pains to defend it; and no truth appears to me more evident, than that beasts are endow'd with thought and reason as well as men.

David Hume

It is for instance possible to say that a chicken believes a caterpillar of a certain sort to be poisonous, and mean by that merely that it abstains from eating such caterpillars on account of unpleasant experiences connected with them.

Frank Ramsey

Reasoning is just trial and error in thought. Dennett (1996) calls animals capable of trial and error in thought "Popperian." The reference is to Popper's remark that **it is better to let one's hypotheses die in one's stead***. The Popperian animal is capable of thinking hypothetically, of considering possibilities without yet fully believing or intending them. The Popperian animal discovers means by which to fulfill its purposes by trial and error with inner representations. It tries things out in its head, which is, of course, quicker and safer than trying them out in the world. It is quicker and safer than either operant conditioning or natural selection. One of many reasonable interpretations of what it is to be rational is that being rational is being a Popperian animal. The question whether any non-human animals are rational would then be the question whether any of them are Popperian.*

Ruth Millian

(R) *A belief's truth condition is that which guarantees the fulfilment of any desire by the action which that belief and desire would combine to cause.*

Ramsey

logically omniscient humans?

Suppose that

- p stands for *Pandas live in Washington DC*,

- q stands for *Quine was born in Ohio*,

- r stands for *Rabbits are called* gavagai *at Harvard.*

Suppose that Jill believes that p is true and that q and r have the same truth values. Then she is allowing two truth valuations, $v = (t, t, t)$, and $v' = (t, f, f)$. Given a formula ϕ on p, q, r in disjunctive normal form, she can evaluate $v(\phi)$ and $v'(\phi)$. If both are t she can say that she believes ϕ. If both are f, she disbelieves ϕ, and otherwise she is suspending judgment. Then Jill **will** be logically ominiscient in this domain.

But note that she will actually have to **make the calculations** rather than just sit back and say, "Now **do** I believe ϕ?"

If beliefs are not (always) expressed by sentences, or coincide with propositions expressed by sentences, then we need *another way* of representing belief states, and then relate language-oriented belief states to some specific species of such belief states.

We shall consider two kinds of beliefs. **Non-linguistic beliefs** which may also be possessed by animals, and **linguistic beliefs** which can only be possessed by humans; adults and older children. Of course the last two groups will *also* have non-linguistic beliefs which must be somehow correlated with their linguistic beliefs.

Let $\mathcal{B}$ be the space (so far unspecified) of belief states of some agent. Then the elements of $\mathcal{B}$ will be identified with the choices which the agent makes.

Roughly speaking, if I believe that it is raining, I will take my umbrella, and if I believe that it is not raining, then I won't.

But clearly the choice of whether to take my umbrella or not is correlated with my belief only if I don't want to get wet. So my preferences enter **in addition** to my beliefs.

We assume that the agent has *some* space $\mathcal{P}$ of preferences, and that the choices are governed by the beliefs as well as the preferences.

We use $\mathcal{S}$ for the set of choice situations, and C for the set of choices.

Thus the set $\{U, \neg U\}$ could be a choice situation (with U standing for *take the umbrella*) and both U and $\neg U$ are elements of C.

An agent who does have language can also be subjected to a purely linguistic choice. If asked **Do you think it is raining?** the agent may choose from the set **{Yes,No,Not sure}**.

And it is going to be *usually* the case that the agent will choose U in the situation, $\{U, \neg U\}$ iff she chooses *Yes* in the situation where she hears *Do you think that it is raining?*

An agent **endorses** (agrees with) a sentence ϕ iff she chooses *Yes* when asked, *Do you think ϕ?*, and **denies** (or disagrees with) ϕ iff she chooses *No.* She may also choose *Not sure*, in which case of course she neither endorses nor denies.

An agent may endorse ϕ, $\phi \rightarrow \psi$, and either deny ψ or at least fail to endorse ψ. We will say in the first case that the agent is logically incoherent, and in the second that the agent is an incomplete reasoner – or simply incomplete.

Given the fact that $(\phi \wedge \neg\phi) \to \psi$ for arbitrary ψ is a tautology, it is obvious that an agent who is logically incoherent, but *not* incomplete, and endorses both ϕ and $\neg\phi$, will end up endorsing everything. Fortunately, most of us, though we *are* logically incoherent, tend also to be incomplete. If we endorse the statements that all men are equal, that Gandhi and Hitler are men, and that Gandhi and Hitler are not equal, we are still not likely to agree that pigs fly – which we would if we were also **complete.**

Elements of $\mathcal{B}$ cannot be identified with propositions, for an agent may agree to one sentence expressing a proposition and disagree (or not agree with) another sentence expressing the same proposition.

An agent in some state $b \in \mathcal{B}$ may agree with **Hesperus is bright this evening**, while disagreeing with **Phosphorus is bright this evening**.

Definition 0.1 *A belief state b is* **incomplete** *when there are sentences $\phi_1, ..., \phi_n$ which are endorsed in b, ψ follows logically from $\phi_1, ..., \phi_n$, and ψ is not endorsed in b.*

A belief state b is **incoherent** *when there are sentences $\phi_1, ..., \phi_n$ which are endorsed in b, ψ follows logically from $\phi_1, ..., \phi_n$, and ψ is denied in b.*

An entirely different sort of incoherence arises when an agent's linguistic behaviour does not comport with his choices.

If an agent prefers not to get wet (which we knew somehow), says that it is raining, and does not take her umbrella, she may well be quite coherent in her linguistic behaviour, but her linguistic behaviour and her non-linguistic choices have failed to match.

Deduction changes things. Meno's slave was initially both incoherent and incomplete. He believed that a square of side four had an area of eight. Since he knew some arithmetic and some geometry, his belief state was not actually coherent. At the end of the conversation with Socrates, he came to endorse the sentence, *The square whose side is the diagonal of a square of side two, has an area of eight.*

Beliefs can change not only as a result of deductions, as those Meno's slave did, they can also change as a result of experience, e.g., raindrops falling on your head, or as the result of hearing something, like *It is raining.*

Some technical details

We assume given a space $\mathcal{B}$ for some agent whose beliefs we are considering. The elements of $\mathcal{B}$ are the belief states of that agent, and these are not assumed to be sentences in Mentalese although for some restricted purposes they could be. There are three important update operations on $\mathcal{B}$ coming about as a result of (i) events observed, (ii) sentences heard, and (iii) deductions made. Elements of $\mathcal{B}$ are also used to make **choices**.

Thus in certain states of belief an agent may make the choice to take his umbrella and we could then say that the agent believes it is raining. Many human agents are also likely to make the choice to *say*, "I think it is raining and so I am taking my umbrella" but clearly only if the agent is English speaking. Thus two agents speaking different languages, both of whom are taking their umbrellas, but making different noises, have the same belief in one sense but a different one in another. Both of these will matter. Later on we shall look into the connection.

Deduction is an update which does not require an input from the outside. But it *can* result in a change in $\mathcal{B}$. Suppose for instance that Jane thinks it is clear, and is about to leave the building without an umbrella. She might say, "Wait, didn't I just see Jack coming in with a wet umbrella? It must be raining." The sight of Jack with a wet umbrella might not have caused her to believe that it was raining, perhaps she was busy with a phone call. But the memory of that wet umbrella may later cause a deduction to take place of the fact that it is raining.

Thus our three update operations are:

$$\mathcal{B} \times \mathcal{E} \rightarrow_e \mathcal{B}$$

A belief state gets revised by witnessing an event.

$$\mathcal{B} \times \mathcal{L} \rightarrow_s \mathcal{B}$$

A belief state gets revised through hearing a sentence.

Clearly Lois Lane will react differently to the sentences **Superman flew over the Empire State Building**, and **Clark Kent flew over the Empire State Buildling.**

Similarly, Kripke's Pierre will react differently to the questions, **Would you like a free trip to Londra?** and **Would you like a free trip to London?** Indeed in the second case he might **offer to pay** in order not to go to London!

$$\mathcal{B} \rightarrow_d \mathcal{B}$$

A deduction causes a change in the belief state (which we may sometimes represent as an **addition**).

Here E is the set of events which an agent may witness and $\mathcal{L}$ is some language which an agent understands (i.e., uses successfully). The last map $\rightarrow_d$ for deduction is non-deterministic as an agent in the same state of belief may make one deduction or another.

Finally, we also have a space $\mathcal{S}$ of **choice sets** where an agent makes a particular choice among various alternatives. This gives us the map

$$\mathcal{B} \times \mathcal{S} \rightarrow_{ch} \mathcal{B} \times C$$

An agent with a certain belief makes a choice among various alternatives.

If we want to explicitly include preferences, we could write,

$$\mathcal{B} \times \mathcal{P} \times \mathcal{S} \rightarrow_{ch} \mathcal{B} \times C$$

While $\mathcal{S}$ is the family of choice sets, C is the set of possible choices and $\mathcal{P}$ is *some* representation of the agent's preferences. Thus **{take umbrella, don't take umbrella}** is a choice set and an element of $\mathcal{S}$ but *take umbrella* is a choice and an element of C.

Among the choices that agents make are choices to assent to or dissent from sentences. But there is no **logical** reason why an agent who assents to "It is raining" must take an umbrella or a raincoat. It is just the more pragmatic choice to take the umbrella when one says that it is raining, because otherwise either one gets wet, or one is suspected of insincerity.

Of course an agent may prefer to get wet and in that case, saying "It is raining," and not taking an umbrella are perfectly compatible choices. This shows that an agent's preferences need to be taken into account when correlating the actions which an agent takes and what an agent believes. But we usually do not want to get wet and to make such choices, and usually we do not say what we do not believe. It does not work for us.

Thus our theory of an agent presupposes such a belief set $\mathcal{B}$, and appropriate functions $\rightarrow_e, \rightarrow_s, \rightarrow_d, \rightarrow_{ch}$. We can identify two different spaces $\mathcal{B}, \mathcal{B}'$ if they are bisimilar – they need not be isomorphic.

We can understand an agent (with some caveats) if what we *see* as the effects of these maps conforms to some theory of what an agent wants and what the agent thinks. And we succeed pretty well. *Contra* Wittgenstein, we not only have a theory of what a lion wants, and what it means when it growls, we even have theories for bees and bats.

Many beliefs are expressed (or so we think) by sentences.

We shall now introduce two notions of belief, e-belief (which is choice-based) and i-belief (which is sentence-based and only appropriate for language users). These two notions will be used to sort out various belief puzzles.

The Setting

In our setting we imagine an observer o who is pondering on what some agent i believes. We assume (for convenience) that o thinks of a proposition expressed by a sentence as a set of possible worlds where that sentence is true, but that the observee i need not even have a language or a notion of truth. However, it is assumed that i does have some plans. Even if i is just a dog digging for a bone, o understands that i has a plan and roughly what that plan is. And we shall use this plan to make it possible for o to attribute beliefs to i.

We also assume that there is a context C which is the set of **relevant** possible worlds, and that worlds outside C, even though they are there, are not considered in deliberating about i's belief or beliefs. It is not assumed that i knows that there are worlds outside C; in some sense i lives inside C, but we will assume that o does know that there are possible worlds outside C. The purpose of the context is to avoid considering cases which are possible but strange, like the laws of physics failing, or Bush suddenly joining the Green party. A plan is assumed to work in **normal** circumstances, and an agent i is only required to have the sort of belief which would be enough for the plan to be carried out in normal circumstances.

So let P be i's plan at the moment, and let $\pi(P)$ be the set of worlds w in C such that the plan is *possible* at w.

There are two senses in which the plan may be possible. One is that the plan can actually be carried out. For instance for the dog digging for a bone, that possibility means that the ground is soft enough. The other sense is that the plan actually yields i the benefit which i wants; in this case, the actual presence of a bone.

So formally, $\pi(P) = \{w| \; w \in C \land w \; enables \, P\}$.

Let ϕ be a sentence. Then $||\phi|| = \{w|w \models \phi\}$, the set of worlds where ϕ is true, is the *proposition* corresponding to the sentence ϕ. Note that if ϕ and ψ are logically equivalent, then $||\phi|| = ||\psi||$.

Definition 0.2 *We will say that i **e-believes** ϕ, $B_e^i(\phi)$ if $\pi(P) \subseteq ||\phi||$. We will suppress the superscript i when it is clear from context.*

It is obvious in terms of the semantics which we just gave that the statements "The dog e-believes that there is a bone where he is digging".

It is easy to see that if an agent e-believes ϕ and ψ then the agent also e-believes $\phi \wedge \psi$ and that if the agent e-believes ϕ and $\phi \rightarrow \psi$ then the agent e-believes ψ.

For if $\pi(P) \subseteq ||\phi||$, and $\pi(P) \subseteq ||\psi||$ then clearly $\pi(P) \subseteq ||\phi|| \cap ||\psi|| = ||\phi \wedge \psi||$. The proof for the other case is similar using the fact that $||\phi \rightarrow \psi|| = (C - ||\phi||) \cup ||\psi||$. Since $\pi(P)$ is contained in $||\phi||$, it is disjoint from $C - ||\phi||$. Hence it can be contained in $(C - ||\phi||) \cup ||\psi||$ if and only if it contained in $||\psi||$.

Oddly enough, creatures which do not use language do not suffer from a lack of logical omniscience!

A lot of logic goes along with e-belief, but only within the context of a single plan. For instance, one may drive a colleague to the airport for a two week vacation in Europe and then forget, and arrange a meeting three days later at which this colleague's presence is essential. But within the context of a single (short) plan, consistency and logical omniscience will tend to hold. The situation is more complex with multiple plans. And there is nothing to prevent an agent from having one e-belief in one plan and another contradicting e-belief in another plan. It is pragmatic considerations – the logical counterpart of avoiding Dutch Book – which will encourage the agent i to be consistent and to use logical closure.

Suppose someone has a plan P consisting of, "If ϕ then do α, else do β" and another plan P' consisting of "If ϕ then do γ, else do δ". Now we find him doing α and also doing δ (we are assuming that the truth value of ϕ has not changed). We could accuse him of being illogical, but there is no need to appeal to logic. For he is doing Dutch book against himself.

Presumably he assumed that $u(\alpha|\phi) > u(\beta|\phi)$ but $u(\alpha|\neg\phi) < u(\beta|\neg\phi)$. Thus given ϕ, α was better than β but with $\neg\phi$ it was the other way around. Similarly, $u(\gamma|\phi) > u(\delta|\phi)$, but $u(\gamma|\neg\phi) < u(\delta|\neg\phi)$. And that is why he had these plans. But then his choice of α, δ results in a loss of utility whether ϕ is true or not. If ϕ is true then he lost out doing δ and if ϕ is false, then he lost out doing α.

For a concrete example of this, suppose that on going out I advise you to take your umbrella, but fail to take mine. If it is raining, there will be a loss of utility for I will get wet. If it is not raining, there will be a loss of utility because you will be annoyed at having to carry an umbrella for no good reason. My choice that I advise you to take your umbrella, but fail to take mine, is not *logically* impossible. It just makes no pragmatic sense.

A similar argument will apply if someone endorses ϕ, endorses $\phi \rightarrow \psi$ and denies ψ. If such a person makes plans comporting with these three conditions, then he will make choices which do not maximise his utility.

If we assume that over time, people learn to maximise their utility (they do not always but often do), then they will 'learn' a certain amount of logic and they will make certain obvious logical inferences.

In [4] Bicchieri suggests that co-operation also comes about as a result of such a learning process. Such suggestions have of course also been made by many others. Since we are only considering the one-agent case here, we shall not go into this issue any further. See, however, our [24].

A Second Notion of Belief – language enters

We now define a second notion of belief which does *not* imply logical omniscience. This is a more self-conscious, language-dependent notion of belief.

For agents i who do have a language (assumed to be English from now on), their plan may contain linguistic elements. At any moment of time they have a finite stock of currently believed sentences. This stock may be revised as time passes. These agents may perform atomic actions from time to time, and also make observations which may result in a revision in their stock of believed sentences.

Thus Lois seeing Superman in front of her will add the sentence "Superman is in front of me", to her stock, but, since she does not know that Clark Kent is Superman, she will *not* add the sentence "Clark Kent is in front of me". Someone else may add the sentence "I see the Evening Star", but not the sentence "I see the Morning Star" at 8 PM on a summer night. A person who knows that $ES = MS$, may add the sentence, "Venus is particularly bright tonight." In any case, this stock consists of sentences and not of propositions.

The basic objects in the agents' **plans** are atomic actions and observations which may be active (one looks for something) or passive (one happens to see something). These are supplemented by the operations of concatenation (sequencing), *if then else*, and *while do*, where the tests in the *if then else* and *while do* are on *sentences*. There may also be recursive calls to the procedure: *find out if the sentence ϕ or its negation is derivable within the limits of my current resources, from my current stock of beliefs*. Thus if i's plan has currently a test on ϕ, then, to be sure, the stock of sentences will be consulted to see if ϕ or its negation is in the stock. But there may also be a recursive call to a procedure for deciding ϕ. If someone asks "Do you know the time?", we do not usually say, "I don't", but look at our watches. Thus consulting our stock of sentences is typically only the first step in deciding if some sentence or its negation can be derived with the resources we have.

Sentences and Propositions

Suppose for instance that Lois Lane has invited Clark Kent to dinner but he has not said yes or no. So she forms the plan,

While I do not have a definite answer one way or another, *if* I see Clark Kent, I will ask him if he is coming to dinner.

Here *seeing Clark Kent* is understood to consist of an observation followed by the addition of the sentence "I am seeing Clark Kent" to her stock.

Suppose now that she sees Superman standing on her balcony. She will *not* ask him if he is coming to dinner as the sentence "I am seeing Clark Kent" will not be in her stock of sentences. And this is the sense in which she does *not* know that when she is seeing Superman, she is also seeing Clark Kent. If she *suspects* that Clark Kent is Superman, then it may happen that her recursive call to the procedure "decide if I am seeing Clark Kent" will take the form of the question, "Are you by any chance Clark Kent, and if so, are you coming to dinner?" addressed to Superman.

Definition 0.3 *If an agent a comes to a point in her plan where her appropriate action is* If ϕ then do α else do β, *and she does α, then we will say that she* i-believes ϕ. *If, moreover, ϕ is true, and we believe that in a similar context she would judge it to be true only if it is true, then (within the context of this plan) we will say that she* i-knows ϕ.

A common example of such a plan is the plan to answer a question correctly. Thus if an agent is asked "Is ϕ true?", the agent will typically call the procedure "decide if ϕ is true", and then answer "yes", "no", or "I don't know" in the appropriate cases.

We no longer have the law that if the agent i-knows ϕ and ϕ implies ψ then the agent of necessity i-knows ψ. But if the agent has the resources to decide ϕ and the proof of ψ from ϕ is easy, then she might well also know ψ. But her notion of "easy" may be different from ours, and how much effort she devotes to this task will depend on her mood, how much energy she has, etc.

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